ALTERNATIVE INVESTMENT RULES

Net Present Value (NPV)

\[
NPV = \sum_{t=1}^{n} \frac{C_t}{(1 + k)^t} - C_0
\]

\[
= PVCF - \text{Initial Investment}
\]

Decision Rule: Accept if NPV > 0

Economic Meaning of NPV?
Example — Consider investments A and B (k=10%)

<table>
<thead>
<tr>
<th>Year</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>($1,000)</td>
<td>($1,000)</td>
</tr>
<tr>
<td>1</td>
<td>500</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>400</td>
<td>200</td>
</tr>
<tr>
<td>3</td>
<td>300</td>
<td>300</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>400</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>500</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>600</td>
</tr>
</tbody>
</table>

\[ NPV_A = 500(PVIF_{10,1}) + 400(PVIF_{10,2}) + 300(PVIF_{10,3}) + 100(PVIF_{10,4}) + 10(PVIF_{10,5}) + 10(PVIF_{10,6}) - 1000 \]

\[ = 1091 - 1000 \]

\[ = 91 \]

\[ NPV_B = 403 \]

Decision:

\[ NPV > 0 \quad \text{Accept} \]

\[ NPV > 0 \quad \text{Accept} \]
OTHER CONTENDERS

- Accounting Return on Investment (AROI)

\[
\text{AROI} = \frac{\text{Accounting Profit}}{\text{Accounting Investment}}
\]

Accounting Profit = forecast of expected profit from project after depreciation and taxes

Accounting Investment = book value of investment

- Decision Rule: Accept if AROI > Benchmark (e.g. firm’s ROA or industry ROA)

- Problems
  - not based on CFs
  - ignores TVM
  - ignores risk
  - ambiguous
  - ad hoc benchmark
Payback Method

Number of years until cumulative forecasted CFs equal initial investment

\[
PB = (T - 1) + \left( \frac{C_0 - \sum_{t=1}^{T-1} C_t}{C_T} \right)
\]

\(T =\) number of years until investment is completely recovered by cumulative CFs.

- Decision Rule: Accept if \(PB <\) Benchmark

\[
PB_A = (3 - 1) + \left( \frac{\$1000 - \$900}{\$300} \right) = 2 \frac{1}{3} \text{ years}
\]

\[
PB_B = (4 - 1) + \left( \frac{\$1000 - \$600}{\$400} \right) = 4 \text{ years}
\]

- Advantages
  - easy, inexpensive, intuitive
  - some indication of risk and liquidity
Disadvantages

- ad hoc PB benchmark
- ignores risk of CFs
- Ignores CFs beyond PB period

Example: \( C_0^E = C_0^D = $10,000 \)

<table>
<thead>
<tr>
<th>Year</th>
<th>( CF^C )</th>
<th>( CF^D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$5000</td>
<td>$10,000</td>
</tr>
<tr>
<td>2</td>
<td>5000</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>20,000</td>
<td>1,000</td>
</tr>
</tbody>
</table>

\( PB_C = 2 \) yrs.

\( PB_D = 1 \) yr.

- Ignores TVM (equal weight gives to each CF)

Example: \( C_0^E = C_0^F = $1,001,000 \)

<table>
<thead>
<tr>
<th>Year</th>
<th>( CF^C )</th>
<th>( CF^D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>± 0</td>
<td>$1,000,000</td>
</tr>
<tr>
<td>2</td>
<td>1,000,000</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1,000</td>
<td>1,000</td>
</tr>
<tr>
<td>4</td>
<td>500,000</td>
<td>500,000</td>
</tr>
</tbody>
</table>

\( PB^E = PB^F = 3 \) yrs.
Discounted Payback

- discounts each CF at cost of capital
- finds "how many years CFs must be received from the investment for it to make sense from an NPV standpoint"
- overcomes "equal weighting" and risk problems to some extent
- still ignores CFs beyond PB
- benchmark is still ad hoc
Internal Rate of Return (IRR)

Measures rate of return that an investment earns

- IRR defined as "rate of return that makes PVCFs equal to initial costs"

- Single Period

\[
r = \text{rate of return} = \frac{\text{Payoff}}{\text{Cost}} - 1 = \frac{C_1}{C_0} - 1
\]  

(1)

Rearranging (1) gives

\[
\frac{C_1}{(1 + r)} - C_0 = 0
\]  

(2)

Remember

\[
\text{NPV} = \frac{C_1}{(1 + k)} - C_0
\]  

(3)

So IRR is the return level, r, that would make NPV = 0.

Note that if \( r < k \), then NPV < 0.

