Knowledge for Teaching School Algebra:
Challenges in Developing an Analytic Framework

Joan Ferrini-Mundy
Robert Floden
Raven McCrory
Michigan State University
211 North Kedzie Lab
East Lansing, MI 48824-1031
Telephone: (517) 432-1490
Fax: (517) 432-5653
jferrini@msu.edu

ABSTRACT
The problem of knowledge for teaching mathematics is of growing concern to practitioners and researchers in ongoing efforts to improve mathematics education. Algebra is of particular importance since it functions as a gatekeeper for later mathematics courses. In this paper we describe our developing framework for understanding and assessing teachers’ knowledge for teaching algebra. Using literature on teacher knowledge and student learning along with empirical evidence from our own research, we propose a two dimensional framework (tasks of
teaching x categories of knowledge) with three additional overarching categories
(decompressing, trimming, and bridging) to conceptualize knowledge for algebra teaching.

KEY WORDS

algebra knowledge for teaching
conceptual framework
mathematical knowledge for teaching
teacher knowledge

1. Introduction

Calls for teacher quality and improved U.S. student achievement in mathematics reinforce the need for continued research and theoretical work to support programmatic and policy shifts in the preparation and continuing professional development of teachers. Students’ performance in algebra is particularly worrisome (Blume, 2000; RAND Mathematics Study Panel, 2003.) Today’s accountability-oriented policy environment produces pressure both for the improvement of students’ performance in mathematics and for the validation of teachers’ mathematical credentials necessary to demonstrate that they are “highly qualified teachers.” In light of current recommendations for the mathematical preparation of teachers (Conference Board of Mathematical Sciences, 2001), as well as growing attention among researchers to the mathematical knowledge used in teaching (Ball et al., 2000a, 2000b) we investigate mathematical knowledge for teaching school algebra.

Why do we need research on teachers’ mathematical knowledge for teaching algebra? Although it is widely believed that teachers’ mathematical knowledge is important and is connected to students’ learning, research linking teacher knowledge to student achievement has produced ambiguous and sometimes contradictory results. Lack of clarity about the impact of
teacher knowledge results from inadequate theory about particular mathematical knowledge or kinds of knowledge that might be important for teaching. Lacking theory, researchers have used proxies for knowledge (e.g., number of mathematics courses, quality of undergraduate institution) rather than direct measures of knowledge; or they have used direct measures of knowledge that fail to take practice into account. In the research reported here, we aim to address the theoretical and conceptual aspects of the problem of teachers’ mathematical knowledge for teaching algebra, and to lay the groundwork for empirical research that probes the connections between teachers’ mathematical knowledge and student outcomes in algebra.

In our research, we have used theoretical and empirical literature in the area of teacher knowledge, research studies about students’ difficulties in learning algebra, and discussions of various curricular perspectives in the area of secondary school algebra to develop a framework of algebra knowledge for teaching. In addition, we draw on examples from three lines of empirical work that were interactive with our framework development: interviews of algebra teachers, analysis of algebra instructional materials, and observation of videotaped algebra lessons. The framework is discussed, applied, and elaborated in other project papers that describe these three lines of work. The framework is intended to help support the design of instruments for assessing teacher knowledge, to inform the development curricula for teacher learning at the preservice and in-service levels, and to serve as a conceptual tool in related research about the mathematical knowledge involved in teaching at the secondary school level.

In the following sections, we outline previous research on teacher knowledge, explain our perspective on the importance of research on teachers’ mathematical knowledge and our particular focus on algebra, and, finally, we present our preliminary knowledge for algebra teaching (KAT) framework.
2. Characterizations of Teacher Knowledge by Others

We see our work on mathematical knowledge for teaching algebra within the broad context of research on teacher knowledge. To set the stage for our focused analysis, we sketch that broad context, with comments on how the work has been developing over the past decades.

With the publication of the *Handbook of Research on Teaching* (Gage, 1963), the field of research on teaching gained a visible position as a distinct area of study. The 1963 *Handbook* had many chapters devoted to teaching in particular subject areas, including a chapter on secondary mathematics teaching. Many of its chapters, however, address research topics and methods about teaching in general, with little attention to differences that might exist across subject areas. Getzels and Jackson (1963) for example, look at research on teacher personality and personal characteristics.

Gage was a leading figure in the development of main programs of research on teaching in the following decade, supported by a large infusion of federal funding. The dominant model for research on teaching in that period was the “process-product” approach, which sought to identify the patterns of teaching most strongly associated with student learning, especially as measured by achievement on tests of basic skills in elementary school reading and mathematics. Large-scale studies used surveys and systematic classroom observations to gather data on teaching processes, then used correlation and regression analyses to estimate the effects of particular patterns in teaching on student learning. The focal processes included student “time on task,” type and frequency of teacher questions, proportion of time teachers and students talked, and frequency of instructional disruptions. The research sought to identify teaching process and teaching skills that would be widely applicable, so little attention was paid to
possible differences across subject areas in what approaches or skills would be most effective for promoting student learning.

In this same period, philosophers of education drew on analyses from epistemology to suggest that discussions of knowledge, whether teachers’ knowledge or pupils’ knowledge, would be clearer if distinctions were made among various forms of knowledge, including propositional knowledge (knowing that something is the case) and procedural knowledge (knowing how to do something). Following Gilbert Ryle’s (1949) argument that these were two distinct types of knowledge, philosophers of education examined their meanings and their connections to teaching (see Green, 1971; Macmillan et al., 1968). Within the propositional realm, philosophers distinguished mere beliefs from knowledge, noting that calling something “knowledge” connotes a claim to truth (Scheffler, 1965). In the realm of procedural knowledge, Chomsky’s (1965) distinction between competency and performance was introduced, providing a way for thinking about the difference between what teachers were capable of doing (competence) and what they actually did on a given occasion (performance). That distinction helps to explain difficulties in assessing teachers’ skills through observations of classroom practice. A teacher might, for example, have the skills needed to gauge a pupil’s understanding with a series of probing questions, yet choose not to exercise the skill.

Ryle (1949) may also be the source of the distinction between beliefs, as psychological states, and dispositions, as descriptions of tendencies to act in a particular way. That is, a teacher might believe that she should treat all student comments with respect, yet frequently dismiss many student comments. It might be that the teacher’s beliefs, as experienced psychologically, were different from the teacher’s tendency to act. Katz and Raths (1985) brought this distinction into discussions of teaching and teacher education, arguing that
dispositions deserved a place, alongside knowledge and skills, in the intended outcomes of teacher education. More recently, Raths (2001) argued that having teacher education focus on developing dispositions, rather than beliefs, would help avoid some of the confusion he sees around the “dicey topic” of beliefs.

Within research on teaching, these distinctions were picked up, often in a formulation that uses the list, “knowledge, skills, and dispositions” to cover a range of teacher characteristics (e.g., Kennedy, 1991), suggesting blurred separations among what teachers can talk about, what they can do, and what they are inclined to attempt. In our work, we also wish to look across these categories, with a similar intent to separate the three, while recognizing that the dividing lines are far from sharp.

In his chapter in the Third Handbook of Research on Teaching, Shulman (1986a) argued that the research on what teachers of all stripes should know should be supplemented with research on the teaching of particular subjects. He asked, rhetorically, whether knowledge for teaching was the same for history teachers as for mathematics teachers. Beyond the obvious difference in the subject matter knowledge needed, Shulman argued that all aspects of teachers’ knowledge likely had a subject-specific dimension. In the subsequent 18 years, research on teaching has shifted away from the attempts in the 1960s and 1970s to identify general teaching practices that effective in all subject areas, toward research set in the context of particular subject matters – mathematics, English, history, and so on. Beginning in the last 1970s, research also shifted from exclusive attention to teachers’ behaviors and action to address teachers’ knowledge, again contextualized within particular subject areas (see chapters in Richardson, 2001). We extend this trend, focusing on a sub-domain of mathematics.
In the same year, Shulman (1986b) argues that, in addition to the distinction between teaching knowledge that is generic and teaching knowledge that is subject-specific, subject-specific teaching knowledge should be separated into subject-matter content knowledge, curricular knowledge, and pedagogical content knowledge. Of these, pedagogical content knowledge (PCK) has probably received the most scholarly attention. In our interpretation, pedagogical content knowledge involves understanding where students are likely to have difficulty, and determining how to address those difficulties by drawing on a stock of “the most powerful analogies, illustrations, examples, explanations, and demonstrations” (p. 9). Making use of this stock to address particular student learning obstacles may well make additional mathematical demands of the teacher, because the teacher must see where a particular analogy, for example, accurately represents a mathematical idea and where the analogy breaks down. The teacher must also be able to modify an explanation if, despite the power it may have for some students, it does not seem to make sense to this particular class of students.

