Part of the impetus for examining teachers’ knowledge is that enhanced teacher knowledge would seem like one way to improve student achievement. Yet, some studies of teacher knowledge are not designed to study such connections, but rather simply to assess levels of teacher knowledge. Such studies often assume that the notion of knowledge itself when it comes to teachers is unproblematic and that the particular knowledge identified is important for teachers to possess and must relate to student achievement. Past examination of links between teacher knowledge and student achievement have not been fine-grained enough to bear out this connection (see, for example, Begle 1972). So, even though it seems obvious that the nature of a teacher’s knowledge of subject matter should have an influence on student achievement, there is little empirical confirmation of such a link.

As a way into this set of issues, in the Algebra Knowledge for Teaching Study, we have made a particular bet. Our bet is that one way to produce more fine-grained understandings of relationships between teacher knowledge and student achievement is to concentrate on a particular curricular area. We are addressing questions about relationships between teacher knowledge and student achievement in a particular contested curricular area, school algebra. Since this curricular area is in flux, examinations of teacher knowledge in the context of school algebra have their own unique challenges. Below, we will illustrate how our work has been shaped by these challenges.

Parenthetically, even though the name of this project includes the word “study,” the focus of the work is on producing a framework for studying teacher knowledge in the context of school algebra. Thus, our empirical work with teachers is not a study, but rather attempts to collect empirical data that will inform the construction of the project’s framework. Therefore, it is not designed to establish in its own right links between teacher knowledge and student achievement. Yet, nonetheless, we seek to create a framework that would support projects aimed at establishing such links.

In the design of our work, we aim to address relationships between teacher knowledge and student achievement by working through the following chain of questions:

- What about school algebra is important for students to learn? This question speaks to notions of desired competencies for students.
- What are the particular difficulties that students face in learning this algebra? This question relates to research on student learning.
- What practices and approaches are being advocated as promising to help students learn this important content successfully? This question examines how the field has thought about unpacking skilled knowledge and performance in algebra and helping students move from novices to experts. Algebra is a curriculum that is in
flux. There are quite different approaches to school algebra and to the question of unpacking skilled performance that are available in current curricula. This diversity in thinking about what constitutes school algebra and how it might be taught is an important challenge in examining teacher knowledge in the context of school algebra.

- What knowledge do teachers need to have in order to be able to implement these practices and approaches? Again, it is important to recognize that the diversity of approaches to school algebra makes this question have multiple dimensions.
- And, finally, to what degree do teachers seem to possess the knowledge that would allow them to carry out practices that have been identified as responsive to important areas of student difficulty?

In this paper, we will describe how we have taken up these questions and the challenges that accompany them related to one topic in algebra, with some discussion of implications for our ongoing work and perhaps that of others.

**One way to illustrate assumptions about teacher knowledge**

The use of research on teacher knowledge to inform policies about matters like undergraduate mathematics preparation of mathematics teachers is built on commonsense notions about links between teacher knowledge and student achievement. These assumptions can be pictured as follows:
Proficient student performance

Classroom where teachers act

Teacher

Supports instruction

Inform

Teachers’ Knowledge

Curriculum materials
Preservice mathematics courses
Technology
Professional development
Preservice methods courses
Own schooling

Should produce

Is not producing

Proficient student performance
This schematic is meant to suggest the following assumptions:

a. Teachers possess something called knowledge.
b. Teachers acquire this knowledge both from teaching and from other experiences outside of the classroom.
c. This knowledge influences how teachers act with students in the classroom as they engage students in studying mathematics.
d. When teachers possess the right sorts of knowledge, their interventions in the classroom lead to greater student achievement (and when teachers do not possess this sort of knowledge, their students’ achievement suffers).

Complexities Related to Knowledge

As other papers in this session suggest, this sort of view of teachers’ knowledge can be critiqued from a variety of perspectives. A situative perspective on knowledge calls into question the degree to which knowledge can be thought of something that is possessed by individuals and acquired outside the context of particular practices (ref Educ Psych paper from Hilda et al’s last project). In the context of teaching, this general critique suggests that teachers may know differently in collegiate mathematics and methods classrooms than they do in the practice of teaching. Thus, the notion of examining what teachers know outside of the practice of teaching may not adequately inform how they influence student achievement.