Decision rule: Accept if \( r > k \).
For multi-year investments \( r \), is defined as

\[
\frac{C_1}{(1 + r)} + \frac{C_2}{(1 + r)^2} + \cdots + \frac{C_n}{(1 + r)^n} = C_0
\]

or

\[
\sum_{t=1}^{n} \frac{C_t}{(1 + r)^t} - C_0 = 0 \quad \text{NPV}(r) = 0
\]

Finding IRRs

- **Annuities**

  Investment cost = $10,000
  CFs at end of years 1-10 = $1,627.45

  \[
  \text{PMT}(\text{PVIFA}_{r,10}) = 10,000
  \]

  \[
  C(\text{PVIFA}_{r,10}) = C_0
  \]

  $1,627.45(\text{PVIFA}_{r,10}) = 10,000

  \[
  \text{PVIFA}_{r,10} = 6.1446 \quad \text{(table value)}
  \]

  \[
  r = 10\%
  \]

  Accept if 10\% > k
Uneven streams of CFs

Trial and error

- Find PVCF at \( \hat{r} \) (start at 10%)
- If PVCF > I then ↑ \( \hat{r} \)
- If PVCF < I then ↓ \( \hat{r} \)
- Repeat until PVCF = I at which \( \hat{r} = r \)

Example: Investment A

\[
PVCF_A = 500(PVIF_{10,1}) + 400(PVIF_{10,2}) + 300(PVIF_{10,3}) + 100(PVIF_{10,4}) + 10(PVIF_{10,5}) + 10(PVIF_{10,6}) \\
= $1091
\]

(PVCF\(_A\)(10\%) = $1091) = (Cost = $1000)

Try \( \hat{r} = 15\% \)

(PVCF\(_A\)(15\%) = $1000) = (Cost = $1000)

\( r_A = 15\% \) (IRR\(_A\))

(IRR\(_B\) = 20\%)

Example: PVCF\(_B\)(20\%) = $1,000

\( r_b = 20\% \) (IRR\(_B\))
Profitability Index (PI)

\[
\text{PI} = \frac{\sum_{t=1}^{n} \frac{C_t}{(1+k)^t}}{\frac{C_0}{C_0}} = \frac{\text{PVCF}}{I}
\]

Decision Rule: Accept if \( \text{PI} > I \)

\[
\text{PI}_A = \frac{\$1091}{\$1000} = 1.091 > 1 \quad \text{Accept}
\]

\[
\text{PI}_B = \frac{\$1403}{\$1000} = 1.403 > 1 \quad \text{Accept}
\]
CRITERIA FOR CHOOSING INVESTMENT
DECISION RULE

Decision Process

- Identify new strategies and opportunities
- Estimate expected cash flows
- Apply decision rule to choose strategy

Decision Rule Should Maximize Value

- All CFs should be considered
- CFs should be weighted by opportunity cost of investment funds
- rule should select value maximizing strategy from a set of mutually exclusive investments
- rule should allow projects to be considered independently
COMPARISON BETWEEN NPV, IRR, AND PI

Remember the decision rules

<table>
<thead>
<tr>
<th>Investment Rule</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>NPV = 0</td>
<td>r = k</td>
</tr>
<tr>
<td>NPV &gt; 0</td>
<td>r &gt; k</td>
</tr>
<tr>
<td>NPV &lt; 0</td>
<td>r &lt; k</td>
</tr>
</tbody>
</table>

Each rule appears to select the same strategy. However, we’ll see that IRR and PI have some weaknesses, the biggest of which is that they don’t always maximize value.

Definitions

- Independent Project – Projects CFs are independent of other project’s CFs
- Mutually Exclusive Projects – Project’s CFs are related to other project’s CFs
Consider Investment A: Invest $100 today and receive $130 next year (r=30%)

Alternative: Invest $100 at k

Suppose k = 25% : $100 \times (1 + .25) = $125
k = 35% : $100 \times (1 + .35) = $135

So IRR decision rule (r > k) makes sense

In terms of NPV:

\[
NPV_A(k=25\%) = \frac{130}{1 + .25} - 100 = $4.00 > 0 \text{ (Accept A)}
\]

\[
NPV_A(k=35\%) = \frac{130}{1 + .35} - 100 = -$3.70 < 0 \text{ (Reject A)}
\]

Graphically

IRR Decision
Rule: Accept if r > k or whenever .3 > k

Financing Type Problems
Now consider Project B: Receive $100 and payout $130 next year (r=30%)

Alternatives:

Suppose $k = 25\% : $100 \times (1 + .25) = $125 \text{ payout}$

$k = 35\% : $100 \times (1 + .35) = $135 \text{ payout}$

In terms of NPV:

$\text{NPV}_B(25\%) = \frac{-130}{(1 + .25)} + 100 = -$4.00 \quad \text{(Reject B)}$

$\text{NPV}_B(35\%) = \frac{-130}{(1 + .35)} + 100 = $3.70 \quad \text{(Accept B)}$

Decision rule is "opposite" for financing type problems

IRR Decision Rule: Accept if $k > r$
Project is actually a substitute for borrowing. If you can borrow elsewhere for less, then reject project. Haven’t really addressed risk issue

Result:

Investment-type projects: Accept if \( r > k \)

Financing-type projects: Accept if \( r < k \)

Decision rule is reversed but not a problem if understood.
Multiple IRRs

Oil-well pump problem

An oil company is trying to decide whether or not to install a high-speed pump on a well that is already in operation. Cost of capital is 10%.

