How much of commodity price behavior can a rational expectations storage model explain?

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Received 13 May 2002; received in revised form 4 November 2003; accepted 19 March 2004

Abstract

A rational expectations competitive storage model was applied to the U.S. corn market, to assess the aptness of this framework in explaining monthly price behavior in an actual commodity market. Relative to previous models, extensive realism was added to the model, in terms of how production activities and storage costs are specified. By modeling convenience yield, “backwardation” in prices between crop years did not depend on the unrealistic assumption of zero ending stocks. Our model generated cash prices that were distributed with positive skewness and kurtosis, and mean and variance that increased over the storage season, comparable to the persistence and the occasional spikes observed in commodity prices. Futures prices were generated as conditional expectations of cash prices at contract maturity, and the variances of futures prices exhibited realistic time-to-maturity and seasonal patterns. Model realizations of cash and futures prices over many “years” were used to demonstrate the wide variety of price behaviors that could be observed in an efficient market with a similar market structure, implying that economic and policy implications drawn from short, historical samples of prices could be misleading.

JEL classification: Q11

Keywords: Agricultural commodity prices; Storage; Rational expectations; U.S. corn market

1. Introduction

Agricultural commodity prices have systematic behavior due to the biological nature of production, even if their markets are efficient. A typical price series is autocorrelated with seasonal and perhaps cyclical components. Net of seasonality and cycles, these prices may be mean reverting to some long-run average. Occasionally, prices jump abruptly and temporarily to a high level relative to their long-run average, creating spikes. Thus, distributions of prices are skewed to the right and often display kurtosis (e.g., Deaton and Laroque, 1992; Myers, 1994).

Many models of the behavior of agricultural prices have been specified, but both time series and structural models have required \textit{ad hoc} assumptions to capture all the features of commodity price series (see Brorsen and Irwin, 1996; Tomek and Myers, 1993; Tomek and Peterson, 2001). In particular, little attention has been paid to the question, can the model parameters that depict behavioral response (e.g., demand elasticity) reproduce the parameters of the probability distributions of prices (e.g., first and second moments)? Knowledge about such distributions is important \textit{inter alia} for evaluating the consequences of economic and policy decisions, and if agricultural and trade policies continue to be liberalized, a deeper understanding of commodity price behavior is essential to better answer questions about managing risk.

A major impediment to estimating the parameters of econometric models or price distributions is the limited number of observations. Unlike financial markets, high-quality, high-frequency observations on cash prices are typically not available for agricultural commodity markets. Some questions relevant to decision makers in agriculture may be answerable using monthly or quarterly observations, but given frequent structural changes in farm policies, the number of such observations available from a single structural regime is often small (for an unusual example of the construction of a long sample, see Voituriez, 2001). With a small sample, the estimated parameters are fragile, and it is not possible to draw precise conclusions regarding outcomes of economic decisions. One way to overcome the short-sample problem is to develop a model that can be used to simulate observations that are consistent with the probability distributions of observable prices.

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The objective of this article is to examine how much of a commodity’s price behavior can be understood in terms of a structural model. We follow a calibration procedure broadly defined (Dawkins et al., 2001). We develop a model, select model parameters that are consistent with recent history, and compare the price distributions generated from the model equilibrium with those observed in a recent time period. Our point of departure is the nonlinear rational expectations storage model, which emphasizes nonlinearity in storage, i.e., the quantity stored cannot be negative (Gustafson, 1958), and incorporates Muth’s (1961) rational expectations. Williams and Wright (1991) synthesize the modern theory of competitive storage, which applies supply, demand, and market clearing conditions to the intertemporal arbitrage equation of the classic model (Working, 1949).1

Our model is the first example of a monthly model that permits carryover of an annual crop into the following year,2 and considerable effort is spent on adding realism to the model. We model the U.S. corn (maize) market, since it satisfies the basic characteristics of a commodity for which the storage model was originally conceptualized. Corn is storable and relatively homogeneous. Moreover, the volume of trading for futures contracts for corn is the largest among agricultural commodity futures traded in the United States. Simulated prices from the model permit analysis of strategies used to market seasonally produced commodities such as corn, which is a relevant economic issue in the U.S. grain sector. Thus, this market should be a useful example that is potentially generalizable to other storable commodities and locations to analyze policy or managerial issues pertinent to the specific market.

The next section describes characteristics of a recent corn price series that serves as a benchmark for our model to explain. Then, the conceptual model is developed, the numerical model is specified, and the equilibrium solutions are reported. The subsequent section compares price behavior implied by the model to the characteristics of the benchmark prices. Monthly patterns in the means and variances were approximately replicated, while the degree of skewness and higher-order autocorrelation differed. The simulations demonstrated that an efficient market with a constant structure can generate a wide range of outcomes. For example, in an individual crop year, prices can spike (with large returns to storage) or alternatively decline (with negative returns to storage). Based on a long sample that can only be obtained from simulations, our analysis casts doubt on the value of economic and political implications drawn from short, historical samples, offering a step toward a fuller understanding of the complexities involved in commodity price behavior. These ideas are developed in the concluding section.

2. Benchmark cash and futures price behavior in the U.S. corn market

Commodity prices at any point in time vary because of differences in location, quality, and delivery terms. We considered monthly cash prices for No. 2 yellow corn at a Central Illinois market as a representative cash price series for U.S. corn. Such a series is highly correlated with transaction prices in other Corn Belt locations in the United States. Given the changes in loan rates (support levels) and in supply management programs for corn in the 1980s, our benchmark sample included monthly observations from crop years 1989/1990 through 1997/1998, i.e., September 1989 through August 1998. The benchmark period characterized a market-oriented situation while embodying most, if not all, standard features of commodity price behavior. There was no obvious trend over time; the prices were variable with positive autocorrelation; and there was a prominent spike in early 1996 (Fig. 1). For these 108 observations, the sample mean, standard deviation, skewness, kurtosis, and autocorrelation coefficients are reported in Table 1.

As a description of the price series, a gamma distribution was fitted to the benchmark prices by method-of-moments estimation, to obtain monthly distributions allowing for skewness and kurtosis. These distributions for selected months are plotted in Fig. 2. The estimates provide sensible approximations of intra-year price behavior for U.S. corn. The modes increased over the storage period through May reflecting higher storage costs, and reverted toward the harvest level during the growing season. The probability masses of prices during the harvest and immediate post-harvest periods were relatively centered. Prices became more dispersed as planting approaches, and this trend

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1 This partial equilibrium framework has simulated some of the generalized, time series features of commodity cash prices (e.g., Chambers and Bailey, 1996; Deaton and Laroque, 1992, 1996; Routledge et al., 2000; Rui and Miranda, 1995). Deaton and Laroque (1992), for example, replicated the degree of skewness and kurtosis in the observed price distributions, but failed to account for the degree of autocorrelation. They used deflated annual prices, which may have introduced autocorrelation that did not exist in the nominal series. Applications of the rational expectations storage framework to commodity-specific markets have focused on simulating policy scenarios in annual framework (Gardner and López, 1996; Lence and Hayes, 2000; Miranda and Helmberger, 1988; Miranda and Glauber, 1993).