And, there are other complexities. For example, teachers draw on more than knowledge in their teaching. “Knowledge” is too narrow a term for getting at the cognitive complexity of teaching and learning algebra. Often, the literature addresses this by talking about knowledge and beliefs. But belief is an overarching term. There are knowledge-like beliefs and those beliefs that represent values, relative priorities, etc., what one might call commitments. Certain beliefs are “knowledge-like” in the sense that one would say that they are right or wrong, and they are beliefs rather than knowledge either because they are wrong or because one does not have a warrant for them. Rather than saying that a commitment is right or wrong, we tend to talk about commitments in terms of whether they are ones that we do or do not share. Both knowledge and commitments are important in teaching, and examining teachers’ knowledge without attention to commitments may also lead to an incomplete picture. Teachers may have the knowledge to implement suggested practices, but due to competing commitments or a lack of commitments not implement those practices in particular teaching contexts. And, then, as a result, there might not be expected connections between knowledge and student learning.

A different sort of complexity suggests that teachers’ knowledge needs to be qualified by adjectives in order to compare important qualitative aspects of the nature of what teachers know. For example, in an unpublished manuscript that has influenced in important ways the work of Ball and Bass, David Cohen argues that: “Because teachers must be able to work with content for students in its growing, not finished, state, they must be able to do something perverse: work backward from mature and compressed understanding of the
content to unpack its constituent elements” (p. 98). From this perspective, it isn’t solely what a teacher knows, but how they know what they know. Is their knowledge compressed? Do they have the capacity to “unpack” their knowledge?

The logic of an empirical strategy for refining a framework for studying teachers’ knowledge

These critiques, as well as others, suggest that our notions of teachers’ knowledge require stronger conceptualizations than we currently hold as a field. In our efforts to examine issues of teacher knowledge in algebra empirically and support the development of a stronger theoretical framework, we have worked toward a design that might help us speak, perhaps critically, to the informal theory we outlined above, though at this point we are NOT designing a study that will allow us to examine quantitatively relationships between teacher knowledge and student achievement.

Our attempt to link teacher knowledge and student achievement in school algebra conceptually starts from focusing on particular mathematical topics or issues. Since we are interested in ways that teachers’ knowledge has an impact on student achievement, we seek topics or issues that are considered important to student achievement and with which students are known to have difficulties. If teachers’ knowledge is going to have an impact on student achievement, teachers who possess the knowledge are capable of teaching in ways more likely to help students understand the topic, while teachers without the knowledge would be unable to teach in these helpful ways. In other words, our talk of knowledge being important for teaching a topic suggests that some ways of teaching the topic are more effective than others, and that the knowledge is required for the more effective forms of teaching. (Although this talk of “ways of teaching a topic” may conjure the image of specific “methods,” we recognize that the differences in teaching might be more general and diffuse features of teaching, such as the mathematical “practices” referred to in the RAND report.) Thus this research strategy requires specifying difference in ways of teaching these topics, so that we can try to connect these ways of teaching to required knowledge for teaching. As will be outlined below, school algebra curricular materials and approaches suggest some candidates for topics that are difficult for students and important in the sense outlined above.

Having identified candidates for different ways to teach topics, we then seek to identify the mathematics knowledge required to teach with them. We do this through some combination of the following:

- Carrying out a logical analysis of what seems important to know in order to teach this approach.
- Finding teachers who use each approach and see what they do and don’t know.
- Asking teachers to teach each approach, seeing which approaches they have difficulty pulling off, and identifying the knowledge needed to be successful.

Thus, this overall approach to the study is based on the following argument:
Some topics in high school algebra are important foci for study because they are both mathematically important and often difficult for students to learn. Student success with these topics is due, in part, to differences in what teachers do to help students learn these topics. Differences between what teachers do in the classroom to teach these topics are due, in part, to differences in the mathematical knowledge for teaching these teachers possess. Logical and empirical research can be used to identify these links between this knowledge, what is done in the classroom, and student achievement.

**School algebra as an arena for the study of teacher knowledge**

Algebra is a curricular arena in flux. The $x$’s and $y$’s of algebra that were traditionally unknowns first and foremost are pattern generalizers or variables in reform-minded curricula. And, graphing calculators provide students access to solution methods for equations that were not usually feasible twenty years ago. Students can now pursue strategies for solutions either with tables or with graphs in a way that was simply not possible with paper and pencil. Thus, new curricula treat content differently than traditional curricula, though all curriculum developers share a commitment to help students “learn algebra.” All curricula seek to help students, for example, to solve equations, though they might differ on what that means and how it is to be accomplished.

While new curricula share the following departures from traditional curricula, they vary widely in how these changes are enacted:

- a shift in the meaning of $x$ from unknown to variable,
- opportunities and problems with graphing calculators,
- a goal of avoiding mindless manipulation,
- the use of key models and metaphors for learning specific topics,
- a change in the role of practice on problems grouped by type,
- the use of different names and definitions of key terms.