<table>
<thead>
<tr>
<th>Year</th>
<th>CF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1600</td>
</tr>
<tr>
<td>1</td>
<td>+10,000</td>
</tr>
<tr>
<td>2</td>
<td>-10,000</td>
</tr>
</tbody>
</table>

\[
NPV = \frac{10,000}{(1 + .10)} + \frac{-10,000}{(1 + .10)^2} - 1600 = -773.55
\]

To find IRR

\[
\frac{10,000}{(1 + r)} + \frac{-10,000}{(1 + r)^2} - 1600 = 0
\]

Note both \( r = 25\% \) and \( r = 400\% \) satisfy IRR definition

\[
\frac{10,000}{(1 + .25)} + \frac{-10,000}{(1 + .25)} = -1600 = 0
\]

\[
\frac{10,000}{(1 + 4)} + \frac{-10,000}{(1 + 4)} = -1600 = 0
\]
Multiple IRRs are the result of sign changes in the CF stream.

Let’s look again at the definition of IRR for this problem

\[
\frac{10,000}{(1 + r)} + \frac{-10,000}{(1 + r)^2} - 1600 = 0
\]

Multiply both sides by \((1 + r)^2\)

\[
10,000 (1 + r) - 10,000 - 1600 (1 + r)^2 = 0
\]

or

\[
10,000 - 10,000 (1 + r) + 1600 (1 + r)^2 = 0
\]

The IRR is the solution to the quadratic equation

\[ax^2 + bx + c = 0\]

where \(a = 1600\), \(b = -10,000\), \(c = 10,000\), and \(x = (1 + r)\)

The solution is given by the quadratic formula

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
(1 + r) = 10,000 \pm \frac{\sqrt{10,000^2 - 4(1600)(10,000)}}{2(1600)}
\]

\[
(1 + r) = \frac{10,000 \pm 6000}{3200}
\]

\[r = 25\% \text{ or } 400\%
\]

IRR decision rule no longer maximizes NPV.

Decarte's rule of signs implies \(m\) sign changes yields \(m\) possible IRRs.

Only safe to use IRR method with 1 sign change.
### Modified Decision Rule for IRR

<table>
<thead>
<tr>
<th>CFs</th>
<th># of IRRs</th>
<th>IRR Decision Rule</th>
<th>NPV Decision Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_0 &lt; 0 ) ( C_1, ..., C_n &gt; 0 )</td>
<td>1</td>
<td>( r &gt; k ) Accept</td>
<td>( NPV &gt; 0 ) Accept</td>
</tr>
<tr>
<td>( C_0 &gt; 0 ) ( C_1, ..., C_n &lt; 0 )</td>
<td>1</td>
<td>( r &lt; k ) Reject</td>
<td>( NPV &lt; 0 ) Reject</td>
</tr>
<tr>
<td>( C_0 \geq 0 ) ( C_1 \geq 0, ..., C_n \geq 0 ) (Multiple sign changes)</td>
<td>( \geq 1 )</td>
<td>no valid IRR</td>
<td>( NPV &gt; 0 ) Accept</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( NPV &lt; 0 ) Reject</td>
</tr>
</tbody>
</table>
Reinvestment Rate Assumption

Both NPV and IRR make explicit assumptions about how cash flows are reinvested.

Consider the Net Future Value (NFV) of an investment defined as the amount of cash available at period n in excess of the cash you could have received investing elsewhere.

\[
NFV = [CF_1 (1 + k)^{n-1} + C_2 (1 + k)^{n-2} + \ldots + C_n] - C_0 (1 + k)^n
\]

This NFV is the net "profit" or "value" of the investment at time n assuming the CFs are invested at k each period.