Our analysis focuses on subject-matter content knowledge for teaching mathematics, together with aspects of pedagogical content knowledge that embody understanding of mathematics. Analyses of subject-matter content knowledge have suggested several distinctions that cut across subject areas, such as substantive vs. syntactic knowledge (Schwab, 1961), or knowledge about the subject vs. knowledge of the subject (McDiarmid, 1994). In addition, analyses of subject matter knowledge for teaching have posited various distinctions: knowledge for teaching vs. knowledge for oneself (Borko et al., 1992) and lesson structure knowledge vs. subject matter knowledge (Leinhardt et al., 1985). Specific to mathematics teaching, Brousseau’s (1997) discussion of the “didactical transposition” (p. 21) has also influenced this work. He suggests that teachers use mathematical knowledge in the work of “transposing”
mathematics from its origins and formal meanings to mathematics that will be learnable by students. Drawing on such distinctions and considering how they might be applied in the development of a framework for knowledge for mathematics teaching, we have chosen to look also for important subject-specific distinctions that are particular to knowledge of mathematics. It may be that analogous distinctions will prove useful for looking at the teaching of other subjects, but we do not require such generality in our effort to focus on issues in algebra.

For the sake of clarity and simplicity, we use the terms “subject matter knowledge” or more specifically, “mathematical content knowledge” in place of Shulman’s term “subject-matter content knowledge.” As explained below, we use the term “mathematical knowledge for teaching” to signify that subset of mathematical knowledge that may be especially important for teachers.

3. Research on Teacher Knowledge and Student Achievement

Underlying previous research on teacher knowledge is the seemingly self-evident claim that teachers’ knowledge plays a significant role in what students learn. This claim is the basis for significant education policy decisions as well as many research studies. Studies of teachers’ mathematical knowledge, however, have found weak and contradictory evidence of connections between larger “amounts” of teacher subject matter knowledge and greater student achievement (see Begle, 1979). Rather than direct measures of teacher knowledge, early studies typically used proxies for teacher knowledge, such as the number of math courses taken, college grade point average, certification or lack thereof, or whether the teacher majored or minored in mathematics. In his survey of the literature, Begle (1979) concluded, “There are no experts who can distinguish the effective from the ineffective teacher merely on the basis of easily observable
teacher characteristics. Similarly, the effects of a teacher’s subject matter knowledge and attitudes on student learning seem to be far less powerful than most of us realized” (p. 54).

Reviews and studies done in the past decade have begun to uncover connections between teacher knowledge and student achievement in mathematics, as well as to highlight the complexity of teacher knowledge and background issues. Monk (1994) reported that mathematics teachers’ knowledge has a positive impact on student knowledge. Rowan, Chiang, and Miller (1997) found significant effects of teacher knowledge on student achievement, with the greatest effects occurring in combination with teacher preparation (majoring in mathematics) and in low performing schools. In a large scale study of comprehensive school reform, Hill and colleagues (2004a) found a positive relationship between teachers’ mathematical knowledge and student achievement in the first and third grades.

Work in this area also has produced counterintuitive findings: Rowan, Correnti, and Miller (2002) report that in grades one through six, “in mathematics… students who were taught by a teacher with an advanced degree in mathematics did worse than those who were taught by a teacher not having a mathematics degree” (pp. 13-14). Monk found that advanced mathematical coursework by secondary school teachers beyond a set of five courses added little value in terms of student gains. In short, evidence for the claim that teachers’ knowledge relates directly to students’ learning remains inconclusive.4

At least two complexities contribute to this mixed and inconclusive evidence. First, the interaction between teachers’ knowledge and student learning is not direct; rather, it is mediated by the acts of teaching and learning, as well as curriculum and instructional materials. That is, we expect teachers’ knowledge to influence student learning because we assume that teachers’ knowledge influences what teachers do -- both in and out of the classroom -- to support student
learning. In addition, what individual students actually learn from the contexts created by teachers’ choices depends as well on what the students bring to the situation.

Second, measuring teacher knowledge and student learning is difficult, both in terms of determining what should be measured and deciding how to measure it. For example, although many kinds of knowledge (e.g., mathematical content knowledge, general pedagogical knowledge, pedagogical content knowledge) would seem to influence a teacher’s teaching, some kinds of teacher knowledge may be more influential in student learning than others. Even within a single category of knowledge for teaching, say mathematical content knowledge, different kinds of knowledge (e.g., knowledge of how to carry out a procedure and knowledge of why it works) may differentially support student learning. Interactions among the different kinds of knowledge would also seem to play a role.

Wilson, Floden, and Ferrini-Mundy (2001) point out that one serious problem in studying these relationships is the lack of sufficiently sensitive measures for examining teacher knowledge. Rowan and colleagues (1997, 2002) and Monk (1994) allude to the same issue. Relationships between teacher knowledge and student performance might become more visible if the indicators of teacher knowledge were sufficiently sensitive to measure the kind of knowledge that is most likely, from a theoretical standpoint, to relate to student learning. Although Ball and colleagues (Ball et al., 2004; Hill et al., 2004a; Hill et al., 2004b) are making headway in conceptualizing and measuring mathematical knowledge for teaching at the elementary level, appropriate theoretical frames and instruments to measure such knowledge on a large-scale at the secondary school level are missing. The measures that have typically been used -- proxies such as those found in Begle’s literature survey and subject matter tests such as those used for licensing -- do not directly address the question of knowledge for teaching. That is, proxies do
not measure knowledge directly, while licensure tests typically have measured knowledge divorced from practice (Ahn & Choi, 2004; Wilson et al., 2002).  

Similarly, on the student end, achievement tends to be measured in terms of standardized test scores. Yet, these tests do not measure all that society, as well as experts in mathematics, believes is important for students to learn in mathematics. Not only are some areas of learning in mathematics difficult to measure, and thus typically omitted from such assessments (e.g., being able to compare the adequacy of two different mathematical justifications), but even within the narrower scope of subject matter competence, there are significant questions about how well these tests measure conceptual understanding – much less knowledge of the nature of mathematics and mathematical practice – versus knowledge of facts and procedures. In the area of algebra, there are no standardized exams in wide use that assess students’ understanding or ability to apply advanced algebra ideas and skills. Despite these complexities, recent evidence using measures that attempt to delineate mathematics used in teaching (Ball et al., 2004; Hill et al., 2004a; Rowan et al., 1997) suggests a relationship between teacher and student learning.

In order to make more effective decisions in both policy and practice, we need to understand better how teachers effectively draw on knowledge in teaching, what kinds of knowledge seem most important for teachers to use, and how to assess this knowledge. This may be especially important in at the middle and secondary school levels. At the secondary level, the prevalent background of teachers of mathematics includes a major or the equivalent in mathematics, and so assumptions about the adequacy of teachers’ mathematical knowledge for teaching have rarely been challenged. In fact, because the domain of mathematical knowledge for teaching at the secondary level is under-conceptualized, it is impossible to know the relationship between that knowledge and teachers’ effectiveness in supporting student learning.
The situation is even more complex for teachers of mathematics at the middle grades who, in the US, may be certified as elementary school teachers in many jurisdictions. Thus they are challenged to teach beginning high school level mathematics with minimal mathematical training. Our purpose in delineating a framework is dual: we expand the conceptualization of knowledge for teaching algebra at the secondary level, and we provide a tool for studying that knowledge.

4. Mathematical Knowledge for Teaching

While Shulman’s influential idea of pedagogical content knowledge is foundational to our research, the focus here is on mathematical knowledge for teaching, whether that knowledge might be categorized as subject-matter, pedagogical, or curricular content knowledge in Shulman’s framework. The theme is mathematics: mathematical knowledge for teaching is mathematics used in teaching and valuable for teachers to know. It is this body of knowledge—and particularly, knowledge for algebra teaching, that we aim to delineate, describe, and study.

Ball and Bass (2000a) have taken a similar approach to mathematical knowledge for teaching elementary school mathematics. They describe mathematical knowledge for teaching as a “kind of understanding [that] is not something a mathematician would have, but neither would be part of a high school social studies’ teacher’s knowledge” (p. 87). In conceptualizing and framing questions about mathematical knowledge for teaching at the elementary level, Ball and colleagues (Ball, 1988; Ball et al., 2000a, 2000b) examine mathematics teaching to generate hypotheses about how teachers access and deploy mathematical knowledge in their teaching. They contend that a special kind of mathematical work is entailed in teaching, and that teachers are more likely to engage in this kind of mathematical work than others who work in mathematically intensive careers.
Other scholars have addressed related issues about teachers’ subject matter knowledge. Ma (1999) describes “knowledge packages” that “reveal the teachers’ understanding of the longitudinal process of opening up and cultivating such a field in students’ minds” (p. 113-114). She puts forward the idea of “profound understanding of fundamental mathematics” as an area of mathematical knowledge used for teaching. Usiskin (2000) introduces the idea of “teachers’ mathematics -- a substantial body of mathematics that arises from teaching situations” (p. 9). He goes on to specify three categories of teachers’ mathematics: mathematical generalizations and extensions, concept analysis, and problem analysis.7 Cuoco (2001), in his paper entitled “Mathematics for Teaching”, pleads for ways of helping prospective teachers “experience the doing of mathematics … develop mathematical taste. …give them a sense for the really important questions that have led to the breakthroughs… develop the skill of seeing what to emphasize with their high school classes. …develop a passion for the discipline that lives on beyond college graduation…. ” (p. 170). There is clearly substantial current interest in gaining clarity about the meaning of mathematical knowledge for teaching at the secondary level.