2 Although most research specifies annual models, a few quarterly models exist (Pirrong, 1999; Williams and Wright, 1991). Chambers and Bailey (1996) propose a monthly framework, but assume a monthly harvest for seven commodities, one of which is soybeans.
Table 1
Basic statistics of benchmark and simulated monthly price series

<table>
<thead>
<tr>
<th></th>
<th>Benchmark (Sept. 1989–Aug. 1998)a</th>
<th>Simulated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of observations</td>
<td>108</td>
<td>119,952b</td>
</tr>
<tr>
<td>Mean (US$/bushel)</td>
<td>2.595</td>
<td>2.696</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.575</td>
<td>0.571</td>
</tr>
<tr>
<td>Skewnessc</td>
<td>2.325</td>
<td>1.310</td>
</tr>
<tr>
<td>Kurtosisc</td>
<td>5.795</td>
<td>4.333</td>
</tr>
<tr>
<td>First-order autocorrelation</td>
<td>1.483</td>
<td>0.862</td>
</tr>
<tr>
<td>Second-order autocorrelation</td>
<td>−0.557</td>
<td>0.018</td>
</tr>
</tbody>
</table>

bThe total of 120,000 observations were simulated, but the first 48 were deleted to eliminate the impact of the initial conditions.
cThe skewness measure is \( \mu_3 / (\mu_2)^{1.5} \) and the kurtosis measure is \( (\mu_4 / \mu_2)^{0.5} - 3 \), where \( \mu_r \) is the \( r \)th central moment.

dried until August, when it was reversed in anticipation of the new crop. These seasonal patterns, along with the summary statistics reported in Table 1, should be reproduced by a useful model.

In addition, a model of commodity futures price behavior must take into account the theoretical relationship between future and cash prices over time. The current futures contract price equals the expected cash price at contract maturity assuming no risk premium and no basis risk. In other words, futures prices for any individual contract in an efficient market do not mean revert, since they are the expected values of a particular month’s price. Previous analyses, which report mean reversion in futures prices, may suffer from statistical weaknesses (Irwin et al., 1996). Also, analyses of mean reversion based on time series constructed from a sequence of prices from nearby futures contracts prices (i.e., using the current maturing contract prices until near the maturity date and then switching to the subsequent maturing contract prices) likely draw misleading results, since the series is analogous to cash prices.

The variance of futures prices for an individual contract is influenced mainly by the flow of information and its uncertainty (see Streeter and Tomek, 1992, for other factors). Variability is expected to increase as contract maturity approaches (Samuelson, 1965), and is affected by the seasonality in the information flow in the underlying market. For a December corn futures contract, for example, the time-to-maturity effect implies larger price variability in December than in May, ceteris paribus, which has been confirmed in empirical studies (e.g., Fackler and Tian, 1999; Goodwin et al., 2000). Due to uncertainty about the expected harvest during the growing season, the variance of futures contract prices will also have a seasonal component, ceteris paribus.

A descriptive analysis confirmed these features of futures price behavior during the benchmark period. Monthly futures price observations were constructed by averaging the settlement prices of the first and third Wednesdays (Thursday, if Wednesday was a holiday) of every month at the Chicago Board of Trade. December and May corn contract prices for the 12 months prior to maturity for the years 1990 through 1997 and 1991 through 1998, respectively, were fitted to a gamma distribution by method-of-moment estimation. In Figs. 3 and 4, these distributions in selected months are plotted for each contract. For both contracts, the time-to-maturity effect was clearly observed—the longer the time to maturity, the less dispersed the price distribution, and the dispersion increased monotonically over time. Comparing the distributions at maturity, May contract prices had a larger dispersion than December contract prices, consistent with the seasonal differences in the variances of cash price. A good model should capture these features. The monthly modes of December futures prices appeared to follow a

Note: Estimated from 1989/1990 to 1997/1998 observations by method-of-moments estimation. Parameter estimates (\( \alpha, \beta \)) for a gamma distribution \( f(x | \alpha, \beta) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} \), \( x < \infty \), are (35.453, 0.070), (39.630, 0.063), (27.797, 0.097), (11.864, 0.233), and (11.046, 0.240), for September, December, March, May, and July distributions, respectively.

Fig. 2. Estimated gamma distributions of monthly cash prices, selected months.

Note: Estimated from 1989/1990 to 1997/1998 observations by method-of-moments estimation. Parameter estimates (\( \alpha, \beta \)) for a gamma distribution \( f(x | \alpha, \beta) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} \), \( x < \infty \), are (150.431, 0.018), (63.444, 0.042), (37.907, 0.070), (45.158, 0.057), and (38.851, 0.066), for March, May, July, September, and December distributions, respectively.

Fig. 3. Estimated gamma distributions of December futures contract price, selected months.
seasonal pattern, which would be inconsistent with an efficient market but was likely due to the short sample.

3. Conceptual model

Minimum key features of the corn sector in the United States during the 1990s were incorporated in the model. Namely, the crop is planted in April and harvested from September through November by producers who are assumed to be expected profit maximizers. Once producers decide how much to plant, the expected crop size evolves, reflecting randomly varying growing conditions between planting and harvest. During months preceding harvest, news arrives regarding how much of the annual crop will be harvested next month. A larger-than-average proportion of the annual crop may be harvested in September of an “early” year, or in November of a “late” year. In April, producers decide how large of a crop (measured in aggregate bushels of corn expected at harvest) to plant. Due to uncertain growing conditions, the planted crop size is an estimate of crop size at harvest. During the months between planting and harvest, the expected crop size, or crop estimate (H), follows a random walk:

\[ H_{m+1} = H_m + \varepsilon_{m+1}, \quad m = 5, \ldots, 10, \quad H_m > 0 \quad \forall m, \quad (1) \]

where \( \varepsilon_{m+1} \) is a mean zero independent disturbance. In the model, planting is completed at the end of April, and the May crop estimate equals the planted crop size, i.e., \( H_5 = H_4 \).

Following Williams and Wright (1991), producers are assumed to be price takers and identical with homogenous land, and to hold rational expectations. Thus, they ignore how their individual planting decisions impact prices, but formulate price expectations that are consistent with the prices generated by their collective behavior. The state of the world in April relevant to the producers’ decisions is defined by the beginning inventory and supply shock, the latter representing unsystematic changes in costs of production. Production technology, which can be described by a convex, increasing cost function, is assumed to be fixed. The producer’s problem is to maximize the expected, discounted profit by choosing crop size to be planted (\( H_4 \)), which represents the market-wide crop size if appropriately scaled, given the homogeneity of producers,

\[
\max_{H_4} E_4 \left[ \sum_{m=9}^{11} \alpha_m \frac{H_m P_m}{(1 + r)^{m-4}} - C_{\theta_4} [H_4] \right]
\]

The first term is expected discounted revenue, computed as the weighted average of crop size (H) times price (P) in September