Again, assuming we can earn k elsewhere,

\[
NPV = \frac{NFV}{(1 + k)^n} = \left[ \frac{C_1 (1 + k)^{n-1}}{(1 + k)^n} + \frac{C_2 (1 + k)^{n-2}}{(1 + k)^n} + \ldots + \frac{C_n}{(1 + k)^n} \right] - \frac{C_0 (1 + k)^n}{(1 + k)^n}
\]

\[
= \left[ \frac{C_1}{(1 + k)} + \ldots + \frac{C_n}{(1 + k)^n} \right] - C_0
\]

NPV assumes CFs and NPV can be invested at k.
Similarly, the IRR can be thought of in terms of NFV:

\[
\text{NFV}(r) = [C_1 (1 + r)^{n-1} + ... C_n] - C_0 (1 + r)^n = 0
\]

or

\[
\text{NPV} = \frac{\text{NFV}}{(1 + r)^n} = \left[ \frac{C_1 (1 + r)^{n-1}}{(1 + r)^n} + ... \frac{c_n}{(1 + r)^n} \right] - \frac{C_0 (1 + r)^n}{(1 + r)^n} = 0
\]

\[
= \left[ \frac{C_1}{(1 + r)} + ... \frac{C_n}{(1 + r)^n} \right] - C_0 = 0
\]

So IRR assumes CFs are reinvested at \( r \).

Assuming reinvestment at \( k \) is more realistic

\( k \) is a market determined return for a given risk level

IRR assumes each project with a different \( r \) can earn a different return \( (r) \) on reinvested CFs, even though the projects may have the same risk level.

IRR doesn’t discount CFs at opp. cost of capital.
Modified IRR — we can modify IRR rule by improving assumptions about reinvestment.

Assuming CFs are reinvested at k gives a future value of the CFs at n of:

\[ FV_n = [C_1 (1 + k)^{n-1} + \ldots + C_n] \]

Modified IRR is the return on \( C_0 \) that makes

\[ FV_n = C_0 (1 + r)^n \]

\[ \sum_{t=1}^{n} C_t (1 + k)^{n-t} - C_0 (1 + r)^n = 0 \]

\[ r = \left[ \frac{\sum_{t=1}^{n} C_t (1 + k)^{n-t}}{C_0} \right]^{\frac{1}{n}} - 1 \]

Eliminates problem with multiple IRRs.

Consider again the oil-well pump example

\[ r = \left[ \frac{10,000(1 + .10) + (-10,000)}{1600} \right]^{\frac{1}{2}} - 1 = -.2094 \]

Reject ⇒ now consistent with NPV
Problems specific to mutually exclusive projects

- Scale problem

Money machine example-use once per day

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_0$</td>
<td>-$1</td>
<td>-$20</td>
</tr>
<tr>
<td>$C_1$</td>
<td>+2</td>
<td>30</td>
</tr>
</tbody>
</table>

$r_A = \frac{2}{1} - 1 = 1.00$ or 100%
$r_B = \frac{30}{20} - 1 = .5$ or 50%

$NPV_A = 2 - 1 = $1
$NPV_B = 30 - 20 = $10$

Ignores scale: high return on A is more than offset by ability to earn lower return on a larger investment in B.
Example 2 — C + D are mutually exclusive

<table>
<thead>
<tr>
<th>Time</th>
<th>C</th>
<th>D</th>
<th>Incremental CF (D - C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1000</td>
<td>-2500</td>
<td>-1500</td>
</tr>
<tr>
<td>1</td>
<td>4000</td>
<td>6500</td>
<td>2500</td>
</tr>
</tbody>
</table>

NPV (k = .25) $2200 $2700

IRR 300% 160%

Note that IRR gives incorrect decision.

We can still use IRR to evaluate mutually exclusive investments if we evaluate the "incremental" CFs (D - C).

We know C is acceptable r = 300% > k = 25%

Now consider if the "incremental" investment to obtain D’s CFs provides an accepted IRR.

IRR of incremental CFs: $r = \frac{2500}{1500} - 1 = 0.6667$ or $66.67\%$

$r_{(D-C)} = 66.67\% > 25\% \Rightarrow$ Make incremental investment, accept D.
\[
NPV_{(D-C)} = \frac{2500}{(1 + .25)} - 1500 = $500
\]

Three ways to compare mutually exclusive projects

1. Compare NPVs of projects
2. Calculate NPVs of incremental CFs
3. Compare IRR of incremental CFs to k

We can’t compare IRRs of projects with different investment sizes
Timing Problem — A and B are mutually exclusive

<table>
<thead>
<tr>
<th>Time</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1000</td>
<td>-1000 (no scale problem)</td>
</tr>
<tr>
<td>1</td>
<td>500</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>400</td>
<td>300</td>
</tr>
<tr>
<td>3</td>
<td>300</td>
<td>400</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>600</td>
</tr>
</tbody>
</table>

Find the NPV at different values of k

<table>
<thead>
<tr>
<th></th>
<th>k = 0</th>
<th>k = 5%</th>
<th>k = 10%</th>
<th>k = 15%</th>
<th>k = 20%</th>
</tr>
</thead>
<tbody>
<tr>
<td>NPV&lt;sub&gt;A&lt;/sub&gt;</td>
<td>$300</td>
<td>$180.42</td>
<td>$78.80</td>
<td>($8.33)</td>
<td></td>
</tr>
<tr>
<td>NPV&lt;sub&gt;B&lt;/sub&gt;</td>
<td>400</td>
<td>206.50</td>
<td>49.15</td>
<td>(80.13)</td>
<td></td>
</tr>
</tbody>
</table>

