Investigating the Development of Students’ Knowledge of

Standard Algorithms in Algebra

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Objectives

The purpose of the present study was to investigate the development of students’ knowledge of standard algorithms in algebra. Of particular interest were whether various forms of problem-solving practice were more likely to lead students to discover a standard algorithm, what developmental learning trajectories were involved in learning (by discovery) a standard algorithm, and whether students who discover and use a standard algorithm had metacognitive awareness that they have done so.

Perspective and Educational Importance

An algorithm is defined here as a procedure that is guaranteed to lead to a solution, when all steps are applied correctly and in a predetermined order. The term “standard algorithm” is used to refer to algorithms that have come to be widely used by members of particular communities to solve certain types of problems (e.g., Morrow & Kenney, 1998). Different communities may have different standard algorithms; for example, US standard algorithms for adding two-digit numbers are not necessarily identical to those widely used in other countries (Morrow & Kenney, 1998). Also, for some types of problems, multiple standard algorithms exist and are widely used.

The teaching and learning of standard algorithms is a flashpoint issue in US mathematics education at present. The “math wars” have led to discussion and disagreement concerning whether standard algorithms should be explicitly taught, discovered by students, or even learned at all. Such conversations about standard algorithms have not been
accompanied by careful research investigating the development of students’ knowledge of standard algorithms, particularly at the middle and high school levels, which is the aim of the present work.

On the one hand, standard algorithms are a quite powerful and useful tool for mathematical problem solving (Morrow & Kenney, 1998). Such algorithms often become “standard” because they have historically been viewed as reasonably (if not maximally) efficient techniques for solving certain kinds of problems. Solvers with knowledge of standard algorithms can apply them automatically to complete a range of problems. However and on the other hand, the learning of and reliance on standard algorithms has sometimes come at the expense of students’ flexibility and conceptual understanding (National Council of Teachers of Mathematics, 2000; National Research Council, 2001). Solvers who only know a standard algorithm may be derailed when they encounter unfamiliar problems. In addition, students who memorize a standard algorithm may not know why the algorithm works (or is an efficient strategy); such rote knowledge of strategies is easily forgotten and inflexible (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986).

In sum, we want students to know how to execute efficient algorithms for solving problems, but they also need to have the conceptual knowledge that underlies flexible use of such strategies (National Council of Teachers of Mathematics, 2000; National Research Council, 2001; Skemp, 1978). At the elementary school level, a variety of activities have been successfully used to approach this goal, including the use of concrete manipulatives, the generation of non-standard algorithms by students, and the in-class sharing and comparing of student-generated algorithm (Carpenter et al., 1998; Fennema et al., 1996; Fuson et al., 1997;
Hiebert et al., 1996). Surprisingly, despite the promise of these activities, less effort has been devoted to applying them to the learning of more advance mathematics such as algebra.

Arguably, flexible knowledge of standard algorithms is paramount in algebra. US students’ tendency to develop rote knowledge of algebra algorithms has been well documented in national and international assessments (e.g., Beaton et al., 1996; Blume & Heckman, 1997; Lindquist, 1989; Schmidt et al., 1999). Given the prevalence of symbolic procedures in algebra, as well as algebra’s role as a critical "gatekeeper" course (earning a passing grade has become a de facto requirement for many educational and workplace opportunities), research on the development of students’ knowledge of standard algorithms in algebra is particularly needed.

The topic area of linear equation solving was chosen as the algebra subdomain for the present study, for three reasons. First, a standard algorithm exists in this subdomain. Most linear equations can be solved using the following steps: Distribute any parenthetical terms such as 3(x + 2). Combine like constant terms and like variable terms on each side of the equation. Next, move variable terms to one side of the equation and constant terms to the other side. And finally, divide both sides by the coefficient of the variable term. This algorithm (henceforth referred to as the standard algorithm or SA for linear equation solving) is commonly taught in many traditional algebra textbooks in the US (e.g., Dolciani et al., 1992; Foerster, 1990) and widely used by students and mathematicians to solve linear equations (Bernard & Bright, 1984; Carry et al., 1979; Gagné & Paradise, 1961). Second, many in the mathematics education community view linear equation solving as a “basic skill,” one where students should know efficient algorithms but also be able to use them flexibly (Ballheim, 1999). And third, although there has been research on students’ errors in
this subdomain (Matz, 1980, 1982; Sleeman, 1984, 1985), little is known about how students
develop flexible knowledge of standard algorithms to solve linear equations.

In addition to our more general focus on students’ discovery of standard algorithms,
we were also interested in whether students who received direct instruction on the SA for
solving linear equations would use this strategy in different frequencies than those who
discovered it. Direct instruction on standard problem-solving strategies (e.g., algorithms)
continues to be the norm in U.S. mathematics classrooms (Schmidt et al., 1999), despite
recent reform efforts recommending more student-centered approaches (National Council of
Teachers of Mathematics, 1989, 2000). Numerous studies have found conceptual knowledge
gains amongst students who are allowed to discover, generate, and compare multiple solution
strategies at the elementary school level (e.g., Carpenter et al., 1998), but critiques of student-
centered discovery methods continue to make a case for more teacher-centered approaches
(e.g., Mayer, 2004).