Table 2
Names and definitions of variables in the conceptual model

<table>
<thead>
<tr>
<th>Variable names</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endogenously determined state variables</td>
<td></td>
</tr>
<tr>
<td>( A_m, m = 1, \ldots, 12 )</td>
<td>Available supply at the beginning of month ( m ) (Eq. (4))</td>
</tr>
<tr>
<td>( H_4 )</td>
<td>Planted crop size in April (Eq. (2))</td>
</tr>
<tr>
<td>( H_5 )</td>
<td>Estimated crop size at the beginning of May (( H_5 = H_4 ))</td>
</tr>
<tr>
<td>( H_{m}, m = 6, \ldots, 11 )</td>
<td>Estimated crop size at the beginning of month ( m ) (Eq. (1))</td>
</tr>
<tr>
<td>( h_{m}, m = 9, 10, 11 )</td>
<td>Crop harvested during month ( m ) (Eq. (3))</td>
</tr>
<tr>
<td>Endogenous variables</td>
<td></td>
</tr>
<tr>
<td>( P_m, m = 1, \ldots, 12 )</td>
<td>Price in month ( m ) (Eq. (6))</td>
</tr>
<tr>
<td>( q_{m}, m = 1, \ldots, 12 )</td>
<td>Consumption in month ( m ) (Eq. (5))</td>
</tr>
<tr>
<td>( s_{m}, m = 1, \ldots, 12 )</td>
<td>Quantity stored from month ( m ) to ( m + 1 ) (( s_{m} = A_{m} - q_{m} ))</td>
</tr>
<tr>
<td>Parameters</td>
<td></td>
</tr>
<tr>
<td>( r )</td>
<td>Monthly interest rate</td>
</tr>
<tr>
<td>( u_{m}, m = 1, \ldots, 12 )</td>
<td>Demand shock in month ( m )</td>
</tr>
<tr>
<td>( y )</td>
<td>Supply shock in April</td>
</tr>
<tr>
<td>( \alpha_{m}, m = 9, 10, 11 )</td>
<td>Proportion of crop harvested in month ( m )</td>
</tr>
<tr>
<td>( \hat{\alpha}_{m}, m = 8, 9, 10 )</td>
<td>Expected proportion of crop harvested in month ( m + 1 )</td>
</tr>
<tr>
<td>( \theta_{m}, m = 1, \ldots, 12 )</td>
<td>State variables in month ( m ) (Eq. (8))</td>
</tr>
</tbody>
</table>

*The model equation in parentheses is the primary relationship that determines the corresponding endogenous variable.*
through November, where the weights are the monthly harvest shares ($\alpha$) divided by the discount factor for a monthly interest rate of $r$; $C_{m}$ is the state-dependent total cost to plant and maintain a given amount of crop size, and $\theta_t$ includes available supply at the beginning of April ($A_4$) and an independently distributed random supply shock ($\gamma$). Given that $H_m$ is a random walk, $E[H_m|P_m] = H_4E[P_m]$; thus, solving the first-order condition yields the supply function $S_{\theta_m}()$ that depends on the state of the world in April and the expected (discounted) price at harvest,

$$H_4 = S_{\theta_m} \left[ E_4 \left[ \sum_{m=9}^{11} \alpha_m \frac{P_m}{(1+r)^{m-4}} \right] \right].$$  

(2)

During September and October, random proportions of the expected crop size are actually harvested, reflecting varying growing conditions; the remaining crop is harvested in November, revealing the true total crop size. Thus, the incoming harvest ($h$) is

$$h_m = \alpha_m H_m, \quad m = 9, 10, 11,$$

(3)

where $\alpha_9$ and $\alpha_{10}$ are independent random numbers between 0 and $\alpha_m < 1 (m = 9, 10)$ such that $\sum_{m=9}^{10} \alpha_m H_m \leq H_{11}$, and $\alpha_{11}$ is a number between 0 and 1 such that $H_{11} = \sum_{m=9}^{11} h_m$. Expectations of these random proportions, $\bar{c}_{m-1} = E_{m-1}[\alpha_m]$, are revealed as news a month prior to their realizations during August through October.

Monthly available supply ($A$) is equal to the carryover ($s$) from the previous month, except in months when the carryover is augmented by the incoming harvest:

$$A_m = \begin{cases} s_{m-1}, & m = 1, \ldots, 8, 12 \\ s_{m-1} + h_m, & m = 9, 10, 11. \end{cases}$$  

(4)

Quantity demanded ($q$) is a state-dependent function of the current price,

$$q_m = D_{\theta_m}[P_m], \quad m = 1, \ldots, 12,$$

(5)

where $\partial D_{\theta_m}/\partial P_m < 0$, $\lim_{u_m \to 0} D_{\theta_m}^{-1}[q_m] = \infty$ (i.e., consumption must be positive for the market to clear), and the state ($\theta_m$) includes demand shock, $u_m$. Since the monthly specification of demand functions accounts for any seasonality in demand, demand shocks are specified as independently distributed, residual disturbance around this seasonality.

Storage is carried out by risk-neutral arbitrageurs, whose profit equals the expected appreciation in price, less opportunity and carrying costs associated with storage. Carrying cost consists of the physical cost of storage minus the convenience yield of stocks, which depends on supply and demand for storage (Telser, 1958).\(^4\) Depreciation in quality due to storage is assumed to be unimportant for corn. Assuming a constant monthly discount rate $r$ and denoting the total carrying cost as $K_{\theta_m}()$, the storer’s problem is

$$\max_{s_m \geq 0} \frac{E_m[P_{m+1}][s_m]}{(1+r)} - P_m s_m - K_{\theta_m}[s_m], \quad m = 1, \ldots, 12,$$

with the first-order conditions

$$\frac{E[P_{m+1} | \theta_m]}{(1+r)} = P_m + K'_{\theta_m}[s_m], \quad s_m > 0,$$

$$m = 1, \ldots, 7, 11, 12,$$

$$\frac{E[P_{m+1} | \theta_m]}{(1+r)} \leq P_m + K'_{\theta_m}[s_m], \quad s_m \geq 0,$$

$$m = 8, 9, 10,$$

(6)

where $K'_{\theta_m}()$ is the state-dependent marginal carrying cost. This specification allows for risk-averse storage behavior despite the assumption of risk neutrality, since risk aversion is observationally equivalent to a storage model with a nonlinear storage cost function such as this (e.g., Just, 1975).\(^5\) These conditions guarantee no arbitrage opportunities in equilibrium. Imposing positive consumption removes the possibility of stock-outs except in the months preceding harvest.

Equations (1) through (6) imply an equilibrium characterized by a rational expectations price function, $P = \Phi(\theta_m)$, and a planting equation, $H_4 = \Psi(\theta_4)$, which solve the following discrete-time, continuous-state, functional equations,

$$\Phi(\theta_m) = \begin{cases} E_m[\Phi(\theta_{m+1})]/(1+r) - K'_{\theta_m}[s_m], & m = 1, \ldots, 11, 12 \\ \max \bigg\{ E_m[\Phi(\theta_{m+1})]/(1+r) - K'_{\theta_m}[s_m], D_{\theta_m}^{-1}[s_m-1] \bigg\}, & m = 8, 9, 10 \end{cases}$$

(7a)

$$D_{\theta_m}^{-1}[A_m - s_m] = \Phi(\theta_m), \quad \forall m,$$

(7b)

\(^4\) Brennan (1958) includes a possible risk premium that is associated with the risk from carrying stocks, such as financial loss from an unexpected fall in prices, but it is very small or nonexistent (Leuthold et al., 1989).

\(^5\) The utility maximization problem for a risk-averse storer can be expressed in terms of the certainty equivalent of a risky prospect (Mas-Colell et al., 1995, chapter 6). Denoting the risk premium (the difference between the mean of the risky profit and its certainty equivalent) as $\rho$, and including the opportunity cost and constant marginal physical storage cost $k$ in the profit, the certainty equivalent maximization problem is

$$\max_{s_m \geq 0} E_m[P_{m+1}][s_m]/(1+r) - P_m s_m - k s_m - \rho[s_m], \quad m = 1, \ldots, 12,$$

But, this is identical to the model assuming risk neutrality, replacing the carrying cost $K_{\theta_m}[s]$, with the sum of the physical storage cost and risk premium.