Cross-over point ≈ 7.1%

IRR<sub>A</sub> = 14.5%

IRR<sub>B</sub> = 11.8%
We can proceed as in the scale problem example and use any of the 3 methods

- Compare NPVs at k. Choose highest NPV.
  - If $k < 7.1\%$ Choose B
  - $7.1\% < k < 14.5\%$ Choose A
  - $k > 14.5\%$ Reject both

- Find IRR of incremental CFs ($B - A$)

<table>
<thead>
<tr>
<th>Year</th>
<th>(B - A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-400</td>
</tr>
<tr>
<td>2</td>
<td>-100</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>500</td>
</tr>
</tbody>
</table>

$r_{B-A} = 7.1\% \Rightarrow$ If $k < 7.1\%$ choose B
  - $k > 7.1\%$ choose A

Must make sure A was acceptable to begin with $r_a > k$
Calculate NPV of incremental CFs (assuming base investment is acceptable)

\[
NPV_{B-A}(k=10\%) = \frac{-400}{(1 + .10)} + \frac{-100}{(1 + .10)^2} + \frac{100}{(1 + .10)^3} + \frac{500}{(1 + .10)^4} - 0 = -29.64 \quad \text{(Reject B)}
\]

\[
NPV_{B-A}(5\%) = $26.07 \quad \text{(Accept B)}
\]

"Brian’s Rule" — incremental IRRs

1. Calculate IRR for each investment.

2. If each IRR > k then calculate incremental CFs; find IRR; and compare to k.

3. If one project has IRR > k and the other has IRR < k, accept project with IRR > k.

4. If both projects have IRR < k, reject both.
Value Additivity Property

Choice of two mutually exclusive projects shouldn’t depend on other independent projects under consideration.

Example: A and B are mutually exclusive
C is independent
\( k = 10\% \)

<table>
<thead>
<tr>
<th>Year</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>A+C</th>
<th>B+C</th>
<th>(A+C)-(B+C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-100</td>
<td>-100</td>
<td>-100</td>
<td>-200</td>
<td>-200</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>450</td>
<td>675</td>
<td>-275</td>
</tr>
<tr>
<td>2</td>
<td>550</td>
<td>0</td>
<td>0</td>
<td>550</td>
<td>0</td>
<td>550</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Project</th>
<th>NPV</th>
<th>IRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$354.30</td>
<td>134.5%</td>
</tr>
<tr>
<td>B</td>
<td>104.53</td>
<td>125</td>
</tr>
<tr>
<td>C</td>
<td>309.05</td>
<td>350</td>
</tr>
<tr>
<td>A + C</td>
<td>663.35</td>
<td>212.80</td>
</tr>
<tr>
<td>B + C</td>
<td>413.58</td>
<td>237.50</td>
</tr>
<tr>
<td>(A+C)-(B+C)</td>
<td>204.55</td>
<td>100</td>
</tr>
</tbody>
</table>
In isolation both NPV and IRR:  Choose A over B

In combination with C:  
NPV chooses A  
IRR chooses B  
Incremental IRR chooses A  
Incremental NPV chooses A

General Result:

1. Compare project NPVs
2. Evaluate incremental IRR
3. Evaluate incremental NPV

IRR survives because it gives concise summary of information about project (rate of return) and is easy to interpret

Most deficiencies can be overcome, but is it worth it?

Less easy to use and explain — but can give correct result.
Profitability Index

Independent Projects

\[ NPV = PVCF - I \quad \text{and} \quad PI = \frac{PVCF}{I} \]
give the same accept/reject decision

Mutually exclusive projects

<table>
<thead>
<tr>
<th>Year</th>
<th>A</th>
<th>B</th>
<th>A - B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-20</td>
<td>-10</td>
<td>-10</td>
</tr>
<tr>
<td>1</td>
<td>70</td>
<td>15</td>
<td>55</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>40</td>
<td>-30</td>
</tr>
</tbody>
</table>

PVCFs  
- A: 70.5  
- B: 45.3  
- A - B: 25.2  
(k = 12%)

PI  
- A: 3.53  
- B: 4.53  
- A - B: 2.52

NPV  
- A: 50.5  
- B: 35.3  
- A - B: 15.2

PI \to \text{Accept B, but A has higher NPV.}

Scale problem: PI misses extra gains from larger investment in A, even though benefits per unit of cost are greater in B.

Once again, problem is fixed by using incremental cash flows.
Capital Rationing

Suppose there are limits on the firm that prevent it from taking all acceptable projects (NPV > 0).