4.1 Frameworks for studying teacher knowledge

We have located three frameworks that have been especially generative in our efforts to specify the dimensions of algebra knowledge for teaching. Even (1990) proposed “an analytic framework for subject matter knowledge for teaching a specific topic in mathematics. “ She then applied it to the concept of function. The Even framework contains seven elements:

1. Essential features
2. Different representations
3. Alternative ways of approaching
4. Strength of the concept
5. Basic repertoire

6. Different kinds of knowledge and understandings of the concept

7. Epistemological knowledge about the nature of the concept

In later work, Even (1993) applied the framework as a means of understanding the interrelationships between teachers’ subject matter knowledge and pedagogical content knowledge in the mathematical area of function. Our framework incorporates some of Even’s elements.

Zandieh (2000) developed a framework to “map the territory” of student understanding in calculus. Although not a framework designed for use in categorizing teachers’ mathematical knowledge, we summarize it here because it provides an organizational approach to this kind of mapping of knowledge. Zandieh’s framework has a two-dimensional structure: multiple representations (e.g., graphical, verbal) and contexts (i.e., places in which the concept resides) vs. layers of process-object pairs. Cells in the framework represent an aspect of the concept of derivative.

Researchers at Horizon Research (Smith, personal communication), interested in teachers’ knowledge in science, have also proposed several categories that they label “Dimensions of Content Knowledge for Teaching Science.” Some dimensions of subject matter knowledge for teaching as we are conceptualizing it include:

- knowledge of disciplinary content,
- knowledge that alternative frameworks for thinking about the content exist,
- knowledge of the relationships between big ideas and the supporting ideas in a content area,
- knowledge/understanding of student thinking about the content, and
• knowledge of how to sequence ideas for students to learn the content of interest.

We have drawn ideas and structures from all of these frameworks in formulating our framework. One issue that is problematic for frameworks of this type is the challenge of distinguishing between objectified knowledge and knowledge in use in the practice of teaching – a distinction that we have addressed in our framework. We aimed for a more general framework, one that could both support detail about algebra knowledge for teaching, and provide a tool for use in other mathematical areas.

4.2 Our perspective on mathematical knowledge for teaching

We focus on knowledge for teaching, acknowledging particularly that secondary school teachers of mathematics may have a great deal of mathematical knowledge that they do not use directly as they teach their students. We are not claiming that mathematical knowledge for teaching is the exclusive property of teachers; others -- mathematicians, engineers, and scientists -- might also have and use such knowledge in their work. What distinguishes this knowledge is that it is used in the course of secondary mathematics instruction. The question of whether this is knowledge uniquely possessed by teachers is an empirical question that our research may help answer but that is less important than whether and how the knowledge is used in and impacts teaching, whether and how this knowledge varies across teachers, and whether different areas of mathematics are characterized differently within the mathematical knowledge for teaching domain. If we can specify a body of knowledge that is important for teaching and if teachers’ profiles with respect to this knowledge vary widely despite apparently common formal mathematical background (e.g., majors in mathematics), we see a promising site for the improvement of mathematics instruction and student achievement at the secondary school level.
5. A Focus on Central Algebraic Concepts

We have identified school algebra as the mathematical domain of focus in our work. A recent report by the RAND Mathematics Study Panel (2003) identified the teaching and learning of algebra in kindergarten through 12th grade as one of three focal areas in its proposed research agenda. The RAND report’s synthesis of what various national reports have proposed as algebraic proficiency provides a reasonable description of school algebra as it is conceived currently:

- The ability to work flexibly and meaningfully with formulas or algebraic relations--- to use them to represent situations, to manipulate them, and to solve the equations they represent;
- A structural understanding of the basic operations of arithmetic and of the notational representations of numbers and mathematical operations (for example, place value, fraction notation, exponentiation);
- A robust understanding of the notion of function, including representing functions (for example, tabular, analytic, and graphical forms); having a good repertoire of the basic functions (linear and quadratic polynomials, and exponential, rational, and trigonometric functions); and using functions to study the change of one quantity in relation to another;
- Knowing how to identify and name significant variables to model quantitative contexts, recognizing patterns, and using symbols, formulas, and functions to represent those contexts. (RAND Mathematics Study Panel, 2003, pp. 44-45)

Since the mid-19th century algebra has been a standard high school course in the United States, providing a conceptual foundation and a language for studying abstractions and generalizations of numbers, and serving as a prerequisite to other mathematics courses and as a
requirement for admission to higher education (Jones et al., 1970). In recent decades many educators have argued that algebra should play a more prominent role in school mathematics, and that it should be taught to all students at an earlier age than has been typical (National Council of Teachers of Mathematics, 1989, 2000; Usiskin, 1987). This has led to debate and discussion about what should constitute school algebra, and to a variety of efforts to reform the K-12 algebra curriculum. In the U.S. algebra serves as a filter for future educational opportunities available to students (U.S. Department of Education, 1997).

A crucial equity and policy concern is that students from racial and ethnic minorities have fewer opportunities to learn algebra than students from the majority. Algebra has been called the “new civil right” (Moses, 1995; Moses et al., 2001), and “algebra for all” has become a goal of professional organizations, local and state educational agencies and other national organizations (e.g. Achieve, Inc., 2002; NCTM, 1989, 2000). In a study of 12,500 students, Gamoran and Hannigan (2000) found that all students, including those with very low prior achievement, benefit from studying algebra, where “benefit” is defined by growth in mathematical achievement between grades 8 and 10. In recent years, efforts have been made across the country to ensure that more students have access to algebra. Nationally, the percent of students who graduate from high school having studied algebra was 92% in 1996 (Dossey et al., 2000). A recent report about high school course-taking in the United States (Perkins et al., 2004), with a nationally representative sample confirms that this figure is holding: “Approximately 91.8 percent of the HSTS [NAEP High School Transcript Study] 2000 students took first-year algebra, geometry, and/or second-year algebra courses.” (p. 55). This focus on the importance and centrality of algebra leads to a concurrent challenge – that of ensuring that there is a well-prepared corps of teachers ready to implement the recommendations effectively. Thus a deeper
understanding of the knowledge involved in algebra teaching could be key in the development and continuing education of teachers in this area.

School algebra is in flux as curriculum developers, mathematicians, and mathematics educators continue to offer various proposal and formulations of what is most important. This variation changes the requirements of teacher preparation -- with the content of school algebra currently in a dynamic state, the demands on teachers’ knowledge are correspondingly in flux.

To ground our work, we chose to limit our analysis to a few central algebraic themes that can be argued to be foundational across various approaches to school algebra, and that have been demonstrated to be difficult for students, either through research and assessment findings, or in the experience of teachers.

5.1  Algebraic expressions and equations

The first topic we chose was algebraic expressions and equations. By “algebraic expression” we mean, “a combination of letters and operation symbols such that if numbers are substituted instead of the letters and the operations are performed, a number results” (Sfard, personal communication). Note that an algebraic expression can be seen as a string of symbols, a computational process, or as a representation of a number. And, if the context changes, an expression can become a function, representing change (Sfard et al., 1994). Sfard and Linchevski also clarify a potential issue with teachers’ and students’ understanding of algebraic expressions when they note “…the difficulty lies … in the necessity to imbue the symbolic formulae with the double meaning: that of computational procedures and that of the objects produced…. Once we manage to overcome this difficulty, it is quickly forgotten. To those who are well versed in algebraic manipulation (teachers among them) it may soon become totally imperceptible” (pp. 198-199).
An equation is a combination of letters, operations and an equal sign such that if numbers are substituted for the letters, either a true or false proposition results. The role of the symbol “=” in an equation to be solved is different from its role in the definition of a function or in a claim of identity (a mathematical statement considered as true). These different meanings are sometimes distinguished by using *defining expression* or *defining formula* relative to the function use. Others use “equation” for any string containing an equal symbol. In the design of this framework, analyses of instructional materials, and interviews with teachers, we have examined expressions, equations, inequalities, and functions that are linear, quadratic, exponential and logarithmic in form. We do not examine expressions or equations involving higher order polynomial, trigonometric or rational relationships.

The use of expressions and equations appears across various conceptions of school algebra, although the emphasis on the manipulation of the symbols, and the relationship to the concept of function, varies by treatment. Kieran (T. Kieran, 1992) distinguishes between *procedural* and *structural* operations within algebra, where *procedural* “refers to arithmetic operations carried out on numbers to yield numbers”, and *structural* refers to “a different set of operations that are carried out, not on numbers, but on algebraic expressions.” (p. 392). This distinction appears later in our discussion of the framework.

Aspects of student difficulty within this topic are well documented in the literature on students’ algebra knowledge (see C. Kieran, 1992). For instance, Booth (1984) highlights the need for students to build on their knowledge of arithmetic so that they are able to see the expression a+b as either an object or a single number. Kuchemann (1978, 1981) found that very few 13-15 year old students were able to see letters in algebraic expressions as generalized numbers (e.g., representing, or being able to take on, several values rather than just one, C.
Kieran, 1992, p. 396), despite classroom experiences that emphasized this meaning. Fey (1989) points to the multiple uses of variables, representing numbers, any one of the elements of a given set, or measurable quantities that change as a situation changes.