More specifically, recent evidence suggests that, despite the promise of student-
centered discovery approaches, there may be a place for direct instruction on problem solving
strategies. Schwartz and Bransford (1998) found that college students who actively compared
contrasting examples before hearing a lecture transferred their knowledge much more readily
than students who actively compared contrasting examples twice without hearing the lecture
or those who read and summarized a text before hearing the lecture. Schwartz and Bransford
(1998) emphasize that the initial comparison of contrasting examples guides students’
attention to important features in a domain, leading to the development of differentiated
domain knowledge. Once learners have sufficient differentiated knowledge in a domain, they
are in a position to deeply understand an expository explanation that is presented to them,
and thus may benefit from direct instruction (e.g., Beitzel & Derry, 2004). Schwartz and Bransford’s (1998) work suggests that an instructional routine optimal for the development of flexibility in the use of problem solving strategies would begin with discovery-oriented problem solving, allowing students to develop differentiated knowledge of possible solution methods in a domain. Once basic knowledge of strategies existed, students may benefit from more explicit instruction on using multiple strategies. The present study assesses this prediction. After a period of initial discovery-oriented problem solving, some students were provided a brief period of direct instruction on three alternative strategies. (This condition is referred to here as DSI or direct strategy instruction.)

Research Questions

We have five main research questions. First, did students discover and use the SA? Second, for students who did not discover the SA, what kinds of strategies did they use to solve equations? Third, did students who received direct strategy instruction use the SA more frequently than those who did not receive DSI? Four, how did students who discovered and used the SA compare to those who did not use the SA in terms of problem solving accuracy, conceptual knowledge, and flexibility? And fifth, what kinds of developmental trajectories did students experience as they discovered or failed to discover the SA?

Method

The present study was part of a larger project that aimed to look at the development of students’ strategies for solving linear algebraic equations (Star, 2004; Star & Madnani, 2004; Star & Rittle-Johnson, 2004). During the summer between their 6th and 7th grade
years, 153 students (94 female, 59 male) volunteered to participate in this research. The students had not received formal instruction on equation solving in school. Students were given a $50 gift certificate to a local bookstore in return for their participation.

All students attended one-hour sessions for five consecutive days (Monday through Friday) during the summer. Students sat individually at tables for the duration of each session, which was held in large seminar rooms on the campus of a local university.

A pretest was administered to all students during the first session on Monday. Students were given 10 minutes for the pretest; all students finished in the allotted time. The pretest assessed students’ knowledge of formal linear equation solving strategies and conceptual knowledge of equation solving. Immediately following the pretest, students received a scripted 20-minute benchmark lesson on linear equation-solving transformations. In this lesson, students were introduced to the four basic transformations used to solve linear equations: combining like variable or constant terms (COMBINE), using the distributive property (EXPAND), adding or subtracting a constant or variable term to both sides of an equation (MOVE), and multiplying or dividing both sides of an equation by a constant (DIVIDE). Students were not given any strategic instruction on how transformations could be chained together to solve an equation. Rather, the focus in instruction was strictly on pattern recognition: identifying which transformation could be used for particular patterns of symbols, and how that transformation was correctly applied. For example, students were shown the symbols, $2x + 3x$, and instructed that the COMBINE step allowed this expression to be rewritten as $5x$. The intent in this brief period of instruction was to provide novice algebra learners with sufficient or prerequisite (Zhu & Simon, 1987) knowledge to enable them to begin to solve very straightforward equations. The lesson provided students both
guided and independent practice at using each of the four transformations. At the conclusion of instruction, students were provided with an 8.5” by 11” card that summarized the instructed material. Students were told that they could refer to the card at any time during problem solving.

For the next three one-hour problem-solving sessions (Tuesday, Wednesday, and Thursday), students worked individually at their own pace through a booklet containing 31 linear equations to be solved (see Appendix A). The booklet was divided into three sections, and students completed only the problems in one section during each class. If a student finished the day’s problems before the end of the class, he/she was instructed to close his/her booklet and sit quietly.

If a student became stuck while attempting a problem, a researcher answered the student’s questions in a semi-standardized format. Specifically, the researcher corrected the student’s arithmetic mistakes (e.g., if the student multiplied 2 by 3 and got 5, the experimenter pointed out this error) or reminded him/her of the four possible transformations and how each was used. Researchers never gave strategic advice to students, such as suggesting which transformation to apply next or whether one method of solution was any better than another method. Students independently came up with their own choices for which transformations to apply.

After solving problems on their own for one session on Tuesday, students in the DSI condition received a brief, eight-minute period of direct strategy instruction on Wednesday. A researcher solved three equations on a blackboard. Each problem was solved using a different strategy and in the most efficient manner. (One problem was solved using the SA but the researcher did not label it as such for students.) Students observed but were not
allowed to take notes during the instruction. The instructor framed his solutions by saying, “This is the way that I solve this equation.” Students were not offered any qualitative judgments about the instructor’s solution methods (e.g., that they were good, efficient, or better than other strategies). The instructor ended his presentation by telling the class that there were many different ways that an equation could be solved, and that, “Maybe you’ll find it helpful to see how I solved each of these problems.” The solutions were removed from view before students began solving additional problems.