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\(^3\) According to Garcia et al. (1997, p. 561), “errors [of crop forecasts by the U.S. Department of Agriculture] are quite small by the time of November announcements.”
\[ \Psi[\theta_4] = S_0[\hat{P}[\Psi[\theta_4]]], \]  
\[ \hat{P}[H_4] = E_4 \left[ \sum_{m=9}^{11} \alpha_m \Phi[\theta_m] \right] (1 + r)^{m-4} | H_4, \]  
where \( \hat{P}[\cdot] \) represents the expected harvest price, given a planted crop size of \( H_4 \), and the state variables in each month are

\[ \theta_m = \begin{cases} 
{m, A_m, u_m, y}, & m = 4, \\
{m, A_m, u_m, H_m}, & m = 5, 6, 7, \\
{m, A_m, u_m, H_m, \hat{a}_m}, & m = 8, 9, 10, \\
{m, A_m, u_m}, & \text{otherwise}. 
\end{cases} \tag{8} \]

Equation (7a) states that current price must equal the discounted expected future price, less the marginal carrying cost \( K' \), except in months prior to harvest. During August through October, stock-outs, although improbable, could occur in which case the price equals the level implied by consuming all the carryover from the previous month, \( D^{-1}_s[s_{m-1}] \). In all months, the equilibrium price depends on inverse demand evaluated at available supply that was not stored (Eq. (7b)). Equation (7c) is the planting function; the crop planted is the state-dependent supply function evaluated at expected harvest price. Expected harvest price, in turn, depends on planted crop size, because crop size influences price received at harvest (Eq. (7d)). Three states exist in all months: the month of the year, the available supply level, and the realized demand shock. In April, a supply shock conditions the planting decision; during May through October, the expected size of the incoming crop is relevant; and expected proportions of crop harvested in the subsequent month are revealed during August through October. In November, the information of the realized crop is incorporated into available supply. Consumption and storage are determined by price, and hence are functions of the state variables.

The existence of price functions in a competitive storage model with seasonal production was proved by Chambers and Bailey (1996). The assumption of rational, forward-looking behavior among the agents implies that stochastic dynamic programming or a similar recursive method must be employed to solve the model. The possibility that the nonnegativity constraint on storage may bind adds further complications. Since the functions cannot be solved in closed form, the problem is solved numerically.

4. Numerical model
4.1. Parameter specification

Six state variables (the month of the year, available supply, crop estimates, harvest timing, demand shocks, and supply shock) determine monthly prices and annual planting. The dynamics and distributions of the state variables were parameterized to represent observed market conditions during the benchmark period from 1989/1990 to 1997/1998 crop years, as were the coefficients of the demand, supply, and convenience yield functions. The specifications of model parameters are summarized in Table 3.

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Parameter specification</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Crop estimate</strong></td>
<td></td>
</tr>
<tr>
<td>( H_5 \sim N(0.86490, 0.06190) ), ( H_{m+1} = H_m + s_{m+1} ), ( m = 5, \ldots, 10 ), ( s_6 \sim N(0, 0.02378), s_7 \sim N(0, 0.02614), s_8 \sim N(0, 0.04081), s_9 \sim N(0, 0.01651), s_{10} \sim N(0, 0.02202), s_{11} \sim N(0, 0.02784) ).</td>
<td></td>
</tr>
<tr>
<td><strong>Demand</strong></td>
<td></td>
</tr>
<tr>
<td>( q_m = a_m (p_0)^{y_m} u_m, m = 1, \ldots, 12 ), ( a_1 = 0.05258, a_2 = 0.05275, a_3 = 0.04594, a_4 = 0.04584, a_5 = 0.04567, a_6 = 0.03726, a_7 = 0.03692, a_8 = 0.03668, a_9 = 0.06134, a_{10} = 0.06085, a_{11} = 0.06901, a_{12} = 0.05239, b_m = -0.25 \forall m ), ( u_1 \sim N(1, 0.08679), u_2 \sim N(1, 0.08998), u_3 \sim N(1, 0.1014), u_4 \sim N(1, 0.11503), u_5 \sim N(1, 0.12409), u_6 \sim N(1, 0.08322), u_7 \sim N(1, 0.08934), u_8 \sim N(1, 0.08580), u_9 \sim N(1, 0.10233), u_{10} \sim N(1, 0.10327), u_{11} \sim N(1, 0.10129), u_{12} \sim N(1, 0.08626) ).</td>
<td></td>
</tr>
<tr>
<td><strong>Harvest timing</strong></td>
<td></td>
</tr>
<tr>
<td>( a_9 \sim N(0.55737, 0.15369), \ln(a_8) \sim N(-2.11864, 0.49733) ).</td>
<td></td>
</tr>
<tr>
<td><strong>Supply</strong></td>
<td></td>
</tr>
<tr>
<td>( H_4 = a(E[p])^y, a = 1.11198, b = 0.2, y \sim N(1, 0.00846) ).</td>
<td></td>
</tr>
<tr>
<td><strong>Convenience yield</strong></td>
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</tr>
<tr>
<td>( CY_m = c_m(s_m)^{y_m} \exp[\zeta_m(H_m - \bar{H})], m = 5, \ldots, 9 ), ( a_{15} = 0.00261, a_{16} = 0.00284, a_{17} = 0.00230, a_{18} = 0.00276, a_{19} = 0.00221, \zeta_m = -1, \forall m = 5, \ldots, 9 ), ( c_5 = 3.60861, \zeta_6 = -6.30824, \zeta_7 = -4.71034, \zeta_8 = -8.07896, \zeta_9 = -8.90749 ).</td>
<td></td>
</tr>
<tr>
<td><strong>Physical storage cost</strong></td>
<td></td>
</tr>
<tr>
<td>( k = 0.003 ).</td>
<td></td>
</tr>
<tr>
<td><strong>Discount rate</strong></td>
<td></td>
</tr>
<tr>
<td>( r = 0.00833 ).</td>
<td></td>
</tr>
</tbody>
</table>

Note: The quantity and price variables are scaled so that 1 equals 10 billion bushels and US$10 per bushel, respectively.  
\( \sim N(\mu, \sigma) \) denotes "follows a normal distribution with parameters \( \mu \) and \( \sigma \)."
Available supply \((A)\) was specified to evolve according to the storage and consumption decisions, but was restricted to lie within monthly bounds that are specified to include the observed range. Observations of available supply were imputed from quarterly total supply and ending stocks during the benchmark period reported by the Economic Research Service (ERS) at the U.S. Department of Agriculture (USDA).

The expected crop size in May \((H_5)\) assumed values between the minimum and maximum annual harvested crop in the benchmark period (6.3 to 10.1 billion bushels). The values were associated with probabilities assuming a normal distribution with a data-consistent mean and standard deviation. Subsequent crop estimates were specified to be revised at the beginning of each month from June through November (when, by assumption, the actual crop size was revealed) with Markov transition probabilities. The standard deviations of the differences between the adjacent months’ crop estimates during the benchmark period (World Agricultural Outlook Board, USDA) were used as the standard deviations of the error terms in Eq. (1).

Harvest timing was defined by the proportions of crop harvested during September, October, and November \((\alpha)\), which were taken from the USDA National Agricultural Statistics Service’s crop progress report. Since 91.7% of the U.S. annual crop was harvested during the three months of an average year in the benchmark period, the percentages were adjusted so that the average of the three-month sum equaled one. The adjusted September, October, and November mean proportions were 0.12, 0.55, and 0.32, respectively. Based on the benchmark evidence, normality was assumed for the logarithms of September percentages, while October percentages were assumed to be normally distributed; the two distributions were independent. November percentages were determined by the total crop size in November.