Several researchers have noted the difficulties students have with expressions, including a tendency to introduce equal signs inappropriately (see Chalouh et al., 1988; C. Kieran, 1983; S. M. Wagner et al., 1984) and then to solve resulting equations, suggesting a need for closure. A variety of research studies have indicated that students’ understandings of the varied uses of the equals sign are fragile and limited (see Wagner & Kieran, 1989). Other work has explored students’ understanding of the equal sign, noting the inclination to interpret equal signs as a “do something signal” (Behr et al., 1976; C. Kieran, 1981), or as a symbol indicating separation, rather than any kind of balance or equality (Byers et al., 1977).

Researchers have also examined the area of equation solving, finding that students use a range of different techniques for solving equations (e.g., use of number facts, use of counting techniques, cover-up, undoing, trial-and-error substitution, transposing, and performing the same operation on both sides, T. E. Kieran, 1992, p. 400); some of these are procedural, while some are structural. Kieran notes, on the basis of research in this area, that “students have generally been found to lack the ability to generate and maintain a global overview of the features of an equation that should be attended to in deciding upon the next algebraic transformation to be carried out,” (p. 401).

Various conceptual frameworks and models for organizing the subject of school algebra have been developed by researchers and theorists concerned with issues of student learning (see Nathan et al., 2000; Sfard et al., 1994), as well as by professional organizations providing
standards (see National Council of Teachers of Mathematics, 2000). We have considered these frameworks and conceptual organizations in developing our framework for teacher knowledge.

5.2 Linear relationships

The second focal area that we chose was linear relationships, an area that is central in any treatment of school algebra, and of course deeply interwoven with algebraic expressions and equations as described in section 5.1. The foundations of linear relationships are typically introduced to U.S. students in the middle grades beginning with the idea of proportional relationships and extending to straight lines (although it is not clear that teachers necessarily make this connection to the concept of linear functions). This category includes representations (tables, graphs, equations, verbal descriptions), properties, and behaviors of linear relationships and functions as well as elements that comprise them, relationships among them and the situations in which they are used or appear.

We argue that attention to the concept of linear relationships in our exploration of teacher knowledge is important, for several reasons. First, linear functions are used as the basis for the broader study of functions, and provide the first context in which students explore the function concept concretely, as well where they encounter such concepts as rate of change and graphs of relationships. Second, many physical phenomena are modeled using linear relationships. Finally, linear functions can serve as approximations for phenomena that are not linear. Ideas of modeling and curve fitting are developed from an analysis of linearity. Linear functions are found in different approaches to school algebra and are treated in different ways, ranging from emphasis on the concept of direct variation or proportionality as a precursor to linear functions, to emphasis on graphical representations (Chazan, 2000; National Council of Teachers of mathematics, 2000; Usiskin, 1988). Some treatments emphasize the concept of slope as rate of
change, while others relate slope to geometric concepts such as similarity. And, because functions-oriented approaches to algebra feature parameters and families of functions, the notion of linear functions becomes foundational. Thus from a mathematical perspective, together with the fact that the curricular issues around this topic are somewhat in flux at this time, we argue that the topic merits attention in our consideration of teacher knowledge.

Experience in teaching as well as research findings suggest that on the basis of given information, some students have trouble forming equations for straight lines, relating graphs to symbols, treating the line as an object itself, making qualitative judgments about graphs, and interpreting the meaning of the concepts involved, such as slope. Research literature that directly examines student understanding of linear functions is not as abundant as the literature about symbols, expressions, and equations. The available work, which is often about functions more generally, does provide important insights about where student difficulties emerge. For instance, Markovits, Eylon, and Bruckheimer (1986) found that students had difficulty with constant functions, concepts and representations of images and preimages, and transferring from graphical to algebraic form. Goldenberg and colleagues (Goldenberg et al., 1990, 1992) have discussed students’ perceptions about the effects of changing the y-intercept of a linear graph, and whether the result is a vertical or horizontal translation. In their review, Leinhardt, Zaslavsky, and Stein (1990) discuss issues of interpretation of graphs in terms of local and global process, similar to Monk’s use of the pointwise and across-time distinction (G. S. Monk, 1989). They report that treatment of the global features of graphs is “overlooked in the earlier grades of the curriculum” (p. 11). They highlight the under representation of the qualitative aspects of graphing in the mathematics curriculum. They also discuss research that has focused on the idea of “construction,” by which they mean creating a graph or plotting points from given data, or
creating a symbolic representation from a graph. Research indicates that constructing a function that represents a given graph can be difficult (Markovits et al., 1986; Stein et al., 1989; Yerushalmy, 1988).

Another aspect of function addressed in this review is the notion of prediction. Stein and Leinhardt (1989) created a task in which students were given a number of points lying on a straight line, and an x-coordinate. The students were then asked to plot a point that would follow the same pattern as the other points, and found this task difficult.

Work of these researchers and others (e.g., Moschkovich, 1990; Schoenfeld et al, 1993) has demonstrated clearly that this most fundamental area in the study of algebra – linear relationships – is challenging for students in ways that might not at first seem obvious. The issues for teacher knowledge in this area are likewise important to consider. Taken together, a careful look at student learning, issues of teaching, and teacher knowledge might lead to much stronger interventions in the area of algebra.


Our approach to the framework development began with literature review in the areas that have been discussed here. We chose areas of algebra that are foundational to school algebra, key to learning algebra in any of the various approaches taken in U.S. curricula and standards. We chose areas that pose difficulties and challenges for students, on the assumption that it would be especially fruitful to better understand teacher knowledge demands for such problematic areas. We then examined approaches to teaching those areas, including curricular treatments and video of teaching, against early versions of our framework. We conducted interviews with teachers using items from our focal domains. This iterative process resulted in the framework described here.
We propose a conceptual framework for knowledge of algebra for teaching intended as a template for organizing this domain relative to the teaching of middle and secondary school mathematics. Because the design of the framework is the effort of a research group whose members have substantial experience in teaching secondary school algebra, our experience as practitioners also influenced the framework. Examples used to illustrate the framework have been drawn from the literature, from our experience, and from ongoing instrument design in our research. Following the presentation of the framework and the illustrative examples, we will describe briefly how we have used the framework, and how it might be used in the future.

The framework is organized as a two-dimensional matrix (Figure 1) with a third aspect that permeates the matrix. The rows of the framework are categories of knowledge of algebra for teaching. The columns are tasks of teaching that identify “sites” or actions in which teachers use mathematical knowledge. Finally, we define three overarching categories -- decompressing, trimming\(^\text{10}\), bridging -- which are mathematical practices infused through all elements of knowledge of algebra for teaching. The two dimensions of the matrix, as well as the overarching categories, are explained below.

-- Insert Figure 1 about here --

Figure 1: Knowledge for Algebra Teaching Framework

6.1 Categories of knowledge of algebra for teaching

These categories emerged from our review of the literature and from some of our empirical work. They describe elements of the domain of mathematical knowledge, applied specifically to algebra.
6.1.1 Core content knowledge

This includes the main ideas and concepts of the domain, the commonly applied algorithms or procedures, the organizing structures and frameworks that undergird the mathematical domain. In the case of our focal areas, we would include such concepts as variable, equation, expression, slope, and linear function; such procedures as solving linear equations, factoring, or simplifying expressions; and such connections as the relationship between slope and similar triangles. Using core content knowledge involves finding structural similarity and equivalence, or underlying mathematical form from particular situations. For example, recognizing that the pattern of the units digits in whole number powers of two is related to congruence modulo four is an example of finding underlying structure.

Some of what we list in this category would, on the surface, match what appears in standards or expectations for middle and secondary school students. However, when analyzing the specific mathematical work done in the various elements of teaching practice, we note that teachers need to understand these concepts, algorithms, and procedures in ways that extend beyond, and are deeper than, what students need to know. Ma’s notion of “profound understanding of fundamental mathematics” evokes this kind of knowledge.

Even (1990) has a similar category called “essential features”, and talks about the “essence of a concept.” This is part of substantive knowledge of the subject Ball, Lubienski, and Mewborn (2001) discuss “knowledge of a web of ideas” (p. 438). This may be similar to what Hill, Rowan and Ball (2004a) are currently calling “common content knowledge” in the development of measures for the Study of Instructional Improvement.
6.1.2  Representation

In this category we place various forms and models for concepts and procedures, and various means of recording, organizing, and communicating concepts and procedures in the domain. Kaput (1985) describes a representation as inherently “involving some kind of relationship between symbol and referent…” (p. 383). Most definitions of representation refer to some kind of correspondence that preserves structure or meaning (e.g., Cuoco, 2001); that is, representations are packages that relate objects and their transformations to other objects and their transformations.