During the final session on Friday, students completed a posttest. Students were given 50 minutes to take the posttest. The posttest contained several types of measures, including isomorphic or familiar equation solving, transfer equation solving, flexibility, and conceptual knowledge of equations. Isomorphic equations were problems that were very similar to those that were solved during the problem solving sessions. Transfer equations were novel linear equations to assess transfer of solution method; each had a feature that would have been unfamiliar to students (see Appendix A for a listing of isomorphic and transfer equations). Conceptual knowledge was assessed with a series of multiple-choice questions relating to the concepts of equation, variable, and equivalency. Flexibility was assessed using two subscales taken from prior studies by the project team (Star, 2002, 2004; Star & Madnani, 2004; Star & Rittle-Johnson, 2004), one which measured knowledge of multiple strategies and the second which assessed adaptive use of multiple strategies. The knowledge of multiple strategies subscale included questions on acceptance of multiple solution paths and identification of multiple next steps. The adaptive use subscale included questions on implementing a non-standard step and identifying and evaluating a non-standard step.
A randomly selected subset of participants (23 students; 12 males and 11 females) were videotaped and interviewed in a one-on-one setting (in a different room) while they engaged in algebra problem solving. At regular intervals, the interviewer asked each of these students a pre-determined series of questions about his/her strategies. At the end of each day’s work, the researcher asked each student several concluding questions, including the following relating to standard algorithms: *Suppose your teacher asked you to talk to the class about how to solve equations. What would you say? Is there a way to solve equations that you use a lot? Is there a way that always works?* Each student was asked these questions at least twice during the study. Note that none of the interview students were in the DSI condition.

**Analysis.** Students’ solutions to all posttest problems were graded on correctness of solution. Students’ solutions to post-test problems were analyzed by two independent analysts (who met to resolve all disagreements) to determine whether the SA had been implemented. Students’ responses to interview questions relating to standard algorithms were transcribed for analysis.

**Results and Discussion**

**Early users of the SA.** An initial issue of interest was whether students discovered the standard algorithm. This determination was made by looking at students’ strategies on post-test familiar equations. A student was coded as having discovered the standard algorithm if he/she chose to use this strategy on at least one familiar equation on the post-test. Our analysis indicated that 77 students (50% of all participants) used the SA on the post-test.
However, a more careful investigation of these 77 students indicated that a few demonstrated the ability to use the SA consistently, from the very beginning of the first problem-solving session and continuing throughout the study. We termed these students early users or EU; there were 15 such students. While these 15 EU students did not use the SA on the pre-test, they showed knowledge of the SA on the first problem on which it was possible to use SA. These EU students may have been extremely quick learners, or they may have received supplemental instruction from their parents on Monday evening, after the pre-test but prior to the first problem solving session (anecdotal reports indicate that parental instruction was definitely a factor for two EU students). As a result, these 15 EU students were excluded from further analysis. (Three of these 15 EU students were interviewed.) With EU students removed, 62 students discovered the SA, or 41% of all participants.

*Common inefficiencies among those who did not use the SA.* Before looking at the learning of the 62 students who did discover the SA, we first examine the 76 students who did not. Two of these students were excluded from this analysis because they left a large number of problems blank. Of the remaining 74, almost all (70 of the 74) solved familiar post-test equations using a strategy that was less efficient (e.g., required the application of more steps) than the SA. What kinds of strategies were these students using that resulted in a less efficient solution method as compared to the SA? There were two main sources of inefficiency.

First, many students implemented what we have termed “redundant moves.” The SA dictates that variable terms should all be “moved” (by adding or subtracting variable or constant terms to both sides) to one side, while constant terms should be moved to the other side. At some point during the execution of the SA, the student identifies a side to which
constants will be moved and a side to which variables should be moved. A common inefficiency occurred when students moved a constant or variable term to one side and then back to the other side. This is illustrated in Table 1. In the solution strategy on the left, which was taken from student #1042’s work on post-test problem #8, the student first moved the 4x from the left to the right side. For a student who was using an efficient strategy, this would indicate that variable terms were being collected on the right side equation. However, in the next step, the student moved the 4 from the left to the right side, suggesting that constants were also being collected on the right side of the equation. One of these moves is redundant.

The solution strategy on the right avoids this redundant move: The variable 4x is first moved from the left to the right side, indicating that all variables would end up on the right side, and the constant 8 is moved from the right to the left side.