Suppose A, B, and C are independent projects

<table>
<thead>
<tr>
<th>Project</th>
<th>$C_0$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>NPV@10%</th>
<th>PI@10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-10</td>
<td>30</td>
<td>5</td>
<td>21</td>
<td>3.1</td>
</tr>
<tr>
<td>B</td>
<td>-5</td>
<td>5</td>
<td>20</td>
<td>16</td>
<td>4.2</td>
</tr>
<tr>
<td>C</td>
<td>-5</td>
<td>5</td>
<td>15</td>
<td>12</td>
<td>3.4</td>
</tr>
</tbody>
</table>

Each project is desirable (NPV > 0, PI > 0). However, capital constraints limit total investment to $10 million.

Choices: Invest $10 million in A
        Invest $5 million in B and $5 million in C.

<table>
<thead>
<tr>
<th>Investment</th>
<th>NPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>21</td>
</tr>
<tr>
<td>B + C</td>
<td>28</td>
</tr>
</tbody>
</table>

Best to choose B + C

Note: PI gain correct ranking at project, while NPV failed to do so.
Using NPV we must evaluate all possible combinations projects and select the ones with the highest combined NPV.

With PI, we just select the highest ranking projects.

Unfortunately, PI ranking doesn’t always work. Consider the case when there are capital constraints in more than 1 period.

<table>
<thead>
<tr>
<th>Project</th>
<th>C₀</th>
<th>C₁</th>
<th>C₂</th>
<th>NPV@10%</th>
<th>PI@10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-10</td>
<td>30</td>
<td>5</td>
<td>21</td>
<td>3.1</td>
</tr>
<tr>
<td>B</td>
<td>-5</td>
<td>5</td>
<td>20</td>
<td>16</td>
<td>4.2</td>
</tr>
<tr>
<td>C</td>
<td>-5</td>
<td>5</td>
<td>15</td>
<td>12</td>
<td>3.4</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>-40</td>
<td>60</td>
<td>13</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Capital constrained to be $10 at t=0 and t=1.

Strategies: 1) Accept B and C  
2) Accept A this year and D next year

NPV_{B+C} = 28  \quad NPV_{A+D} = 34

PI breaks down
SIMPLE CAPITAL RATIONING MODEL

- In perfect markets, no capital rationing exists
- In practice, firms often imposed capital rationing to
  - reduce transaction costs
  - reduce bias in CF estimates
  - manage growth
- Information asymmetries may result in "effective" capital rationing

Consider the previous problem. Suppose we can make fractional investments in each projects (A, B, C, D) subject to our $10 budget constraint this period and next period.

Our objective is to maximize NPV st. constraints

\[
\text{Max} \quad \text{NPV} = 21X_a + 16X_B + 12X_c + 13X_d
\]

\[
x_a, x_b, x_c, x_d
\]

s.t. \[
10X_a + 5X_b + 5X_c + 0X_d \leq 10
\]

\[
-30X_a - 5X_b - 5X_c + 40X_d \leq 10
\]

\[
0 \leq X_a \leq 1, \quad 0 \leq X_b \leq 1, \quad 0 \leq X_c \leq 1, \quad 0 \leq X_d \leq 1
\]
Solve using linear programming software (or trial and error)

Solution: \( X_a = .5, \ X_b = 1, \ X_c = 0, \ X_d = .75 \)

NPV = $36.25 million \quad \text{(Increase of $2.25 million in NPV over integer solution)}

☐ Sometimes fractional projects make sense (building size)

☐ Sometimes project sizes need to occur in fixed increments — solve using integer (zero-one) programming
Cash Carry Forward — Suppose unspent budget in t=0 can be carried forward and earns rate k

\[
\text{Max } \text{NPV} = 21X_a + 16X_b + 12X_c + 13X_d \quad j = a, b, c, d
\]

\[
s.t. \quad 10X_a - 5X_b + 5X_c + 0X_d + s = 10
\]
\[
-30X_a - 5X_b - 5X_c + 40X_d \leq 10 + (1 + k)
\]
\[
S \geq 0 \quad 0 \leq X_j \leq 1
\]

Mutually Exclusive projects (B and C)

Constraint: \( X_b + X_c \leq 1 \)

\[\begin{align*}
X_b &= 0 \text{ or } 1 \\
X_c &= 0 \text{ or } 1
\end{align*}\] Integer investment
Contingent projects — Suppose D is attached to A. Can’t accept D without A.

\[ X_a = 0 \text{ or } 1 \]
\[ X_d = 0 \text{ or } 1 \]
\[ X_d - X_a \leq 0 \]

So,

if \[ X_a = 1 \Rightarrow X_d = 0 \text{ or } 1 \]
\[ X_a = 0 \Rightarrow X_d = 0 \]

Nonfinancial resource constraints

Suppose each project requires services of a 12 person support technical design team

\[ A = 3 \text{ FTE, } B = 2 \text{ FTE, } C = 8 \text{ FTE, } D = 3 \text{ FTE} \]