For our focal areas within algebra, this would include graphs of lines and functions, concrete models such as algebra tiles, tabular or verbal descriptions of relationships between variables, etc. The category includes relating various representations, such as connecting the symbols in the equation of a linear function with the related components of the graph. Many authors have written about the importance of multiple representations in mathematics learning (see Tall, 1992; Vinner, 1983). NCTM’s Principles and Standards for School Mathematics (2000) observes “the term representation refers both to process and to product – in other words to the act of capturing a mathematical concept or relationship in some form and to the form itself.” They note the importance of selecting, applying, and translating among mathematical representations. Ma (1999) argues for something similar: “to promote understanding, it is necessary for teachers to be able to make connections between manipulatives and mathematical ideas explicit” (p. 6).

6.1.3  Content trajectories

This category of subject matter knowledge extends beyond what might be expected for student understanding, and is one that we hypothesize is important for teachers.
trajectories we mean understanding both the origins and extensions of core concepts and procedures – knowing the basis for ideas in the domain, and understanding how those ideas grow and become more abstract or elaborated. For example, knowing that the concept of slope builds on ideas from ratio and proportion, and that it leads to the more general concept of rate of change and derivative, would be knowledge of a content trajectory in algebra. Here also is knowing how ideas “fit into the larger landscape” (see Cuoco, 2001, p 169-170). Cuoco gives an example, using DeMoivre’s Theorem, about how, without having a sense of where mathematical ideas are situated in a larger scheme, a teacher may emphasize a low-level application rather than exploit important mathematical connections.

Zandieh (2000) in her framework development, raised the question of whether students would learn more efficiently if various aspects of a concept were introduced to them in a particular order (p. 103); such ordering would fall into our definition of trajectories. Usiskin’s (2000) ideas about being able to find extensions and generalizations of familiar theorems are consistent with knowing trajectories. Usiskin, Peressini, Marchisotto and Stanley (2003) discuss knowing the relationships of ideas studied in school to ideas that students may encounter in later study. Included here would be understanding the mathematical and pedagogical costs and benefits of selecting various sequences and approaches to particular topics – e.g., some sequences that might be mathematically elegant might present challenges for learning for various reasons. A content trajectory may be characterized by a mathematical ordering or rationale as well as a sense of how ideas might be best organized to support student learning.

Cuoco (2001) talks about “the ‘flatness’ syndrome” (p. 169), something he has observed in classrooms, where distinctions about the different levels of importance and role of topics are not made. We posit that a part of teacher knowledge involves being able to distinguish big ideas
from less important ideas, and to locate them within the trajectory. Greeno’s (1994) concept of understanding as navigation of a landscape is also consistent with what we are trying to express here.

The notion of alternate approaches is a component of content trajectories. This includes knowing that the main ideas and concepts in a domain can be organized according to a particular perspective on that content, and with various emphases, and knowing in particular some of the sequences and emphases that instantiate the perspective. Chazan (2000), for example, discusses various “mathematical points of view” or approaches, that might be represented in school algebra: e.g., a points of view based in arithmetic, theory of equations, mathematical models, real analysis, or abstract algebra. He points out that discussions and debates about the alternative approaches to algebra are “not an everyday concern” for high school teachers (p. 75), yet given the range of available instructional materials and tools that teachers might choose to use in teaching high school algebra, it is conceivable that awareness of the alternatives and the concomitant emphases and de-emphases might be useful. Our colleagues Senk et al. (in preparation), in an area that they have labeled “curricular knowledge”, include a related idea: “awareness that content can be packaged in different ways that are mathematically correct.”

Finally, examples has been included in this category because we feel that knowing canonical examples is closely related to knowing about different approaches to a topic, and to different sequencing. Even (1990) calls for teachers to know “powerful examples that illustrate important principles, properties, theorems, etc.”, and contends that these “should be well known and familiar, and readily available for use” (p. 525). Usiskin discusses the notion of alternative ways of approaching problems; knowing these alternatives for certain “standard” problem types.
in the school curriculum, and being able to make good choices among those alternatives on the basis of student needs and mathematical considerations, would fall into this category. This category relates to one of the components of pedagogical content knowledge proposed by Shulman: “powerful analogies, illustrations and examples” (Shulman, 1986b).

6.1.4 Applications and contexts

Within this category we include knowledge of problems that arise from situations, contexts, or circumstances outside of algebra, or within a different part of algebra. We draw on the ideas of Freudenthal’s (1991) “realistic mathematics”, which advocated that a real-world situation or authentic context should serve as the starting point for learning mathematics. The solution of such problems involves using the concepts and procedures of algebra. Such problems are used in the school algebra curriculum as sites both for introducing new concepts and for applying concepts in various “real world” or contextualized situations. The school algebra curriculum has a long history of including what some have called “traditional word problems” including problems about age, rate-time-distance, work, mixtures, coins, etc. Researchers and curriculum developers have examined the various techniques typically used, most prominently the translation of these contextually-based problems into equations or systems of equations which carry the substantive relationships embodied in the problem, and can be solved using the manipulative techniques of algebra without attention to the contextual meaning. Here we also include the types of contextual problems that have resulted from the interpretation of the 1989 NCTM standards’ call for more real world applications, and for problems that might be more meaningful for students. These sorts of problems might be more consistent with the kinds of items represented in PISA, where a “mathematisation cycle” is used to describe the steps
involved in solving these problems that are “situated in reality” (Programme for International
Student Assessment, 2003, p. 38). These steps include:

• Organizing it according to mathematical concepts and identifying the relevant
  mathematics;
• Gradually trimming away the reality;
• Solving the mathematical problem; and
• Making sense of the mathematical solution in terms of the real situation… (p. 38).

Mathematical modeling, described in NCTM’s *Principles and Standards for School
Mathematics* (2000) as “identifying and selecting relevant features of a real-world situation,
representing those features symbolically, analyzing and reasoning about the model and the
characteristics of the situation, and considering the accuracy and limitations of the model.”
(NCTM, 2000, p. 303) is central in this category.

For teachers, it seems that knowing some of the useful contexts for motivating particular
concepts in algebra (e.g., situations that will give rise to linear relationships vs. situations that
will be modeled with linear growth), and knowing interesting applications and for working with
contextualized problems, may be a part of knowledge of algebra for teaching.

6.1.5 *Language and conventions*

This category falls within syntactic knowledge of the discipline of mathematics, and in
particular, the syntactic knowledge of school mathematics. It involves understanding aspects of
the nature of mathematics, including what is arbitrary, what is based on convention, and what
necessary in terms of logical and axiomatic structures. Within knowledge of language we would
include such things as knowing that the meanings of terms commonly used in school algebra
(such as equal, equivalent, simplify, solve, variable, function, and expression) may be ambiguous
for students, are not used consistently across instructional materials, and are possibly conflated with students’ everyday meanings for these words. School algebra involves both definitions (e.g., definition of the slope of a line) and language conventions (e.g., let x equal the unknown.). Within his construct of teachers’ mathematics, Usiskin includes the need for teachers to be able to analyze alternate definitions.

Notational conventions are a part of this category as well, such as knowing the distinctions among $2x$, $2^x$, and $x^2$. The mnemonics and abridged reminders common in school algebra in the United States fall into this category. Examples include PEMDAS (parentheses, exponents, multiplication, division, addition, subtraction) for order of operations or FOIL (first, outside, inside, last) for multiplication of two binomial expressions.

The talk that occurs in mathematics classroom draws on knowledge that is mathematical. Adler (2002) building on the ideas of David Pimm, makes a distinction between two types of talk in the mathematics classroom: “Exploratory talk which is informal, and a necessary part of talking to learn because learners need to feel at ease when they are exploring ideas (Barnes, 1992, p. 126); and (ii) discourse-specific talk which is part of learners’ apprenticeship into the discourse genres of subjects in the school curriculum (Wells, 1992, p. 291)” p. xx. Mathematics and school mathematics make significant use of conventions, deriving in large part from the compactness of mathematics (e.g., the use of notation, the introduction of new terms that capture what otherwise would have been expressed with a phrase or more). These conventions often pose difficulties for students (see, for example, Booth, 1984), particularly in the two focal areas of algebra that we have selected. Consider the notation $f(x)$ for function. Students may read this as “$f$ times $x$” until they learn the convention that is it read “$f$ of $x$” and represents a function. Even the notion of $2x$ involves the convention of suppressing the multiplication symbol, while
the other three arithmetic operation symbols are preserved in algebraic notation (e.g., $a + b$, $a - b$, $a/b$). As students move from arithmetic to algebra, there are interesting notational transitions that occur; for instance, the notation for the mixed number $3\frac{1}{2}$ is understood to mean $3$ plus $\frac{1}{2}$. Then in algebra, seemingly similar notation such as $3x$ means $3$ times $x$. The work of teaching associated with knowing conventions overlaps some with the work associated with coordinating mathematical and everyday language.

6.1.6 Mathematical reasoning and proof

In school algebra knowledge of reasoning and proof includes such things as knowledge of the specialized vocabulary of reasoning (e.g. terms such as contrapositive), the ability to find examples and counterexamples of statements, the ability to use analogies or geometric arguments to justify statements, and the ability to use various proof techniques within an axiomatic system to make convincing arguments. Knowledge of reasoning and proof also includes being aware of the role of definitions and axioms (assumptions) in algebra, appreciation of the consequences of changes in definitions or assumptions, and the equivalence of certain axioms, such as the Principle of Mathematical Induction and the Well-Ordering Principle.