Table 1: An example of a strategy with a redundant move, from student #1042

<table>
<thead>
<tr>
<th>Strategy with a redundant move</th>
<th>No redundant moves</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable</td>
<td>variable</td>
</tr>
<tr>
<td>4x + 4 = 8x + 8</td>
<td>4x + 4 = 8x + 8</td>
</tr>
<tr>
<td>- 4x</td>
<td>-4x</td>
</tr>
<tr>
<td>constant</td>
<td>constant</td>
</tr>
<tr>
<td>4 = 4x + 8</td>
<td>4 = 4x + 8</td>
</tr>
<tr>
<td>- 4</td>
<td>- 8</td>
</tr>
<tr>
<td>constant</td>
<td>constant</td>
</tr>
<tr>
<td>0 = 4x + 4</td>
<td>-4 = 4x</td>
</tr>
<tr>
<td>- 4</td>
<td>- 4</td>
</tr>
<tr>
<td>-4 = 4x</td>
<td></td>
</tr>
</tbody>
</table>
The use of redundant moves can be interpreted to mean two different things. First, this kind of inefficiency might indicate that a student has failed to learn one of the sub-goals (or intermediate goals) of the standard algorithm – that constants should be moved to one side and variables to the other side. Alternatively, it may be the case that a student who uses this kind of inefficiency is following a heuristic such as, “Get variables terms together on the same side and constant terms on the same side, and when moving terms, try to move the smaller term to the side of the bigger term.” According to this heuristic, in the left-most solution shown in Table 1, the student chooses to move the 4 to the side where the 8 is, because $8 - 4$ is an easier calculation for a 6th grade student to do, as compared to $4 - 8$. This type of inefficiency was quite common in our data. However, our data (even from interviews) does not allow us to distinguish between these two explanations for this strategy. We aim to pursue this kind of error more carefully in future studies.

A second common inefficiency for students who did not discover the SA is the active avoidance of the “combine like terms” transformation. Many students chose to only infrequently use the COMBINE step and preferred instead to combine like terms by moving them. This is illustrated in Table 2, which shows the work of student #1071, booklet problem 18. Instead of combine like variable terms on the left side ($3x, 6x, 9x$), the student uses the MOVE step instead. By subtracting $3x$ from both sides and calculating the result, the $3x$ can be eliminated or removed from the equation without combining like terms. This student moved rather than combining three times in this solution, in fact – with the $3x$, $6x$, and then the $9x$. This kind of inefficiency indicates that students did know that reducing the number of terms in an equation indicates progress toward the solution state of $x = a$, but they were either
uncomfortable with or did not know how to use the COMBINE step as a more efficient way to accomplish this sub-goal.

Interestingly, most of the students who used this kind of inefficient strategy did use the COMBINE step sometimes. For example, in Table 2, once all variable terms have been moved individually from the left to the right sides, the student does combine like constant terms \((6 + 12 + 18)\). In only very rare cases did we find a complete avoidance of this step. However, of all of the equation-solving transformations, our data does indicate that COMBINE was the one that students seemed most reluctant to use and/or had the most difficult time determining how it could be used to solve an equation efficiently.

Table 2: Avoiding the COMBINE step, from student #1071

\[
\begin{align*}
3x + 6 + 6x + 12 + 9x + 18 &= 36x \\
-3x & \quad -3x \\
\hline
6 + 6x + 12 + 9x + 18 &= 33x \\
-6x & \quad -6x \\
\hline
6 + 12 + 9x + 18 &= 27x \\
-9x & \quad -9x \\
\hline
6 + 12 + 18 &= 18x \\
& \quad 36 = 18x
\end{align*}
\]
Direct strategy instruction and the use of the SA. Another issue of interest was whether students who received direct strategy instruction (DSI) used the SA more frequently than those who did not receive DSI. Recall that generally students were not taught the SA nor did they see any worked-examples of solved equations. However, students in the DSI condition did see a demonstration of three equations, one of which was solved using the SA (but was not labeled as such). Interestingly and perhaps counter-intuitively, the relationship between the direct instruction on the SA and eventually use of the SA was not straightforward. Among students who did use the SA, significantly more were from the DSI condition: Of the 62 SA users, 38 (61%) were in the DSI condition. However, among students who did not use the SA, 33 were in the DSI condition while 37 were not. In other words, a student who used the SA on the post-test was more likely to have received direct instruction on this strategy, while a student who did not use the SA was no more likely to be in the control condition.

To some extent these results about the effect of DSI on use of the SA are consistent with the findings of Schwartz and colleagues (Schwartz & Bransford, 1998; Schwartz & Martin, 2004). As discussed above, Schwartz has found that direct instruction can be particularly effective when it is provided at a time when students are ready to receive it and integrate the new information into their existing knowledge base. For Schwartz and colleagues, the issue is not discovery versus direct instruction, but rather whether there is a time in which direct instruction is more or less effective. In the present study, participants in the DSI condition had one hour of problem-solving practice prior to receiving direct instruction on the SA. It appears that DSI students were able to integrate this new information effectively and thus were more likely to use the SA as compared to those who
did not receive DSI. (It should be noted that the present study did not test whether DSI provided at the very beginning of the study (prior to any problem-solving practice) was more or less effective than DSI provided in the middle of the study.) Schwartz and Bransford (1998) did test this prediction and found that DSI provided after some basic fluency had been achieved was more effective.