Monthly demands were specified as constant elasticity functions with multiplicative demand shocks, with the price elasticities set at \(-0.25\) for all months.\(^6\) The demand functions were calibrated to the modes of the estimated monthly price distributions (Fig. 2) and monthly means of total disappearance (quarterly total disappearance reported by the ERS divided by three). Demand shocks \((u)\) were specified as mean one, normal random variables. Standard deviations were imputed from the ratio of consumption and predicted consumption, assuming the constant-elasticity demand functions with calibrated coefficients.

The supply equation was also assumed to take a constant-elasticity form with a multiplicative, normal supply shock \((y)\) with the supply elasticity set at \(0.2^7\). The function was calibrated to the average harvest price during the benchmark period discounted to April (US$2.30 per bushel) and the mean of annual production (8.29 billion bushels), with the mean of the supply shock set to \(1.8\). Because expected harvest prices are not observed, the variance of \(y\) was inferred from the benchmark sample. The calibrated supply curve was evaluated at the minimum and maximum observed harvest prices and compared to the minimum and maximum observations of planted crop, respectively. The average of squared deviations between predicted and observed planting was used as an estimate of the variance of \(y\).

The net carrying charge of storage was specified as the difference between current and expected prices, and consisted of storage costs and marginal convenience yield. If the expected price in the next period differed from the current price only by the costs of storage, prices would increase indefinitely, on average, relative to the previous month. Yet, observed prices typically increase from October through May and then decrease. This puzzle of price backwardation has been a long-standing issue (Frechette and Fackler, 1999), and a conventional rational expectations storage model has used aggregate stock-outs to account for price backwardation, ignoring convenience yield. But zero stocks of corn have never occurred in recorded history in the United States (though they were very small in 1934 and 1936).

Thus, we modeled marginal convenience yield, despite the fact that it is unobservable, as a source of price backwardation. Observations for marginal convenience yield \((CY)\) were calculated as the remainders of the non-arbitrage equation evaluated at monthly prices: \(CY_m = P_m + k - P_{m+1}/(1 + r)\), where the monthly physical cost of storage \((k)\) and the discount rate \((r)\) were specified as 3 cents per bushel and 0.0083, respectively. The monthly averages of marginal convenience yield were close to zero from October through April, and ranged from 9.5 to 21.2 cents per bushel during the remaining months. To be consistent with the observed price and inventory behavior, marginal convenience yield was assumed to be a decreasing, convex function of the storage level \((s)\), following Rui and Miranda (1995), and a decreasing function of expected crop size \((H)\): \(CY_m = \omega_m \xi_m \exp[\xi_m(H_m - \bar{H})]\), where \(\omega_m, \xi_m, \text{ and } \xi_m\) are parameters, \(\xi_m < 0, \text{ } \bar{H}\) is the benchmark-average crop size. Setting the elasticity parameter \(\xi_m\) to \(-1\), the constants \(\omega_m\) were obtained by calibrating these functions to the monthly average convenience yield, storage level, and historical average crop size. The shift parameters \(\xi_m\) associated with expected crop size were calibrated so that the shift factors \(\exp[\xi_m(H_m - \bar{H})]\) took on their maximum values when crop estimates were the consensus around \(0.2\); a recently published price elasticity of acreage for the 1991–1995 period was 0.293 (Lin et al., 2000). Estimated supply (production) elasticities ranged from 0.28 to 0.39 (Shonkwiler and Maddala, 1985).

\(^6\) Estimates of annual demand elasticities for U.S. corn in the literature varied from \(-0.54\) to \(-0.73\) (Holt, 1994; Holt and Johnson, 1989; Shonkwiler and Maddala, 1985). Subbotnik and Houck (1979) reported ranges of quarterly price elasticities of feed use, of food use, and of exports as \(-0.15\) to \(-0.22, 0\) to \(-0.034, \text{ and } -0.71 \text{ to } -2.0, \) respectively.

\(^7\) Estimated acreage elasticities for U.S. corn with respect to changes in expected price ranged from 0.05 to 1.04 (Chavas and Holt, 1990, 1996; Holt, 1994, 1999; Lee and Helmberger, 1985; Tegene et al., 1988), with an apparent discount rate was based on an annual rate of 0.1 \((r = 0.0083 = 0.1/12)\).
lowest during the benchmark period. The convenience yield was set to zero for the months October through April. This specification effectively eliminated stock-outs.

4.2. Solution method

The recent literature on numerical solution methods has used variations of a functional approximation method to replace the original functional equation problem with a finite-dimensional (discrete) problem (e.g., Deaton and Larroque, 1992; Rui and Miranda, 1995; Williams and Wright, 1991). Miranda (1998) compared the accuracy and efficiency of different numerical strategies for computing approximate solutions to the nonlinear rational expectations commodity market model, and his results supported the superiority of the cubic spline function collocation method over the traditional space discretization, linearization, and least-squares curve-fitting methods.

The cubic spline collocation method involves three steps. First, the solutions to the problem, price, and planting functions, are approximated by finite linear combinations of known cubic b-spline functions. Second, collocation nodes are selected for both functions, and lastly, continuous distributions of random state variables are replaced by discrete distributions. The procedure replaces the original functional equation problem (7a)–(7d) with a finite-dimensional problem, where unknowns are coefficients of the approximating collocation functions, storage levels, planted crop size, and expected harvest price.

Our equilibrium conditions have a recursive structure in that the planting equation does not enter the storer’s arbitrage Eq. (7a). From the storer’s point of view, the harvest is exogenous, and information on expected crop size becomes available only after the crop has been planted and growing conditions become known. Hence, a solution can be obtained in a sequence of independent operations. First, Eqs. (7a) and (7b) are solved to determine the equilibrium market price at all possible states. Second, these price functions are used to determine expected harvest price, respectively. A positive supply shock (i.e., lower cost of production) induced a larger crop to be planted, and harvest timing sequentially. Then, the planting decision was endogenized, and lastly, the uncertainty regarding production costs was added, resulting in what we refer to as the full model. Results from the nested cases allowed us to relate aspects of simulated behavior to certain model specifications. Due to the limited space, we report only the results for the full model below but discuss observations based on the comparisons. The results for all nested cases are documented in Peterson and Tomek (2000).

5. Model solutions

5.1. Equilibrium cash price and planting functions

The equilibrium price functions were monotone-decreasing, convex functions of available supply and crop size estimates, respectively, in all months, with a negative cross derivative. They were increasing in demand shocks and at a faster rate at lower availability levels. Similarly, prices were more responsive to the changes in demand shocks when smaller crops were expected. When a small proportion was expected to be harvested in September (a late harvest), much of the uncertainty regarding total crop size was transferred to the remaining harvest months, causing August prices to increase at a given level of availability, ceteris paribus. The impact was more apparent in the October price function, with November being the terminal harvest month. When a smaller proportion of the crop was harvested in November, the upward shift in the October price function was greater at lower availability levels and with smaller crop estimates, respectively.

Equilibrium planting, a function of available supply in April, was derived from intersections between the supply function and the inverse demand function expected at harvest, which were the increasing and decreasing functions of the expected harvest price, respectively. A positive supply shock (i.e., lower cost of production) induced a larger crop to be planted, and smaller stocks in April shifted the expected inverse demand outward, ceteris paribus. The planting function was decreasing in available supply in April and increasing in supply shocks.