The proliferation of different “flavors” of elementary algebra has many ramifications. Most importantly, it has made the existing state of research on student learning and teacher knowledge vulnerable. Against the backdrop of the traditional algebra curriculum, the main research story that has developed is that algebra, synonymous with symbol manipulation, is hard to learn. As the content of school algebra changes, this story becomes problematic. Because students are introduced to algebra in different ways (via fundamentally different curricula), unlike the early 80s, there is no single curriculum against which to plan studies of student learning. On the one hand, this situation has impeded the progress of research on students’ learning, particularly on concepts central to school algebra of different flavors. Thinking about the content, timing (by grade), supportive technology, and pedagogy of algebra has moved ahead, but research about how key ideas develop over time has not.

On the other hand, the presence of these different curricular approaches potentially provides a fascinating site for exploration of teacher knowledge. Different curricular approaches are in use in part because of disagreements about how to best teach students to exhibit accomplished performance, say in the solving of equations. As teachers carry
out these different approaches, particularly newer ones that may involve different knowledge, perhaps this will allow opportunities to understand more deeply the role of teacher knowledge in supporting student achievement (refer chazan IJCML).

**Views of equations and their constituent elements: A focal topic**

Skilled users of algebraic symbols have much knowledge at their fingertips. Here are some examples:

- They are able to understand the same string of symbols differently in the context of different task settings and treat them accordingly. For example, the equal sign in the string \(3(x-2)=3x-6\) means something quite different if this string is part of a problem to find solutions than if it is a recording process of the multiplication of two factors to create two terms. Similarly, the role of \(x\) in these two settings is different as well.
- They are able to employ the same operations in different mathematical settings. For example, they can add the same quantities to both sides of different sorts of equations and understand the different results that are achieved.

\[
\begin{align*}
\text{Solve:} & & \text{Solve for } y: \\
3y+5 &= 2y+4 & 3y+5 &= x+2y+4 \\
y+5 &= 4 & y+5 &= x+4 \\
y &= -1 & y &= x-1
\end{align*}
\]

By contrast, examination of novices’ experiences with algebra suggests that novices are often tripped up by exactly these sorts of issues. Novices may want to turn expressions into equations at inappropriate moments as opposed to appropriate ones (Georgia study vs. Curt Bennet’s example). They may not understand why their teachers think that the string \(3x^2+3x+3\) can be divided by 3 in some problem settings, but not in others (For other examples of this kind, see Chazan & Yerushalmy, in press). An, as Zalman Usiskin (ref) and Anna Sfard (ref) point out, in the context of multi-stage problems, they must learn to move between different sorts of meanings of a letter and operate appropriately as these meanings change.

Thus, one suggestion about what makes school algebra itself intrinsically difficult is that students must learn to move flexibly between multiple views of equations and multiple views of their constituent elements.

**A challenge in school algebra: Multiple approaches**

And, different curricular approaches to school algebra differ with one another on just this matter. They unpack the subject differently. They cover topics in different orders and with different emphases. Some of the best-selling textbooks start with letters as unknowns and move from expressions to linear equations in one variable, to linear equations in two variables, to non-linear equations in one and then two variables, some of which will be considered functions. By contrast, a functions-based approach might start
with expressions and explicit equations as representations of functions, then move to
equations in one variable as questions about functions or as comparisons of two functions
of one variable, and then only address implicit equations in two variables later

These differences in approach to school algebra as a curricular topic suggest that teachers
need to understand differences in how different curricula conceptualize even simple
equations like $3x-2=12$. Is this a problem to find the unknown number that will create a
true statement? Or is it a problem about one function where one seeks the input to $3x-2$
that will equal 12? Or is this a task to find the members of the real numbers for which the
two functions $3x-2$ and 12 share outputs?

While all of these views are similar, there are fundamental differences between them that
are easily glossed over by those with mature, compressed understanding. In the more
standard view that underlies the first reading of the equation, letters represent unknowns,
and equations suggest a question as to the value of the unknown that will make the
statement true. Within this view, expressions have no inherent meaning, and students
must reconceptualize letters as representing variable quantities and equations as
representing relationships when they encounter equations in two variables and function
rules. Also, graphing calculator methods of solving equations in one variable are
inconsistent with the established definitions.

On the other hand, in the functions-based approach that underlies the latter readings of
the equation, letters represent variable quantities from the outset; expressions represent
functions; and equations represent relationships or questions about relationships.
Graphing calculator approaches to solving fit naturally with a functions story line.
However, equations such as $2x+3y=6$ become more problematic within this approach, as
it is most naturally read as a question about a function in two variables, and it makes an
issue of rewriting an equation in two variables as a function in one variable. Further,
when expressions and explicit equations are used interchangeably to represent functions,
then there may be more room for confusion as to whether solutions should be of the form
\{x\} or \{(x, y)\} when problems are presented in pure mathematical contexts.