However, the fact that the students who failed to discover SA were no more likely to be part of the DSI condition is not consistent with the results of Schwartz and colleagues. Possible explanations include that students who failed to use SA but received DSI did not have sufficient time to develop domain knowledge, that the timing of DSI (on the second day of problem solving) was too soon, and/or that the DSI provided was too short to make a difference.

**SA use and measures of flexibility, equation solving, and conceptual knowledge.** Our third research question concerned how students who used SA on the post-test performed on measures of equation solving accuracy, conceptual knowledge, and flexibility as compared to those who did not discover SA. Equation solving accuracy was evaluated by the number of correct answers obtained on familiar and transfer equations on the post-test. Students who used SA on the post-test scored significantly higher on both types of equations. On familiar equations, SA users performed extremely well, averaging 88% correct, while non-SA users only averaged 66% correct; F(2, 138) = 11.206, p < .001. On transfer equations, SA users also outperformed non-SA users, 67% correct versus 52% correct, but this difference was only marginally significant, F(2, 138) = 3.546, p = .062. These results are perhaps not surprising. The SA is a generally applicable problem-solving strategy and can be used successfully without major modifications on both familiar and transfer equations. Knowledge
of the SA is clearly an asset to problem solving, and SA users implemented this strategy on the post-test to outperform non-SA users.

However, SA users did not differ from non-SA users on either flexibility measures or conceptual knowledge measures. Students from all conditions showed significant gains from pre-test to post-test in both their flexibility and conceptual knowledge, but there was not a relationship between knowledge of SA and flexibility and conceptual knowledge. We are currently looking more closely at the interview data to better understand why this might be the case. (Note: Elsewhere we report the effects of DSI on students’ flexibility, conceptual knowledge, and equation solving ability (Star, 2004; Star & Rittle-Johnson, 2004). The focus here is exclusively on SA.)

Trajectories toward discovering the SA. A final area of interest was the developmental trajectories that students’ experienced as they discovered or failed to discover the SA. At present we are investigating this issue by looking at the interview students who did and did not use the SA on the post-test and comparing students’ written strategies with their responses to the interview questions, particularly the question concerning students’ perceptions of whether there was a way that they always used to solve equations. We hypothesized that students who had discovered the SA would be more likely to say that they had discovered a strategy that could solve all equations.

We are in the early stages of this analysis, so here we present a brief discussion of two cases. Both of these students were interviewed (and thus not in the DSI condition). One student (#1003; Table 3) did discover and use the SA on the post-test, while the other (#1018; Table 4) did not.
Table 3
Sample solutions and interview responses for student 1003

<table>
<thead>
<tr>
<th>Strategy annotations and transcript</th>
<th>Problem solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tuesday sample problem solution:</strong></td>
<td>10. (2(x + 5) + 4x = 4 + 8x)</td>
</tr>
<tr>
<td>Student first distributed and the combined like variable terms. He then subtracted 4 from both sides, and then 6x from both sides, and finally divided both sides by 2. This strategy follows the standard algorithm.</td>
<td>10. (2(x + 5) + 4x = 4 + 8x)</td>
</tr>
<tr>
<td><strong>Response to prompt on Tuesday:</strong> Well, it depends on the equation but it should always work, I guess I don’t know. Yeah, you always. eventually I think you will always use the MOVE step to move something probably, and probably you would use the DIVIDE step. That usually.</td>
<td></td>
</tr>
</tbody>
</table>
**Table 3**  
**Sample solutions and interview responses for student 1003 (continued)**

**Wednesday sample problem solutions:**  
15. \(6(x + 2) + 3(x + 2) = 27\)  
The student first distributes and then combines like variable terms (6x and 3x). Then 12 is subtracted from both sides, and then 6 is subtracted from both sides. Finally, both sides are divided by 9. Note that the 12 and the 6 are moved separately, rather than combining them prior to moving.

**Response to prompt on Wednesday:** Um, well, yeah like in EXPAND kind of equations that start out in brackets like that, usually you’ll start out by the EXPAND step and then go on to the MOVE step to get rid of the extra number and then you’ll use the DIVIDE step. And that works for those kinds of equations, usually. It usually follows that same pattern.

\[
egin{align*}
6(x + 2) + 3(x + 2) &= 27 \\
6x + 12 + 3x + 6 &= 27 \\
9x + 18 &= 27 \\
9x + 18 - 18 &= 27 - 18 \\
9x &= 9 \\
x &= \frac{9}{9} \\
x &= 1
\end{align*}
\]
### Thursday sample problem solutions:

26. \[9(x + 2) + 3(x + 2) + 3 = 18x + 9\]

After distributing, student combines like variable terms and then, in a separate step, like constant terms. He subtracts 9 from both sides, then 12x from both sides, and finally divides both sides by 6. This is the standard algorithm.