Constraint: \[ 3X_a + 2X_b + 8X_c + 3X_d \leq 12 \]

Could continue — many possible examples — L.P. is very versatile

Problems — difficult to justify constraints — difficult to deal with risk
### PROJECTS WITH DIFFERENT LIVES

<table>
<thead>
<tr>
<th>Year</th>
<th>C</th>
<th>D</th>
<th>k = 10%</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(40,000)</td>
<td>(20,000)</td>
<td>Mutually Exclusive</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>8,000</td>
<td>7,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>14,000</td>
<td>13,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>13,000</td>
<td>12,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>12,000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>11,000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10,000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \text{NPV}_c = 9281 \quad \text{NPV}_d = 6123 \]

- Replication Method — assume projects can be replicated at a constant scale

- Compute NPV of an infinite stream of constant scale replications

\[
\text{NPV}(N, \infty) = \text{NPV}(N) + \frac{\text{NPV}(N)}{(1 + k)^N} + \frac{\text{NPV}(N)}{(1 + k)^{2N}} + \ldots
\]

Let \( u = \frac{1}{(1 + k)^N} \)

\[
\text{NPV}(N, \infty) = \text{NPV}(N) (1 + u + u^2 + \ldots) \quad (1)
\]

\[
u \text{NPV}(N, \infty) = \text{NPV}(N) (u + u^2 + \ldots) \quad (2)
\]
This is the NPV of an N year project replicated at a constant scale on infinite number of time.

\[
NPV(N, \infty) - uNPV(N, \infty) = NPV(N)
\]

\[
NPV(N, \infty) = \frac{NPV(N)}{1 - u} = NPV(N) \left[ \frac{1}{1 - \frac{1}{(1 + k)^N}} \right]
\]

\[
= NPV(N) \left[ \frac{(1 + k)^N}{(1 + k)^N - 1} \right]
\]

\[
NPV_a(6, \infty) = $9281 \left[ \frac{(1 + .10)^6}{(1 + .10)^6 - 1} \right] = $21,309.86
\]

\[
NPV_B(3, \infty) = $6123 \left[ \frac{(1 + .10)^3}{(1 + .10)^3 - 1} \right] = $24,621.49
\]

B preferred under replication assumption.
Annual Equivalent Method (AE)

Calculate annuity amount a project’s NPV can provide during its life.

\[
NPV = AE(PVIFA_{k,n})
\]

\[
AE = \frac{NPV}{(PVIFA_{k,n})} = \text{perpetual stream of income project could generate with reinvestment}
\]

Gives same answer as NPV \((N, \infty)\) if projects have same risk.

Problem may occur if projects have different risk \((k)\).
Note: \[ AE = \frac{NPV(N)}{PVIFA_{k,n}} = k \ NPV(N,\infty) \]

Remember \[ PVIFA_{k,n} = \frac{1}{k} \left( \frac{1}{(1+k)^N} - 1 \right) \]

so

\[
\frac{NPV(N)}{\left( \frac{1}{1 - \frac{1}{(1+k)^N}} \right)} = k \ NPV(N) \left( \frac{1}{\left( \frac{1}{1 - \frac{1}{1+k^n}} \right)} \right)
\]

\[ AE = kNPV(N_d,\infty) \]

Suppose \( NPV(N_c) = NPV(N_d) \) but \( k_c > k_d \)

It is now possible that \( NPV(N_c, \infty) < NPV(N, \infty) \)

but

\[ k_c \ NPV(N_c, \infty) > k_d \ NPV(N_d, \infty) \]

So AE method may give the wrong result
Example

<table>
<thead>
<tr>
<th>Year</th>
<th>A</th>
<th>B</th>
<th>( k_a = 10% )</th>
<th>( k_b = 40% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-10</td>
<td>-10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>6.55</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>6.55</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6.55</td>
<td>6.55</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ NPV_a = $0.4132 \quad NPV_b = $0.4074 \]

\[ NPV_a (2, \infty) = .4132 \left[ \frac{(1.10)^2}{(1.10)^2 - 1} \right] = $2.3808 \]

\[ NPV_b (3, \infty) = .4074 \left[ \frac{(1.40)^3}{(1.40)^3 - 1} \right] = $0.641 \]

\[ k_a \ NPV_a (2, \infty) = .10 \ (2.3808) = $0.23808 \]

\[ k_b \ NPV_b (3, \infty) = .40 \ (.641) = $0.2564 \]

B gives larger CF stream, but we can’t compare them directly because of different risk levels.
Note — we could have made other assumptions about how to reinvest. Our constant scale assumption look like