5.2. Time series behavior and long-run distributions

Equilibrium price and planting functions alone do not tell us much about the validity of the model, since they have no...
observed counterparts. Rather, the model is “good” if it generates equilibrium price series that are comparable to those observed.\textsuperscript{9} We have used price levels to parameterize demand and supply functions, but higher moments of distributions as well as serial correlation in price series are not guaranteed to be data consistent.

Using the equilibrium functions, equilibria were sequentially generated for 10,000 “years,” by drawing random disturbances consistent with the model. At the beginning of each month, availability was realized as the amount carried over from the previous month plus any incoming harvest, and together with other realized state variables, determined the current price through the price function. Then, the quantity consumed at that price was determined from the demand function, and the carry over to the next month was the difference between availability and consumption.

Table 1 reports the summary statistics for the simulated price series based on the full model, along with their benchmark counterparts. The model successfully simulated the prices at a comparable mean level and variability around the mean. The overall price distribution was positively skewed and more peaked than normal to the extent comparable to the benchmark. Williams and Wright’s (1991) claim that storage induces skewness was confirmed under our specification of random shocks. Deaton and Laroque (1992) noted the inability of their model to replicate the degree of first-order autocorrelation (their model results ranged between 0.08 and 0.48), but our model was more successful regarding this point. However, in terms of the second-order autocorrelation, which Deaton and Laroque did not report, the simulated series could not reproduce the negative second-order autocorrelation exhibited in the benchmark prices.

In terms of seasonal patterns, Fig. 5 depicts probability distributions of simulated cash prices in selected months, which can be compared directly with Fig. 2. The seasonal changes in the modes and dispersion were reproduced yet not as pronounced as in Fig. 2. Table 4 compares the monthly statistics between the benchmark and simulated prices. Since there are only nine observations for each month in the benchmark, only the mean and standard deviations are reported. Simulated average price levels were slightly higher than those observed in the benchmark prices (by 6 to 16 cents per bushel), but the spread between the highest (May) and lowest (October) mean prices of 36 cents corresponded to the benchmark. The seasonal pattern in price means reflected an equilibrium outcome of planting, storage, and consumption decisions. Uncertainty regarding crop size during the growing season produced a monthly pattern in price variability consistent with the benchmark. Calibrated demand shocks helped align simulated variability with that observed during the months before planting. The seasonal differences in variability, however, were not as large as in the benchmark period, as seen in Figs. 2 and 5, and the simulated monthly maximum and minimum prices were higher and lower, respectively, than the benchmark.

Table 4 also reports monthly statistics for simulated storage, consumption, crop estimates (including planted crop size), and incoming harvest. Storage declined steadily through the post-harvest season, smoothing out intertemporal consumption. During the benchmark period, the standard deviation of storage was 10% to 40% of the mean, while that of consumption was 8% to 10%. In the full model, the standard deviation of storage ranged from 12% and 35% of the mean, and that of consumption ranged from 9% to 12%.

The equilibrium planting function generated planted crop size with a similar magnitude and variability as the benchmark. The simulated crop estimates showed no systematic trend over the growing season, reflecting the Markov specification. In contrast, the means of the USDA monthly crop estimates seemed to decline over the growing season in our benchmark period, but the literature has suggested little or no bias in longer time series (Garcia et al., 1997; Sumner and Mueller, 1989). The specification of a supply shock at planting time generated a consistent level of uncertainty in crop estimates throughout the growing season. Standard deviations of crop harvested were brought in line with the benchmark by the random realization of harvest timing calibrated to the benchmark period.

The magnitude of the simulated convenience yield and its relationship with storage levels are illustrated in Fig. 6, for selected months. In May, the simulated observations of convenience yield and storage were tightly distributed along a decreasing convex locus, because the variability of expected crop size was limited. As the growing season proceeded and the variability of expected crop size increased, convenience yield became more dispersed, and its magnitude increased.

\textsuperscript{9} In most of the literature, model validation has focused on the prediction that prices follow a two-regime process, depending on whether or not inventories are held (e.g., Chambers and Bailey, 1996; Deaton and Laroque, 1995, 1996; Michaelides and Ng, 2000; Ng, 1996), which is not applicable to our model where stock-outs are precluded. Interestingly, Beck (2001) compared the asymmetry of price distributions for storable and nonstorable commodities and did not find a significant difference.
Table 4: Simulated outcomes of the full model

<table>
<thead>
<tr>
<th>Month</th>
<th>Price (P) ($/bushels)</th>
<th>Storage (s), (million bushels)</th>
<th>Consumption (q), (million bushels)</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>2.67</td>
<td>2.59</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>(2.56)</td>
<td>(2.49)</td>
<td>(0.44)</td>
</tr>
<tr>
<td>February</td>
<td>2.72</td>
<td>2.64</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>(2.60)</td>
<td>(2.58)</td>
<td>(0.48)</td>
</tr>
<tr>
<td>March</td>
<td>2.77</td>
<td>2.69</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>(2.69)</td>
<td>(2.59)</td>
<td>(0.51)</td>
</tr>
<tr>
<td>April</td>
<td>2.83</td>
<td>2.75</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>(2.74)</td>
<td>(2.50)</td>
<td>(0.67)</td>
</tr>
<tr>
<td>May</td>
<td>2.87</td>
<td>2.72</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>(2.77)</td>
<td>(2.51)</td>
<td>(0.80)</td>
</tr>
<tr>
<td>June</td>
<td>2.84</td>
<td>2.73</td>
<td>0.63</td>
</tr>
<tr>
<td></td>
<td>(2.72)</td>
<td>(2.59)</td>
<td>(0.78)</td>
</tr>
<tr>
<td>July</td>
<td>2.79</td>
<td>2.67</td>
<td>0.63</td>
</tr>
<tr>
<td></td>
<td>(2.65)</td>
<td>(2.34)</td>
<td>(0.80)</td>
</tr>
<tr>
<td>August</td>
<td>2.75</td>
<td>2.63</td>
<td>0.66</td>
</tr>
<tr>
<td></td>
<td>(2.57)</td>
<td>(2.45)</td>
<td>(0.76)</td>
</tr>
<tr>
<td>September</td>
<td>2.59</td>
<td>2.49</td>
<td>0.64</td>
</tr>
<tr>
<td></td>
<td>(2.47)</td>
<td>(2.35)</td>
<td>(0.42)</td>
</tr>
<tr>
<td>October</td>
<td>2.51</td>
<td>2.41</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td>(2.40)</td>
<td>(2.27)</td>
<td>(0.40)</td>
</tr>
<tr>
<td>November</td>
<td>2.57</td>
<td>2.49</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>(2.46)</td>
<td>(2.41)</td>
<td>(0.38)</td>
</tr>
<tr>
<td>December</td>
<td>2.62</td>
<td>2.54</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>(2.51)</td>
<td>(2.42)</td>
<td>(0.40)</td>
</tr>
</tbody>
</table>

Note: Numbers in parentheses are observed (benchmark) values for the period 1989/1990–1997/1998.

The full model was tested for its robustness under alternative specifications of five parameters: the interest rate, storage cost, and elasticity coefficients with respect to demand, supply, and convenience yield. While magnitudes of the moments varied slightly, the seasonal patterns in the moments remained unchanged for all variables. The results of the sensitivity analysis are reported in Peterson and Tomek (2000).