**Responses to prompt on Thursday:** Yes, it depends on the equation, but usually on most equations I have done here, you would use the EXPAND step first, and then you would use the EXPAND step first to get all the EXPANDING out of the problem. Then you would COMBINE the, um, like the remnants of the EXPAND, like the 4x and the 6x, you would COMBINE those. And then you would COMBINE the added numbers, like plus six and plus seven, you would add those to get 13. So then you would have like 6x plus 14, or 13. Instead of having it scattered and it makes it easier of a process. And then after that you would MOVE the number and, yeah.
### Table 4
Sample solutions and interview responses for student 1018

<table>
<thead>
<tr>
<th>Strategy annotations and transcript sections</th>
<th>Problem solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tuesday sample problem solution:</strong></td>
<td></td>
</tr>
<tr>
<td>10.  (2(x + 5) + 4x = 4 + 8x)</td>
<td></td>
</tr>
<tr>
<td>Student first distributed and then subtracted 4x from both sides. She then subtracted 10 from both sides and then stopped, unable to complete the problem.</td>
<td></td>
</tr>
<tr>
<td><em>Response to prompt on Tuesday:</em> Um, I mostly use the MOVE a lot and the DIVIDE.*</td>
<td></td>
</tr>
</tbody>
</table>
### Table 4
Sample solutions and interview responses for student 1018 (continued)

#### Wednesday sample problem solution:

19. \(5(x + 1) + 10(x + 1) + 5(x + 1) = 30x + 10x\)

Student distributes and combines like variable terms on the right side, all in the first line. Then she subtracts 5 from both sides. She then combines like variable and constant terms on the left side of the equation, then subtracts 15 from both side, then 40x from both sides, and finally divides both sides by -20. This is not the standard algorithm, as the 5 was moved from the left to the right side prior to combining it with the other constant terms.

**Response to prompt on Wednesday:** In most problems you will have to use the DIVIDE step to get to the right answer. And you will have to use the MOVE step in most of them. And I don’t use the first step a lot. (What step is that?) The one where you add a 4X plus a 10X. I don’t have many problems like that. I use EXPAND step a lot. (Is there a certain order that you use a lot?) Yeah, it’s mostly the EXPAND step, the MOVE step and the DIVIDE step. Yeah, if I do the DIVIDE step the MOVE step and then the, wait, the EXPAND step, the MOVE step, and the DIVIDE step. (And that’s something that always seems to work?) Yeah.
Table 4
Sample solutions and interview responses for student 1018 (continued)

<table>
<thead>
<tr>
<th>Thursday sample problem solution:</th>
</tr>
</thead>
<tbody>
<tr>
<td>27. (2(x + 3) + 4(x + 3) + 4 = 8x + 6)</td>
</tr>
<tr>
<td>Student distributes first and then subtracts 4 from both sides. She then combines like variable and constant terms on the left and right sides. 18 is subtracted from both sides, and then 8x is subtracted from both sides. Finally, both sides are divided by -2. Because she moved the 4 as an early step prior to combining it with other constants, this is not the standard algorithm.</td>
</tr>
</tbody>
</table>

Response to prompt on Thursday: Yeah the EXPAND, MOVE and DIVIDE step. In that order. And with the longer problems I use the MOVE step once or twice, twice or three times actually, sometimes. (Have you ever done them in a different order?) Yeah sometimes I do the COMBINE, EXPAND, MOVE, DIVIDE step. And sometimes I didn’t even use the EXPAND step because there was no need to, I just did the MOVE and DIVIDE. |
Student #1003. Student #1003 (Table 3) discovered the SA rather quickly in the study. He began the study with very little knowledge of how to solve equations. He did not know how to proceed when given problem #3 (2x + 4 = 10; not shown in Table 3); he first tried to divide both sides by 2 and then crossed out his work. He then tried to subtract 2x from both sides and then crossed out his work again. Finally, he subtracted 4 from both sides and completed the problem. However, by the time he arrived at problem 10 (2(x + 5) + 4x = 4 + 8x; also completed in the first problem-solving session), he was using the SA (see Table 3). He distributed first, then combined like variable terms, then moved constants and variables to opposite sides of the equation, and finally divided both sides by 2.

However, despite his use of the SA on problem 10, 1003 did not consistently use the SA in its most compact form. On problem 15 (6(x + 2) + 3(x + 2) = 27; see Table 3), after distributing and combining like variable terms, 1003 moved the 12 and the 6 to the other side of the equation in two separate steps, rather than the more efficient strategy of combining 12 + 6 first and then moving 18 to the other side. In subsequent problems, 1003 began using the most compact form of the SA again, including problem 26 (shown in Table 3).

Although he was quick to use the SA in his written strategies, 1003’s utterances lagged somewhat behind his solution strategies. At the end of the first problem-solving session, when asked whether there was a way that he used to solve equations that always worked, 1003 was unsure (see Table 3). He seemed to think that one always could use MOVE and DIVIDE, but he did not have much to say about the order that one might apply steps. We infer that despite his work on problem 10, when he did use the SA, he was not cognizant of the fact that he had discovered a general solution strategy for linear equations.
By the end of the second problem-solving session, 1003’s responses to interview questions indicated a more developed sense of the SA (see Table 3). He indicates that one starts by using the EXPAND step, followed by the MOVE step, and then the DIVIDE step. He remarks that this method works “for those kinds of equations, usually.” It should be noted that on some of the equations that students saw, such as problem #22, 3(x + 2) = 6(x + 2), there are not any variable or constant terms to combine. So for this problem, 1003’s description of his strategy that always works is accurate. However, on other problems such as problem #15 (see Table 3), the more general description of the SA (which includes the COMBINE step) is necessary.