<table>
<thead>
<tr>
<th>Year</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(40,000)</td>
<td>(20,000)</td>
</tr>
<tr>
<td>1</td>
<td>8,000</td>
<td>7,000</td>
</tr>
<tr>
<td>2</td>
<td>14,000</td>
<td>13,000</td>
</tr>
<tr>
<td>3</td>
<td>13,000</td>
<td>12,000  + (20,000)</td>
</tr>
<tr>
<td>4</td>
<td>12,000</td>
<td>7,000</td>
</tr>
<tr>
<td>5</td>
<td>11,000</td>
<td>13,000</td>
</tr>
<tr>
<td>6</td>
<td>10,000</td>
<td>12,000</td>
</tr>
</tbody>
</table>

NPV\_c = 9281 \quad NPV\_d = 10,723

Other assumption

- Other investments available in future
- Scale of investment charges as CFs are received (remember capital rationing example)
- making no adjustment means funds are reinvested to earn k (i.e. NPV = 0)
Last thought on Projects With Unequal Lines

Could have made other assumptions about reinvestment opportunities

If we make no reinvestment assumption, we are implicitly assuming reinvestment in projects earning k (NPV = 0)

Example

<table>
<thead>
<tr>
<th>Year</th>
<th>D</th>
<th>E</th>
<th>D + E</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(20,000)</td>
<td>(20,000)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>7,000</td>
<td></td>
<td>7,000</td>
</tr>
<tr>
<td>2</td>
<td>13,000</td>
<td></td>
<td>13,000</td>
</tr>
<tr>
<td>3</td>
<td>12,000</td>
<td>(34,770)</td>
<td>(22,770)</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>3,477</td>
<td>3,477</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>3,477</td>
<td>3,477</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>38,247</td>
<td>38,247</td>
</tr>
</tbody>
</table>

FV of CFs from D

\[ FV = 7000(1 + 10)^2 + 13000(1 + .10)^1 + 12000 = 34,770 \]

\[ NPV_{D+E} = 6123 \]

\[ NPV_D = 6123 \]

\[ NPV_E = 0 \]
Capital Budgeting with Inflation

Start in world without inflation

\[
NPV = \frac{C_1}{(1 + k)} + \ldots + \frac{C_n}{(1 + k)^n} - C_0
\]

Suppose \( C_{1.5} = \$26,500 \), \( k = 9\% \), \( n = 5 \), \( C_0 = \$100,000 \)

\[
NPV_R = 26,500 \times (PVIFA_{9,5}) - 100,000 = 3,075.75
\]

Now suppose the future rate of inflation is 6%/year.

Since investment and security returns are based on expected future CFs, the anticipated inflation rate will be reflected in the required return on the project.

Fischer Effect: \((1 + r) (1 + i) = (1 + k)\)

\[
\begin{align*}
    r &= \text{real rate of return} \\
    i &= \text{anticipated annual inflation} \\
    k &= \text{required return}
\end{align*}
\]
In our case

\[(1 + .09)(1 + .06) = (1 + .09 + .06 + .0054)\]

\[\Rightarrow k = .1554 \text{ or } 15.54\%\]

Suppose you lend $1000 today and want to increase your purchasing power next year by 9%.

Cost of goods and services next year = 1000 \((1 + .06)\) = $1060

To increase purchases by 9% you need $1060 \((1 + .09)\) = $1,155.40

Required return

\[r = \frac{1,155.40}{1000} - 1 = .1554 \text{ or } 15.54\%\]

\[k = .09 + .06 + .0054\]

\[\downarrow \quad \downarrow \quad \downarrow\]

9% on 6% 9% on

$1000 inflation inflation amount
Market data on interest rates and rates of return will include a premium for inflation that is anticipated.

To avoid bias → must include inflation effects in CFs.

Alternative: Don’t include inflation in CFs and remove inflation premium from required return.

Suppose the 6% inflation rate applied to the projects CFs.

\[
\text{NPV}_N = \frac{26,500(1 + .06)}{(1 + .09)(1 + .06)} + \ldots \frac{26,500(1 + .06)^5}{(1 + .09)^5(1 + .06)^5} - 100,000
\]

\[
= \frac{26,500}{(1 + .09)} + \ldots \frac{26,500}{(1 + .09)^5} - 100,000
\]

\[
= $3,078.75
\]

So \(\text{NPV}_N = \text{NPV}_R\) (NPV is both real and nominal)
Consistency:

1) Nominal CFs discounted at nominal rate
2) Real CFs discounted at real rate

In practice, effect of inflation on CFs may differ from effect on required return.

Each of component may be impacted differently by expected inflation.

\[
NPV = \sum_{j=1}^{n} \frac{(R_j(1 + i_R)^j - E_j(1 + i_E)^j - Dep_j)(1 - t) + Dep_j}{(1 + k)^j} - C_0
\]

- Applications later
- Point — account for anticipated inflation in CFs and discount rate.