5.3. Futures price functions

The full model was used to obtain December and May futures price functions, dependent on the state in each month starting a year prior to maturity. Available supply, expected crop size, and percentage of crop harvested, which are observable monthly in reality, were discretized within respective ranges, and each
combination of discrete state variables was regarded as an initial state. The model was simulated from all initial states to both December and May, 10,000 times. For each initial state, the averages of December and May conditional prices over the 10,000 simulations were computed as prices of December and May futures contracts. The functional relationships between state variables and futures prices were formalized by the cubic spline collocation method, analogous to the cash price functions.

Both futures price functions were monotone decreasing with respect to available supply and expected crop size; the cross derivatives were negative. With respect to expected percentage of crop arriving next month, futures price functions were increasing at high levels of available supply, but not monotone at low levels of available supply. Statistics of simulated December and May futures prices, respectively, along with their benchmark counterparts are reported in Table 5. The simulated means, in contrast to the means of the benchmark prices, did not exhibit any trend, except anomalous spikes in September and October. The variability of both benchmark futures price series was small in the early months of the contract and increased toward maturity, a pattern that was also replicated by the simulated prices ignoring September and October.

The anomalies in the September and October distributions were likely caused by the specification of discretized states regarding harvest timing. Despite careful calibration, it placed a larger probability mass than in reality on those expected harvest proportions, implying high prices. If September and October were ignored, the simulated futures prices were plausible representations of futures prices in an efficient market, exhibiting a time-to-maturity effect and positive skewness indicated by observed futures prices.

5.4. Implied price behavior

Equilibrium prices generated by our model allowed for numerous seasonal patterns that are consistent with the specified market structure. The model permits changes in post-harvest prices of various magnitudes; the largest simulated price increase from November to May in 10,000 simulations was US$2.23 per bushel. A lucky farmer could face such highly advantageous price increases. Alternatively, prices do not necessarily increase monotonically, and it is possible (with a small probability) for prices to decline seasonally after harvest. The largest simulated seasonal decrease was US$2.98 per bushel. Thus, marketing strategy recommendations based on short, historical samples could be misleading.

For example, one commonly proposed risk management strategy in the U.S. grain sector is to diversify post-harvest cash sales, under the assumption that diversification can reduce risk, since cash prices are not perfectly correlated from one month to the next. In a typical year, cash prices are lowest immediately following harvest and then increase through the post-harvest season. As expected prices increase, however, so does the expected variance. Thus, it is not necessarily the case that diversifying sales will reduce the variance of returns. Indeed, comparing simulated November prices with those in the following May indicated that when opportunity and storage costs were taken into account, the mean return was reduced by 2 cents per bushel while the variability of returns increased by 20%.

Another proposed marketing strategy involves establishing positions in current marketing year futures contracts when prices are high, and rolling them over to new crop futures in an attempt to obtain a higher return for the next crop. The simulations demonstrated that this was a poor strategy. The relationship between the simulated price level of old crop (May) futures and the spread between new (December) and old crop futures prices in April is plotted in Fig. 7. This relationship is similar to that depicted by historical data (Lence and Hayenga, 2001, Fig. 3). At low levels of May futures prices, the spread was a small, negative number. As the price level increased, the spread decreased at an increasing rate until the
Table 5
Simulated futures prices

<table>
<thead>
<tr>
<th>Month</th>
<th>December futures ($/bushel)</th>
<th>May futures ($/bushel)</th>
</tr>
</thead>
<tbody>
<tr>
<td>December</td>
<td>2.60</td>
<td>2.59</td>
</tr>
<tr>
<td></td>
<td>(2.57)</td>
<td>(2.54)</td>
</tr>
<tr>
<td>January</td>
<td>2.61</td>
<td>2.59</td>
</tr>
<tr>
<td></td>
<td>(2.60)</td>
<td>(2.57)</td>
</tr>
<tr>
<td>February</td>
<td>2.60</td>
<td>2.59</td>
</tr>
<tr>
<td></td>
<td>(2.65)</td>
<td>(2.63)</td>
</tr>
<tr>
<td>March</td>
<td>2.61</td>
<td>2.59</td>
</tr>
<tr>
<td></td>
<td>(2.69)</td>
<td>(2.64)</td>
</tr>
<tr>
<td>April</td>
<td>2.61</td>
<td>2.60</td>
</tr>
<tr>
<td></td>
<td>(2.70)</td>
<td>(2.62)</td>
</tr>
<tr>
<td>May</td>
<td>2.61</td>
<td>2.56</td>
</tr>
<tr>
<td></td>
<td>(2.69)</td>
<td>(2.62)</td>
</tr>
<tr>
<td>June</td>
<td>2.61</td>
<td>2.56</td>
</tr>
<tr>
<td></td>
<td>(2.71)</td>
<td>(2.70)</td>
</tr>
<tr>
<td>July</td>
<td>2.60</td>
<td>2.56</td>
</tr>
<tr>
<td></td>
<td>(2.65)</td>
<td>(2.51)</td>
</tr>
<tr>
<td>August</td>
<td>2.60</td>
<td>2.54</td>
</tr>
<tr>
<td></td>
<td>(2.57)</td>
<td>(2.48)</td>
</tr>
<tr>
<td>September</td>
<td>2.73</td>
<td>2.64</td>
</tr>
<tr>
<td></td>
<td>(2.57)</td>
<td>(2.46)</td>
</tr>
<tr>
<td>October</td>
<td>3.46</td>
<td>3.29</td>
</tr>
<tr>
<td></td>
<td>(2.54)</td>
<td>(2.48)</td>
</tr>
<tr>
<td>November</td>
<td>2.60</td>
<td>2.53</td>
</tr>
<tr>
<td></td>
<td>(2.55)</td>
<td>(2.55)</td>
</tr>
<tr>
<td>December</td>
<td>2.60</td>
<td>2.53</td>
</tr>
<tr>
<td></td>
<td>(2.56)</td>
<td>(2.56)</td>
</tr>
</tbody>
</table>

Note: Numbers in parentheses are observed (benchmark) values for the period 1989/1990 – 1997/1998.

rate of change became unity. Consequently, high prices in May were associated with a large in-absolute-value spread, implying that a high price for the current crop cannot be maintained for the new crop by rolling over a hedge into the December contract.

6. Concluding remarks

This article approached model building from the viewpoint that the parameters of price distributions should be consistent with, and identifiable from, a structural model. A rational expectations commodity storage model of the U.S. corn market, calibrated to the experience of the 1990s, was used to examine the aptness of this framework in explaining price behavior in an actual commodity market. Extensive realism was added to the model in terms of modeling the timing of production activities. Moreover, price backwardation was modeled, by including convenience yield as a component of carrying costs. Thus, unlike standard models in the literature that assume a constant carrying cost, backwardation did not depend on stock-outs, which in reality have never occurred for U.S. corn.

Monthly price series, simulated from the models’ equilibrium price functions, were distributed with similar first, second, and fourth moments as those faced by market participants during the 1990s. The season-high prices occurred in May and the season-low in November, while price variability was the smallest in November and highest during the growing season (May through August), consistent with the benchmark. Simulated December and May futures price distributions exhibited a time-to-maturity effect similar to those estimated from the benchmark.
period. The results were robust with respect to changes in parameter values. On the other hand, the model performance could have been improved in terms of skewness and autocorrelation coefficients.