1003’s answers to the interview questions at the end of the third and final problem-solving session indicate a fully developed awareness of the SA (see Table 3). At this point 1003 not only solved equations using the SA but he also was aware of the existence of this general and widely-applicable strategy.

1003’s case is consistent with findings from the strategy choice literature in at least two ways. First, somewhat different trajectories become apparent when one looks at students’ written strategies and their verbal responses to interview prompts. Students used written strategies before they became fully (and metacognitively) aware of the existence and broad applicability of these strategies (Crowley et al., 1997). Second, even when a student used the SA once or twice, he or she did not consistently use this strategy for all subsequent problems. The development and use of strategies is not a linear process (e.g., Siegler, 1996).

Student #1018. Student #1018 represents a different kind of case. She did not use the SA on the post-test but instead developed her own standard but idiosyncratic approach to solving linear equations. As with 1003, student #1018 did not know how to use symbolic

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methods to solve equations at the beginning of the study. By the time she reaches problem 10 (see Table 4), she has figured out how to use a few steps (EXPAND, MOVE) but still has trouble completing this problem.

Her facility with equation solving improved; she solved problem 19 correctly (see Table 4). However, as her work on problem 19 illustrates, she developed a somewhat unusual strategy. She began by distributing whenever possible. In her second step, she applied the MOVE step to the constant term on the left side of the equation that is closest to the equals sign (in this case, 5). After subtracting 5 from both sides, the remainder of her steps followed the SA: She combines like variable and constant terms, moved variable terms to one side and constants to the other, and then divided both sides by -20. 1018 used this same strategy several more times, including problem #27 (see Table 4). On this problem, she distributed and then subtracted 4 from both sides. After this initial and early MOVE step, she followed the SA for the remainder of the problem.

1018 responses to the interview prompts revealed a similar finding as was noted with 1003. When asked whether there was a strategy that she always used or that always worked for us, she initially could only reply that she used the MOVE and DIVIDE steps a lot (see Table 4). When she was asked these questions for a second time, she had considerably more to say. She indicated that she tended to use the EXPAND, MOVE, and then DIVIDE steps (which, as pointed out above, is a version of the SA for certain problems where combining like terms is not necessary). She also said that she did not use the COMBINE step a lot: “And I don’t use the first step a lot. The one where you add a 4x plus a 10x. I don’t have many problems like that.” We find two things to be interesting about her responses on this day. First, she did not appear to be aware that she had started to use her rather idiosyncratic
strategy involving an early MOVE step. And second, while she said that she did not use the COMBINE steps a lot, in her written solutions she used it in most problems.

Finally, the third time that 1018 was asked the interview questions about her strategies, she was similarly vague about the order in which she applied steps or whether there was a way that always worked. She said that she used the steps EXPAND, MOVE, then DIVIDE a lot, with more frequent use of the MOVE step. Again, there was no mention of her idiosyncratic method involving an early MOVE step.

We draw three conclusions from the case of 1018. First, and similar to 1003, we found a disconnect and/or a developmental lag between this student’s strategy use as compared to her verbal metacognitive awareness of strategy usage. Second, 1018 developed an idiosyncratic (and not particularly efficient) strategy for solving equations and tended to use it reliably (using the steps accurately and reaching the correct solution) and consistently. This is consistent with our prior work on students’ strategy development in this domain (Star & Seifert, 2004). And finally, student 1018 did not exhibit an awareness, either in her written solutions or in her verbalizations, of solution efficiency - how steps might be done in certain orders in order to end up with a more efficient solution. In prior work we have found that many students fail to develop sophisticated conceptions of efficiency and the criteria by which one judges one strategy to be better than another (Star & Madnani, 2004). 1018 is a nice example of this tendency; she chose not to (or was unable to) make modifications to her idiosyncratic strategy (particularly the early MOVE step) to increase its efficiency.

Consistent with the literature on skill acquisition (Anderson, 1982, 1987), 1018 was likely able to execute her own strategy faster and faster, but she failed to spontaneously rearrange some of her problem-solving steps to result in a more efficient strategy.
Discussion and Conclusion

Our results indicate that many students (41% of participants) do discover the SA, largely on their own. Perhaps this suggests that the quest for efficient problem-solving strategies may be present for students at a somewhat intuitive level (Anderson, 1987). Whether because students do not enjoy algebra equation solving (and want to get it over with quickly!) or because they are responding to a hard-wired preference for efficient strategies as opposed to inefficient ones, participants in the present study tended to start with inefficient strategies and, with increased problem-solving experience, develop greater efficiency. Note that this trend toward more efficient strategies by students occurred largely without feedback and was present independently of the effects of direct strategy instruction.