This category is another area of syntactic knowledge, in that it involves knowledge about the nature of mathematics, which includes knowing the forms of argument and justification that are used in mathematics, the means by which truths are established, and the level of rigor that is appropriate for the community (in this case, the class). In particular, for school algebra, this category includes knowing the standard conventions about how algebraic arguments or justifications are made. For instance, the processes typically used in simplifying algebraic expressions rely on the properties of rings of polynomials, including distributivity of multiplication over addition, additive inverses, and associativity.
For the focal area of linear functions, a teacher may need to respond to a student who believes that the equations 0 = 2y - 4x + 1 and y = 2x – 1 will generate different graphs because of the following argument: functions can only be the same if they have the same slope; the slope is the coefficient of x; these two equations have different x coefficients; thus, they must have different graphs. If this student response is unfamiliar to the teacher, it will require making a judgment of the mathematical argument, responding to it, or perhaps producing another argument.

*Principles and Standards for School Mathematics* (NCTM, 2000) includes the following components of reasoning and proof for elementary, middle, and high school students:

- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof (NCTM, 2000, p. 342)

The Performance Expectations dimension of the Third International Mathematics and Science Study (TIMSS) Curriculum Framework (Robitaille et al., 1993) developing vocabulary, developing algorithms, and axiomatizing as part of mathematical reasoning. However, the TIMSS framework did not include evaluating mathematical arguments or proofs (Valverde et al., 2002, p. 186).

**6.2 Tasks of Teaching**

In developing and using the framework as a way of classifying knowledge elements, we found ourselves drawn to the discussion of knowledge in use, and thus arrived at the view that knowledge for algebra teaching, in our two focal domains, is likely to be most visible or most readily hypothesized about when one is considering particular acts or practices of teaching.
Much has been written about ways of parsing or describing elements of the work of teaching (see, for example, Ball et al., 2000b; Chazan, 1999; National Research Council, 2001; Seago et al, 2004). We have drawn on that work in arriving at the following categories, chosen because of their potential as sites for using mathematical knowledge for teaching.

6.2.1  *Analyzing students’ mathematical work and thinking*

This category encompasses central activities of teaching for which we speculate that mathematical knowledge for teaching, in interaction with other kinds of knowledge, such as pedagogical content knowledge and knowledge of students, may be deployed. In listening to and interpreting students’ explanations, interpreting and responding to their questions, “figuring out what students know” (Ball and Bass, 2000a, p. 89), and “determining the mathematical validity of a student strategy, solution, or conjecture” (National Research Council, 2001, p. 134), teachers draw on mathematical knowledge elicited by the students’ mathematical work. They may need to determine what an individual student believes, understands, or has difficulty with; to find ways of making sense of student thinking even when it is incorrect; and to decide whether a surprising idea that a student provides is worth capitalizing on and exploring.

6.2.2  *Designing, modifying and selecting mathematical tasks*

The NCTM *Professional Standards for Teaching Mathematics* (1991) point to “worthwhile mathematical tasks” as essential in effective mathematics teaching. The work of designing, modifying and selecting such tasks is inherently mathematical in its knowledge demands. “Making the task accessible to a range of learners” (NRC, 2001, p. 134) and modifying tasks to fit the needs of particular groups of learners while preserving the mathematical intentions of the tasks are [mathematical activities. Work with mathematical tasks also involves appraising and assessing mathematical elements, organizing tasks in ways that will
help students learn the intended mathematics, and scaling back or modifying tasks to make them accessible to students. Issues of determining and maintaining cognitive demand of tasks (see Stein et al, 1996), determining how to scaffold student work (Shannon, NRC, 2000), and deciding if tasks allow for discussion (Black & Wiliam, 1998) are also part of this category. In particular, Black and Wiliam note: “To begin at the beginning, the choice of tasks for classroom work and homework is important. Tasks have to be justified in terms of the learning aims that they serve, and they can work well only if opportunities for pupils to communicate their evolving understanding are built into the planning.” (p. 143)

Deciding where tasks fit in mathematical trajectories, determining if a particular task is likely to be accessible in mathematically useful ways to a range of students, and planning for how to maintain cognitive complexity are also aspects of this category. Kilpatrick, Swafford, and Findell (2001) discuss the challenges in teachers’ design of tasks: “This process entails judgments about design so that the tasks anticipate students’ responses and are built on appropriate-sized mathematical steps” (p. 350).

6.2.3 Establishing and revising mathematical goals for students

U.S. mathematics teachers today are confronted with various national standards documents and state-level grade-by-grade standards that outline very specific mathematical goals for each school year. It is within the teachers’ purview to consider how those policy documents can fit together with designated instructional materials to support specific mathematical goals for their students. This involves mathematical judgment and knowledge. Deciding what the central ideas are in a given domain, which ideas are most important, what should be emphasized, what sequence to use, and how to approach particular topics is part of this process of establishing goals. This relates to understanding and knowing the mathematical precursors and successors of
key ideas and concepts—the earlier idea of knowing how to work with content trajectories. And, once goals are established, there is mathematical activity involved in refining, rethinking, and reflecting on them.

6.2.4 Accessing and using tools and resources for teaching

A multitude of instructional materials and other pedagogical texts (e.g. technology applications, concrete materials, supplementary treatments of mathematical topics, etc.) are available to teachers. Evaluating their appropriateness for a given mathematical goal and a given set of students, determining how to integrate or coordinate ideas from across a set of such resources, and organizing their use so as to support student learning of mathematics involves mathematical knowledge. In addition, creating correspondences between the tool or manipulative and the mathematical concepts or procedures is also part of secondary school mathematics teaching. For instance, understanding the affordances and limitations of a tool such as algebra tiles for exploring polynomial arithmetic is a type of knowledge in this category. Likewise, knowing what kinds of student difficulties and misunderstandings might emerge when using technological tools such as calculators (see review by Burrill et al, 2002) would fall in this area.

6.2.5 Explaining mathematical ideas and solving mathematical problems

An important aspect of teaching is finding appropriate ways to explain mathematical ideas to students in ways that help them learn. Teachers also find themselves, in the course of teaching, planning, or reflecting on teaching, solving mathematical problems that they did not anticipate – sometimes these problems emerge in class from students’ observations or questions; sometimes they are problems in textbooks. Frequently teachers find themselves doing mathematics problems publicly in their classrooms; modeling the actual work of doing mathematics is part of teaching.
6.2.6  Building and supporting mathematical community and discourse

Finally, we propose that establishing a mathematical community and productive mathematical discourse in a classroom relies on elements of mathematical knowledge for teaching. For instance, deciding what a class will hold as mathematical assumptions, axioms, or starting points for justification and argument might be seen as part of building a mathematical community. Developing definitions as a group and discussing expectations for students’ mathematical explanations also might be part of such community building (see National Research Council (NRC), 2001, p. 134). We conjecture that helping students learn to interact with one another mathematically, respond appropriately to each other’s claims and arguments, and collectively build a body of mathematics knowledge for the classroom, all entail mathematical knowledge for teaching. Modeling the state of the mathematical understanding of the entire classroom relative to a particular concept may be part of this work.¹²

6.3  Overarching categories

We posit that there are mathematical practices that serve as overarching categories within the work of teaching. These are decompressing, trimming, and bridging. As we have indicated in figure one, these overarching categories in a sense “hover” over the two-dimensional framework. Here we both describe these categories and provide some examples as they apply to entries in the framework.

6.3.1  Decompressing

Ball and Bass (2000a) and Cohen (2004) describe the need for teachers to “decompress” their knowledge in the practice of teaching. Cohen argues, “Because teachers must be able to work with content for students in its growing, not finished, state, they must be able to do something perverse: work backward from mature and compressed understanding of the content
to unpack its constituent elements” (p. 98). Ball and Bass suggest that, for elementary school teachers, this involves “deconstruct[ing] one’s own mathematical knowledge into less polished and final form, where elemental components are accessible and visible” and that mathematics’ “polished, compressed form can obscure one’s ability to discern how learners are thinking at the roots of that knowledge.” (p. 98). They go on to suggest that decompressing may involve finding the invisible elements of a mathematical idea that, for someone who is more mathematically mature and fluent, may have been covered up long ago, or perhaps never known.

In considering the role of decompressed knowledge in the teaching of algebra, we focus on our two topical areas. So, for instance, “decompressing” may involve attaching fundamental meaning to symbols and algorithms that are typically employed by sophisticated mathematics users in automatic, unconscious ways. A number of studies indicate that students have difficulty using unknowns and variables, including the inability to distinguish between algebraic letters as generalized numbers or as specific unknowns (see MacGregor et al., 1997). Algebraic letters are interpreted as “abbreviated words” or labels for objects (Quinlan, 2001, p. 512).

Consider, for instance, the different nature of the solutions in the following:

\[3x + 8 = 17, \text{ and } 3x + 8 < 17.\]

Most algebra teachers would have efficient algorithms for solving both the equation and the inequality and could competently show students those algorithms. Yet the first problem yields a single value for \(x\), while the second produces an infinite set. For teachers, being able to “unpack” these different roles of the variable in superficially similar situations might be quite useful in supporting deeper student understanding. Decompressing seems to involve making the situation more complex than it appears on the surface.
Our interviews with teachers include the following task, which would be located in the cell (analyzing students’ mathematical work and thinking, core content knowledge.)