A variety of simulation results suggested that the experience of the 1990s was only a tiny proportion of outcomes that could have occurred under the same market structure and cautioned against inferences on economic and policy decisions drawn from short, historical samples. In our application of the U.S. corn market, for example, analyses of a long sample of simulated prices raised questions about the consequences of using some of the commonly recommended marketing strategies, such as diversification of cash sales, as a way to reduce the variability of returns or to enhance average returns. In addition to an improved understanding of long-run distributions of commodity prices, our model could be used to gain insights into the possible realizations of prices that might be encountered in different finite time periods. If, for example, one looked at simulations in terms of 40-year “lifetimes,” it is clear that a corn producer could face a variety of outcomes over different finite periods. A producer could be “lucky” or “unlucky” in terms of the behavior of prices, and these differences would arise notwithstanding the assumptions of an efficient market with rational decision makers.

The model included six state variables, but a model, no matter how complex, is an abstraction from the actual world. So many factors evidently combine to affect prices in reality that most of them cannot be disentangled from noise. The model’s complexity cannot be easily enhanced because of the “curse of dimensionality”; computational time increases almost exponentially as state variables are added. It should also be noted that the rational expectations framework implies efficient markets, and consequently, the model would need to be modified to consider hypotheses related to market inefficiency.

Nonetheless, our results suggest that a relatively simple model can be developed that produces prices that mimic the complexity of actual commodity price behavior. The virtue of the approach taken in this article is that both the simulated cash and the futures prices are consistent with recent historical experience and reflect an efficient market with rational decision makers. The simulations, therefore, are useful in generating prices that permit empirical analyses of the long-run impacts of economic and policy decisions.

**Appendix: Details of the solution method**

The unknown monthly price function \( P[\cdot] \) (decomposed into 12 monthly component functions \( P_m[\cdot] \), \( m = 1, \ldots, 12 \)), the planting function \( \Psi[\cdot] \), and the expected harvest price function \( \hat{P}[\cdot] \) were approximated by finite linear combinations of cubic b-spline functions, respectively. Function approximants were required to hold precisely at collocation nodes. The collocation nodes for the price and planting functions were specified as discretized values of available supply at the beginning of the month, \( A^m_n, n = 1, \ldots, N \), and discrete levels of crop size \( H^m_n, n = 1, \ldots, N_P \), were specified as those for the expected price function. The collocation nodes were equally spaced within the approximation range, and the same number of cubic spline functions and collocation nodes was chosen.

Demand and supply shocks assumed \( I_m \)-by-1 vector \( u_m \) and \( J \)-by-1 vector \( y \), with probability vectors \( w_m \) and \( x \), respectively. Actual and expected proportions of crop harvested assumed similar \( G_m \)-by-1 vectors \( a_m \) and \( \hat{a}_m \) with corresponding probability vector \( z_m \). Normal random variables were discretized according to the Gaussian quadrature principles (Miranda and Fackler, 2002).

The expected crop size \( (H) \) was discretized to an \( L \)-by-1 vector with elements \( (H_l, l = 1, \ldots, L) \) that were equally spaced. A *priori* probability vector was assigned with it in May. For all adjacent months between May and November, the crop estimate was assumed to follow an \( L \)-state Markov chain with Markov transition matrices with elements

\[
v^m_{k,l} = \Pr \{ H^{m+1} = H_k | H^m = H_l \},
\]

\( l, k = 1, \ldots, L; \ m = 5, \ldots, 10. \)

Following Pirrong (1999), the transition probability was calculated as

\[
v^m_{k,l} = N[h_1] - N[h_2],
\]

where \( h_1 = ((H_{k+1} + H_k)/2 - H_l)/\sigma_m, \ h_2 = ((H_k + H_{k-1})/2 - H_l)/\sigma_m \) is the standard deviation of the error term \( \mu_m \) in Eq. (1), and \( N[\cdot] \) is the standard normal cumulative distribution function.

Hence, the function approximants in vector notation were

\[
P_m[A_m] \approx \Phi_m[A_m]c_m, \ \forall m,
\]

\[
\Psi[A_i] \approx \Phi_5[A_i]c_5, \quad \text{and}
\]

\[
\hat{P}[H_l] \approx \Phi_P[H_l]c_P,
\]

where elements of \( \Phi \) and \( c \) matrices are cubic b-spline functions and collocation coefficients, respectively; \( P_m[\cdot] \) and \( c_m \) are \( N \) by \( \Theta_m \) (the number of discrete combinations of state variables in a given month excluding the one specified as collocation nodes); \( \Phi_m[\cdot] \) and \( \Phi_5[\cdot] \) are \( N \) by \( N \), \( \Psi[\cdot] \) and \( c_5 \) are \( N \) by \( J \), \( \hat{P}[\cdot] \) and \( c_P \) are \( N_P \) by \( 1 \); and \( \Phi_P[\cdot] \) is \( N_P \) by \( N_P \). The results in the article used \( I_m = I = 5, J = 30, G_9 = G_{10} = 3, G_{11} = 9, L = 10, N = 30, \) and \( N_P = 100 \).

Prior to the initial iteration, the monthly collocation nodes and initial values for the collocation coefficient matrix \( c \) were assigned. The iteration began by solving Eq. (7a) for October carryover levels \( s_{10} \) with the November price coefficients \( c_{11} \) fixed using Newton’s method. Once such storage levels were found, Eq. (7b) was solved for the (updated) October price function coefficients \( c_{10} \) using L-U factorization. Using these updated October price coefficients, Eq. (7a) was solved for September carryover levels \( s_9 \), which were used to obtain new
values of the September price coefficients \( \hat{c}_9 \), and the procedure was repeated for the remaining 10 months backward in time. At the end of a 12-month iteration, the norm of the difference between the old and new November price function coefficients, \( c_{11} \) and \( \hat{c}_{11} \), was compared to a pre-determined convergence level. If it was below the level, the new set of coefficients became a part of the solution and the algorithm stopped; otherwise, the iteration resumed with the new set of coefficients replacing the old guess.

Using the equilibrium price functions, the expected harvest price function was solved. Prices during the harvest months were obtained from simulating the corn model from various availability levels and planted crop sizes in April through November for 5,000 times. Expected harvest price function \( \hat{P} \) was calculated as the average over the 5,000 simulations of September, October, and November prices weighted by the ratio between crop harvested in each month and annual crop size. For each availability level, we interpolated a functional relationship between crop size and expected harvest price over the range of crop size as

\[
\hat{P}_n = \Phi_P [H_4]c_{P,n} \quad \forall A_n, \ n = 1, \ldots, N,
\]

by solving for \( c_P \) using L-U factorization, which is Eq. (7d).

Lastly, analogously to the monthly price functions, Miranda's (1998) two-step algorithm was applied to solve for the planting function defined in Eq. (7c). First

\[
\Phi_S [A_4]c_S = S_0[\Phi_P [H_4]c_P],
\]

was solved for planted crop size \( H_4 \) by Newton's method with the coefficients \( c_S \) fixed. Once such a crop size was found, \( c_S \) was updated according to

\[
H_4 = \Phi_S [A_4]c_S,
\]

via L-U factorization. The iteration was repeated until the coefficients converge.

Acknowledgments

The research was supported by U.S. Department of Agriculture, National Research Initiative Competitive Grants Award No. 99-35400-7796. We acknowledge helpful comments from A. Barkley, J. Conrad, J. Crespi, J. Peterson, L. Tauer, seminar participants at Cornell University, and two anonymous referees.

References