Although this trend was largely true in our data, there were some students who failed to develop more efficient strategies. However, even for those students who did not modify their strategies for maximal efficiency (e.g., discover the SA), it was still the case that they developed a personal, often idiosyncratic method for solving equations that was used reliably and consistently (Lewis, 1981, 1988; Star & Seifert, 2005).

The present results lead us to an interesting and important issue – how students who have developed idiosyncratic and inefficient approaches could be prompted to improve their strategies. One promising approach that has emerged both from the cognitive science literature as well as from the mathematics education community is having students explain contrasting examples (Gentner & Loewenstein, 2002; Lampert & Cobb, 2003; Schwartz & Bransford, 1998). For example, students could be asked to share solution strategies for a particular problem and then to discuss the similarities and differences in the different...
procedures. When presented with contrasting examples of solution procedures and then asked to compare them, learners should come to notice critical features of the procedures and to abstract their common underlying structure. In fact, this kind of instructional approach lies at the core of current reform pedagogy in mathematics. At all grade levels, teachers are encouraged to create an environment where students can engage in thinking and communicating deeply about mathematics—in discussing, collaborating, justifying, conjecturing, experimenting, and responding to the ideas of their peers (National Council of Teachers of Mathematics, 2000, p. 18). The development of mathematical understanding is believed to be enhanced by classroom discussions, where students share procedures and evaluate the procedures of others (Lampert, 1986, 1992a, 1992b), including informal and non-standard algorithms (Carroll, 2000; Mack, 1990). We believe that the contrasting cases approach has promise for addressing some students’ tendency to discover a (sometimes inefficient) strategy that works (e.g., solves a problem correctly) but then fail to consider whether the approach is efficient. We are currently engaged in pilot work that investigates this possibility.

Despite the fact that many students discovered the SA on their own, it was also the case that direct strategy instruction, provided at an opportune time, did increase the chances that a student discovered the SA. Consistent with the results of Schwartz and colleagues (Schwartz & Bransford, 1998; Schwartz & Martin, 2004), well-timed direct instruction did yield significant benefits in terms of student learning. Whether or not direct instruction is more effective than discovery, an issue that has re-emerged among some cognitive scientists (Klahr & Nigam, 2004; Mayer, 2004), obscures an important point – that the timing and
content of direct instruction may be more important to student learning than the format (discovery or direct instruction).
References

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Appendix A
Problems solved by participants in the booklet and in the post-test

Booklet problems
B1. \( x + 4 = 7 \)
B2. \( 2x = 12 \)
B3. \( 2x + 4 = 10 \)
B4. \( 3x + 5x = 16 \)
B5. \( 5x + 3 = 2x + 6 \)
B6. \( 3(x + 1) = 15 \)
B7. \( 4(x + 3) = 8x \)
B8. \( 3(x + 2) = 5x \)
B9. \( 4(x + 1) = 2(x + 1) \)
B10. \( 2(x + 5) + 4x = 4 + 8x \)
B11. \( 3(x + 2) + 9x = 3x + 15 \)
B12. \( 2(x + 1) = 10 \)
B13. \( 3(x + 2) = 15 \)
B14. \( 2(x + 3) + 4(x + 3) = 24 \)
B15. \( 6(x + 2) + 3(x + 2) = 27 \)
B16. \( 2(x + 5) = 4(x + 5) \)
B17. \( 3(x + 3) = 5(x + 3) \)
B18. \( 3(x + 2) + 6(x + 2) + 9(x + 2) = 30x + 6x \)
B19. \( 5(x + 1) + 10(x + 1) + 5(x + 1) = 30x + 10x \)
B20. \( 2(x + 4) = 14 \)
B21. \( 4(x + 1) = 12 \)
B22. \( 3(x + 2) = 6(x + 2) \)
B23. \( 4(x + 3) = 2(x + 3) \)
B24. \( 3(x + 1) = 15 \)
B25. \( 2(x + 3) = 12 \)
B26. \( 9(x + 2) + 3(x + 2) + 3 = 18x + 9 \)
B27. \( 2(x + 3) + 4(x + 3) + 4 = 8x + 6 \)
B28. \( 2(x + 1) + 6(x + 1) = 4(x + 1) \)
B29. \( 3(x + 3) + 6(x + 3) = 5(x + 3) \)
B30. \( 2(x + 5) = 14 \)
B31. \( 5(x + 3) = 20 \)

Post-test problems
PT5. \( 2x + 5 = 21 \) [isomorphic]
PT6. \( 2(x + 1) + 3x + 4 = 8 + 3x + 4 \) [isomorphic]
PT7. \( 4(x + 3) = 16 \) [isomorphic]
PT8. \( 4(x + 1) = 8(x + 1) \) [isomorphic]
PT9. \( 3(x + 2) + 9(x + 2) = 6(x + 2) \) [isomorphic]
PT10. \( 0.2x + 0.3 = 0.9 \) [transfer]
PT11. \( 3x + x + 5 = 13 \) [transfer]
PT12. \( 2(3x + 2) + 2x = 20 \) [transfer]
PT13. \( 2(2x + 3x) + 2 = 12 \) [transfer]