Suppose you ask your students to solve \(4(x-1)-x = 3x - 4\).

What questions or difficulties might you expect students to have?

How might you respond if a student asked you to explain why you can get \(0 = 0\) when solving an equation?

Teachers’ responses varied. Several talked about using a few specific numbers as a test to suggest that this is an identity that holds for all \(x\). This suggests to us another case of a need for teachers to be able to decompress their knowledge. When applying a solution algorithm for equations, in this case, it is necessary to attach meaning (or make sense of) the result of an identity such as \(0 = 0\) or \(3x - 4 = 3x - 4\). Basically, the solver has assumed that there exists some \(x\) such that \(x\) is a solution to this equation, and has transformed this equation to a series of equivalent equations (i.e., equations whose solution sets are the same as those of the original equation.) The solution set of \(0 = 0\) is all real numbers; thus the solution of the original equation is also all real numbers. However, students might conclude from this equation that the solution is 0. In addition to realizing that in this case the equal sign conveys an identity, teachers may also need to help students connect the idea of variable as taking on a single, specific value and the notion that a variable may represent a set containing many elements.

It may be that, for secondary school mathematics teachers, decompressing involves developing knowledge that is new for those with sophisticated mathematical training, and that takes something seemingly simple and routine and “complexifies it”. The need to do this may arise in attempts to address directly student learning difficulties of algebra. Sfard and Linchevski (1994, p. 112) observe “Singularities and the things that happen at the fringes of mathematical
definitions are often the most sensitive instruments with which student’s understanding of concepts may be probed and measured.” We would suggest that decompressing may be a useful activity for teachers themselves to undertake in the context of these “fringes of mathematical definitions.” We expect that decompressing comes into play in direct interactions with students around student misunderstandings and questions, as well as in the design of lessons and choice and sequencing of algebra tasks. So, in terms of our framework, we suspect that decompressing may be particularly relevant in the following rows: analyzing students’ mathematical work and thinking; and designing, modifying and selecting mathematical tasks. Empirical work would be necessary to test this assumption.

6.3.2  Trimming

Mathematics teachers of grades nine through twelve typically have studied a substantial amount of mathematics; for more than fifty years, most national recommendations about the mathematical preparation of secondary school teachers have called for the equivalent of a major in mathematics (Ferrini-Mundy et al., 2003) and thus it seems plausible to assume that most have had some exposure to more advanced versions of mathematical ideas that appear in the secondary curriculum. The undergraduate mathematics curriculum that forms the basis for the mathematical preparation of high school mathematics teachers in the United States includes as staples an emphasis on axiomatic systems, formal proof, and mathematical rigor. The high school algebra curriculum includes concepts and processes that can be viewed precursors to ideas that appear in more sophisticated, abstract, or general ways in the undergraduate curriculum.

In school algebra, as well as other areas of the curriculum, we propose that teachers have to trim mathematical or contextual content in a way that is mathematically acceptable but leaves
intact the content to be learned. This trimming entails retaining important mathematical features while reducing complexity in ways that make the content accessible to students.

We include within trimming the notion of “shortcutting”, which is used in explaining what is meant by horizontal mathematisation (Treffers et al., 1985). Horizontal mathematisation is “transforming a problem field into a mathematical problem question” (p. 109). A transformation of the problem field into a mathematical formulation may not preserve all of the information available, but rather that which is necessary to preserve the essence of the situation. Similarly, we see trimming as a transformation of mathematical ideas from a more advanced or rigorous form to a form that preserves the essence but that will be accessible to students, considering their backgrounds, understanding, and knowledge. This may be an element of the didactical transposition as construed by Brousseau, where he notes that teaching involves "recontextualization and repersonalization" (1997, p. 23). Moving from an applied context of some kind to a mathematical formulation also may involve trimming away irrelevant context, and “promoting the mathematical features of the situation” (Programme for International Student Assessment, 2003, p. 27). Ma (1999) discusses the possibly related notion of having a “clear idea of what is the simplest form of a certain mathematical idea” (p. 47).

We see trimming as being related to Bruner’s oft-quoted hypothesis: “… any subject can be taught effectively in some intellectually honest form to any child at any stage of development” (Bruner, 1960, p. 33). Bruner, writing at the onset of the post-Sputnik curriculum reforms in the United States, was promoting the idea of spiral curriculum and of early introduction of fundamental structural ideas in mathematics to young children. In 2004, with the “new math” having come and gone, we are not promoting the idea that secondary school algebra teachers should be teaching some version of Galois theory, but rather we explore the
mathematical knowledge demands required to work with what is currently part of the standard curriculum. We hypothesize that trimming is necessary and involves mathematical work. Bruner quotes David Page: “…giving the material [mathematics] to them in terms they understand, interestingly enough, turns out to involve knowing the mathematics oneself, and the better one knows it, the better it can be taught” (p. 40).

Trimming involves scaling down, and intentionally and judiciously omitting detail and modifying levels of rigor, and also being able to judge when a student, or a textbook presentation, is trimming, and if so, whether the trimming is appropriate. It includes reducing one’s own mathematical understanding to a form that is accessible for a student, or that connects well to the mathematics students bring to the context. Trimming includes the interpretation and judgment a teacher must use in considering a textbook’s treatment of a mathematical concept or process. For example, in calculus (considered a postsecondary area of study in the U.S.), students see how to raise any positive number to an arbitrary power through the definition of the natural logarithm function and then through the definition of the function $e^x$ (Thomas et al., 1992). Yet most secondary school algebra curricula introduce exponential functions, often with care about moving from integer to rational exponents but then with little discussion about why the graphical representations of such functions as they are presented in secondary textbooks (continuous graphs) are not justified, or even why it is possible to operate on such functions. This is trimming that teachers may or may not recognize, with possible consequences for their students.

Whatever mathematical decisions are made in trimming need to anticipate later mathematical ideas that students will encounter. In elementary mathematics, a frequent example is how the adage “multiplying makes bigger” can cause problems for students when they later
encounter multiplication of whole numbers by fractions between zero and one (Graeber et al., 1990). At the secondary school levels, the examples are more complex. In an abstract algebra or number theory course, prospective teachers might see a proof of the theorem that a negative integer multiplied by a negative integer equals a positive integer in the more general setting of rings, a proof dependent on the uniqueness of additive inverses, the distributive property of multiplication over addition in rings, and the multiplicative property of zero (see Burton, 1988, p. 250). Certainly this approach is too abstract for middle school teachers to use in introducing this foundational algebra idea. In the middle grades, when the concept of negative integers is introduced and when rules for multiplication of integers are discussed, teachers can choose from a variety of approaches for helping students understand the rules. While Davis’ (1980) “postman stories”\footnote{Davis, 1980} are compelling, a pattern approach, such as

\begin{align*}
3 \times 4 &= 12 \\
3 \times 3 &= 9 \\
3 \times 2 &= 6 \\
3 \times 1 &= 3 \\
3 \times 0 &= 0 \\
3 \times -1 &= -3 \\
3 \times -2 &= -6
\end{align*}

may be a better example of trimming because key structural elements of the ring of integers are embedded here.

Trimming and decompressing are closely related. Consider the following homework problem in an Algebra I text (Larson, 2001, p. 530):

*Represent the solution graphically. Check the solution algebraically.*
2 \times 2 + 4 \times 6 = 6

The textbook’s intention is that the student graph the functions \( f(x)=2x^2+4x \) and \( g(x)=6 \), find their points of intersection, and note that the \( x \) coordinates of those points are the solutions. However, if a student first “divides through by 2”, then her graph will look different, although her solutions will be correct. For herself, the teacher might need to verify this by thinking in an informal or more formal way: Suppose \( x \) is a solution of \( f(x)=g(x) \), and \( k \) is a non-zero real number, then \( x \) is also a solution of \( kf(x)=kg(x) \). If a teacher can work out for herself why “dividing through by 2” is acceptable, by decompressing the problem, she still faces the mathematical challenge of “trimming” her reasoning back to a level that allows her to explain the matter to a high school algebra student in a mathematically defensible way that is understandable at the student’s level, and perhaps that relates to the curriculum’s graphical treatment of the topic. We would locate this example in the cell (analyzing students’ mathematical work and thinking, representation).

Another example of trimming, in the domain of linear relationships, is suggested by the introduction to slope in the Interactive Mathematics Program (Fendel and Resek, 1999, p. 300):

The concept of slope is an abstraction from the idea of a rate of change. Formally, the slope of the line connecting two points \( (x_1, y_1) \) and \( (x_2, y_2) \) is defined as the ratio

\[
\frac{y_2 - y_1}{x_2 - x_1}.
\]

The numerator in this fraction is sometimes referred to as the change in \( y \) or the rise. The denominator is called the change in \( x \) or the run.