Teachers must therefore decompress their understandings of equations and unpack them
into their constituent elements to recognize the approach of the intended curriculum, to
make decisions about how to enact the curriculum, and to respond to their students’
approaches and difficulties in ways that support student learning.

**Getting at Teacher Knowledge: The design of an interview**

For the teacher, a key question is being able to put oneself in students’ shoes. Given the
curriculum that they have experienced to date, what do they think equations are? What
sorts of experiences will ask them to generalize or expand their notion of what an
equation is? From our perspective, these questions ask a teacher to unpack and
decompress their understandings of equations. They must look back at content that they
already know and use distinctions that may not be important for them as expert users of
algebra in order to have criteria and distinctions to understand the experience of their students and the impact of curricular orderings on students’ understandings.

In an attempt to surface issues of unpacking and decompression on the part of algebra teachers, we are interested in focusing our interview tasks on teachers’ thinking about equations and expressions. With our focus on contributing to the design of a framework, designing the interview, we felt it was important to try to ask questions that would be difficult for teachers and that situate teachers in practice. We recognize that different questions will be difficult for different teachers depending on their view of or approach to teaching algebra. As such, we begin with two diagnostic items that will help the interviewer select a path of subsequent questions to pursue that we predict will be difficult for teachers with a particular perspective.

Early on, teachers are presented with the following nine equations on nine index cards and asked to discuss them in relation to their conception of equations, to compare and contrast them, to order the cards in the ideal order in which they would want students to meet these examples, and to explain the reasoning behind their choices.

- \( y = 3x - 4 \)
- \( 3x - 4 = 12 \)
- \( 9x - 3y = 12 \)
- \( f(x) = 3x - 4 \)
- \( 4(x - 1) - x = 3x - 4 \)
- \( 3(x - 2) + 2 = 4x - 4(x - 3) \)
- \( g(x, y) = 3x - y \)
- \( y = 3x^2 + x - 2 \)
- \( 3x^2 + x - 2 = 0 \)

Traditionally, and in most textbooks today, students will meet equations like \( 3x - 4 = 12 \) before equations like those on the other 8 cards. In a functions-based approach, students would be more likely to meet equations like \( y = 3x - 4 \) first.

Following this first interview item, teachers will be initially categorized as “functions first” or “not functions first”. For those sorted into the latter group, there is a second sorting item that follows:

Suppose you ask your students to solve \( 4(x - 1) - x = 3x - 4 \). What questions or difficulties might you expect students to have? Some students interpret \( 0 = 0 \) as \( x = 0 \). Why do you think students do this? How would you help your students to interpret a solution of \( 0 = 0 \)?

*If the teacher does not suggest a graphical approach, an example of such an approach will be presented, and the following question will be asked:*
One thing some teachers do is to treat each side of the equation as a function. Students can compare tables of values and graphs of both sides. What do you think about this approach to solving this problem or explaining the solution to students—is it algebra?

Following this item, teachers will again be sorted according to whether they express the classic explanation of this problem (calling on properties to justify equivalence of the original equation and \(0 = 0\), stating the this latter equation is true for all values of \(x\), and therefore concluding that the original equation is true for all values of \(x\)), or whether they invoke an explanation that involves thinking of the equation as a comparison of functions (whether or not they use this language).

For each of these 3 groups, we tried to think about what would be hard questions. For example, for those teachers that indicated a functions approach in the initial ordering task, we pose questionst that include the following:

1. Let’s look at the examples:
   - \(y = 3x - 4\)
   - \(9x - 3y = 12\)

   Compare and contrast these two equations. How would you help your students to see that the second example is linear?

   *If the approach does not involve manipulating the into the form \(y = \), then ask:*

   Do you think that it is important for students to be able to move from the second form to the first?

   How do you explain to students that some equations in two variables can be rewritten as functions in one variable?

For teachers who did not initially indicate a functions approach but who appealed to such an approach in explaining \(0 = 0\), we ask questions like the following:

1. How are equations in one variable different from systems of equations? Are the graphs one might use to solve them different in any way?

2. How do you evaluate and compare two students’ solutions to solving an equation: one that solves using algebraic manipulation and one that solves by generating and interpolating from a table? Do the two approaches reveal anything different about what the two students know or understand?

**Conclusion**

In sum, in our examination of teachers’ knowledge in school algebra, we are working on two challenges:
a. Creating possibilities for plausible connections between teacher knowledge and student achievement.

b. Working with teachers of school algebra across the differences in curricular perspective now an important part of that landscape.