What trimming is involved here? First of all, one might pause in reading the first sentence of the explication, to wonder if the concept of slope is instead a particular case (rather than an abstraction) of the idea of rate of change. Recalling the definition of the first derivative, a teacher might note that the rate of change concept for non-linear functions of one variable rests on the
notion of the constant rate of change in lines. So, in considering how to “trim” these ideas back to connect both to the curriculum materials and to student thinking, teachers’ mathematical knowledge comes into play. Key in defining slope as a difference quotient, as this text does, is that this calculation produces the rate of change for straight lines (not for other curves.) It might also be important for teachers to know that the language of “rise” presents a problem (reported by experienced algebra teachers, similar to the “multiplication makes bigger” problem) for some students who interpret rise as a positive quantity, where in this case, rise may be negative. This example could be located in the cell (accessing and using tools and resources for teaching, language and conventions).

6.3.3  Bridging

This category is meant to include various kinds of connecting and linking that mathematics teachers are called upon to do: bridging from students’ understandings to the goals that the teacher is seeking to meet; connecting the ideas of school algebra to those of abstract algebra and real analysis; and linking one area of school mathematics to another. Bridging from students’ understanding to the goals that the teacher is trying to meet requires the ability to assess student understandings and the disposition to work from them as a base for instruction. The connections between school algebra and abstract algebra or real analysis are not necessarily obvious, although some curricular materials have been developed to explore those connections (e.g., Cooney, Brown, et al., 1996). The Mathematical Education of Teachers (Conference Board of Mathematical Sciences, 2001) suggests that all teachers should understand how basic ideas of number theory and algebraic structures underlie rules for operations on expressions, equations, and inequalities. Some experts have also commented on the linkages between one area of school mathematics and another. For instance, it is commonly believed that developing algebraic skills
and concepts is dependent on an understanding of arithmetic (see Wu, 2001). Van Dooren, Verschaffel, and Onghena (2002) suggest that the arithmetic in the elementary grades should be “algebraified” to facilitate the formalizing of the algebra skills as students progress through the curriculum. For example, place value can be linked to polynomial expressions through expression of base ten numbers in expanded form; addition of fractions can be related to the development of an algorithm for adding rational expressions.

We offer an illustrative example in the cell (explaining mathematical ideas and solving mathematical problems, language and conventions). Moving appropriately from exploratory language to discourse-specific talk may involve bridging knowledge, that is, being mindful of how the students talk about the ideas in exploratory language, and then helping them transition to language that is more conventional mathematically. We have heard students say that they solve proportion problems such as \( x/3 = 4/9 \) by “cross multiplying” – a practical but mathematically limiting way of remembering how to do such problems. Helping students make the transition to understanding that they are multiplying both sides of an equation by the same number might be an example of moving from exploratory talk to discourse-specific talk that involves bridging.

In our analyses of various algebra textbooks, we noted that different “content packages” (in the sense of Ma, 1999) are presented by different series. For example, the University of Chicago School Mathematics Project introduces the concept of linear functions together with arithmetic progressions. The Interactive Mathematics Program introduces linear functions with quadratic functions as a way of comparing and contrasting ideas. Teachers need to hold flexible understandings and be proficient at making sense of such alternative ways of organizing the material. This kind of understanding might be located in the cell (establishing and revising mathematical goals for students, content trajectories).
7. Applying the Knowledge for Algebra Teaching (KAT) Framework and Implications for Future Work

The Knowledge of Algebra for Teaching Framework we have developed is intended as a device for organizing hypotheses about knowledge of algebra for teaching. We envision that the cells would be filled with specific examples that fall within our two focal areas, that arise in some task of teaching, that fall into a particular category of mathematical knowledge, and that seem to involve trimming, decompressing, or bridging. In other project work we report our use of the framework as a guide for analyzing instructional materials (Senk et al., in preparation), interpreting teacher interviews (Marcus, 2004), and analyzing videos of teaching (McCrory & Smith, 2005). In a current research project we are using an adapted version of the KAT framework as a blueprint for the development of assessment items to measure teachers’ knowledge.

Other researchers engaged in the study of teacher knowledge (e.g. Ball and Bass, 2000a, b; Borko, Peressini, Romagnano et al., 2000) offer a number of compelling examples and ideas about mathematical knowledge for teaching. Much of this work highlights mathematical processes, such as reasoning or justification, or mathematical practices (RAND, 2003), such as symbolizing or mapping between representations. One of our intentions is to find the affordances and constraints that arise when a mathematical subject area is placed securely in the foreground of an effort to clarify ideas about mathematical knowledge for teaching. We find that the overarching processes that we posit (e.g. decompressing, trimming, bridging) emerged in part from our literature review because they had special resonance with the mathematical demands that we view as particular to the teaching of algebra, and because the examples that we found in our empirical work lent themselves to explanation along these categorical lines. Nonetheless,
keeping a subject matter in the foreground is challenging, and one negative byproduct may be that more general mathematical knowledge demands in high school teaching have not been identified or anticipated in this work.

We also find that the relationships and overlaps among mathematical knowledge, mathematical knowledge for teaching, pedagogical knowledge, curricular knowledge, pedagogical content knowledge, and dispositions and commitments are not at all clear, and that indeed the most useful models ultimately reflect an intertwining and integration of these areas. Thus the distinctions made here are in some sense artificial and are intended to highlight an area of interest for our group. Another distinction difficult to maintain was between the tasks and practices of teaching and the knowledge that might be used in those tasks and practices. It is impossible to discuss knowledge apart from its use, and we addressed this problem methodologically by drawing some of our empirical work from videotapes of classroom teachers, interviews with teachers about tasks grounded in teaching practice, and analysis of curriculum materials.

We have provided a framework that we hope will provide possible clarification of the complex ideas surrounding knowledge of algebra for teaching. Within our own project we intend to use the framework as a way of thinking about assessment tasks, as a tool for analyzing interviews, instructional materials, and instructional practice. These categories and constructs are hypotheses, inferred from instances of practice and from conversations with teachers. As the ideas are refined and elaborated, and as assessment tools are invented to probe in these areas, then empirical work that can examine the categories will lead to refinement and more detail. In addition, such empirical work can test hypotheses about the importance of these areas of knowledge in teaching that supports student learning. The potential contribution of valid and
reliable assessment tools and instruments to serve as measures of teachers’ knowledge of algebra for teaching is also likely to be useful currently. We see potential for this framework and the research to follow as useful in rethinking the mathematical preparation of preservice teachers. We have been struck by the realization that very little of the knowledge of algebra that has been discussed here appears explicitly in typical curricula for the mathematics majors preparing to teach secondary school mathematics. In closing, we look forward to continued exploration of the area of mathematical knowledge for teaching at the secondary school level.

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Although philosophers have subsequently shown that the connection between “justified true belief” and “knowledge” is complex, depending, for example, on what counts as “justified,” on the link between what determines whether the claim is true and the justification, and on the context in which the claim to knowledge is made (Gettier, 1963; Pryor, 2001). Nonetheless, the idea that knowledge and belief are distinct concepts remains as part of the background to the discussions of the particular conditions that set knowledge apart.

For a review of these issues in mathematics, see Ball, Lubienski, and Mewborn, 2001.

There are some exceptions, such as the efforts of the National Board for Professional Teaching Standards.

There are counterexamples; see, for example, the Programme for International Student Assessment (PISA), Organisation for Economic Cooperation and Development, 2003, The PISA 2003 Assessment Framework – Mathematics, Reading, Science and Problem Solving Knowledge and Skills.

Usiskin and his colleagues have produced a textbook intended for use with preservice and in-service mathematics teachers that explores “teachers’ mathematics.” See (Usiskin et al., 2003).

In the U.S., there is great variety in both the organization of algebra courses in secondary schools (e.g., algebra one, first half of algebra one, algebra two, second half of algebra one, etc. See Kher 2005) as well as in the content of these courses, which might emphasize
paper and pencil symbolic manipulation and algebraic structures, or might feature technology for
manipulation and emphasize functions and relationships.

9 We share at a general level the view of Ball and Bass that a productive approach to better understanding teacher knowledge issues is to examine teaching practice. We see the analysis of instructional materials, which was an important component of our framework development, as an alternative site for making inferences about the mathematical demands of teaching practice.

10 This language was suggested to us when it appeared in The PISA 2003 Assessment Framework (OECD, 2003), although its usage there is different from how we define it. There, the developers of the PISA Framework discuss this in relation to solving problems that are “situated in reality”, and describe it as one aspect of “mathematising” such problems: “Gradually trimming away the reality through processes such as making assumptions about which features of the problem are important, generalizing and formalizing (which promotes the mathematical features of the situation and transform the real problem into a mathematical problem that faithfully represents the situation), pp. 26-27.

11 One of the authors came upon such an unanticipated problem while observing a secondary school algebra two lesson. The teacher introduced the absolute value function notation and definition, and showed the students the graph. A student asked, “Can we use that function to get a formula for any angle?”

12 We acknowledge Bill Schmidt, Bill McCallum, and Paul Sally for this idea of modeling the understanding of the classroom.

13 We acknowledge that Hyman Bass called our attention to this issue.
14 If the postman brings checks this is good (+ x + = +); if he takes away bills that is good
(- x - = +), etc.

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