PRESENT VALUE ANALYSIS

Traditional Single Period Profit Maximization

- \[ \max \pi = Pf(x) - c(x) \]

where: \( \pi \) = profit
\( P \) = output price
\( f(x) \) = production function
\( c(x) \) = cost function
\( x \) = input level

\[ f'(x) > 0 \quad c'(x) > 0 \]
\[ f''(x) < 0 \quad c''(x) > 0 \]

- First-order Condition (FOC)

\[ \frac{\partial \pi}{\partial X} = Pf'(x) - c'(x) = 0 \]
\[ Pf'(x) = c'(x) \]
\[ MR = MC \]

- Second-Order Condition (SOC)

\[ \frac{\partial^2 \pi}{\partial x^2} = Pf''(x) - c''(x) < 0 \]
Two Period Profit Maximization

\[ \text{Max} \pi_1 = Pf(x) - c(x)(1 + k) \]

\[ k = \text{opportunity cost (interest rate)} \]

Costs increase by opportunity cost of spending money on input at \( t = 0 \), while returns are received at \( t = 1 \).

FOC

\[ \frac{\partial \pi}{\partial x} = Pf'(x) - c'(x)(1 + k) = 0 \]

\[ Pf'(x) = c'(x)(1 + k) \]

\[ MR = MC \]

Rearranging FOC gives

\[ c'(x) = \frac{Pf'(x)}{1 + k} \]

Initial Investment (I) = Present Value of Future Cash Flow (PVCF)

Decision Rule: Continue to invest as long as PVCF > I

Net Present Value (NPV) = PVCF - I

Decision Rule: Invest if NPV > 0
Example 1:

Suppose you can purchase a 60 acre tract of land that is currently overgrown with small trees and brush for $250 per acre. Clearing the land will take 1 year but can be done at a cost of $300 per acre payable at the day the land is purchased.

You estimate you can sell the land for $650 per acre at the end of 1 year based on sales of comparable land in the area. Assume for a moment the land’s payoff of $650 is a “sure thing.”

If you don’t invest in the land you could invest in US government securities that mature in 1 year. Suppose the return on these securities is 7% per year.

\[
\text{The PVCF} = \frac{$650}{(1 + .07)} = $607.47/acre \quad \text{or} \quad \$36,448.20
\]

Initial Investment = $250 + 300 = $550/acre \quad \text{or} \quad \$33,000.00

\[
\text{NPV} = \text{PVCF} - I = $607.47 - $550 = $57.47/acre \quad \text{or} \quad \$3,448.20
\]
Example 2:

Now suppose, more realistically, that the future value of the land is not known with certainty but instead represents an average value. This changes the amount investors will pay for the land.

Premise: A safe dollar is worth more than a risky dollar and investors will require a higher *expected* return to purchase the claim to the future risk CF.

Suppose you believe the investment is as risky as another investment on *traded asset* which provides an average return of 14%.

\[
NPV = \frac{650}{(1 + .14)} - 550 = \$20.18/acre \text{ or } \$1210.80
\]
Economic Interpretation of NPV Measure

- NPV = PVCF - I
  Invest if NPV > 0

Suppose CF$_1$ = $120
\[ I = $100 \]
\[ K = 10\% \text{ (opp. cost)} \]

\[
\text{NPV} = \frac{\text{CF}_1}{(1 + .10)} - I = \frac{120}{(1 + .10)} - 100 = \frac{$9.09}{10} > 0 \implies \text{Invest}
\]

- You are $9.09 better off in today’s $s if you invest.
- Your alternative is to invest in something that return 10%. So to get the same $120 CF next period you need

\[
($100 + $9.09) (1 + .10) = $120
\]

\[
\uparrow \quad \uparrow \quad \uparrow
\]

cost of NPV \quad return \quad project \quad available \quad elsewhere
• Rate of Return Criterion

We can re-express our criterion in terms of the rate of return the investment generates.

• Remember our profit max FOC

\[ Pf'(x) = c'(x)(1 + k) \]

\[ \frac{Pf'(x) - c'(x)}{c'(x)} = k \]

• Decision Rule: Invest until the marginal return investment equals opportunity cost of the return.

Internal Rate of Return (IRR) = \( \frac{\text{Cash Flow}}{\text{Investment}} \).

• In our example: Return = \( \frac{650 - 550}{550} = .1818 \)

or 18.18% > 14% \( \Rightarrow \) Invest.
- NPV Rule: Accept if NPV > 0

IRR Rule: Accept if IRR > K

- The opportunity cost, \( K \) reflects the rate of return on traded assets that have similar risk characteristics as the CFs of the asset under consideration. (More later.)
SIMPLE MODEL OF CAPITAL MARKETS, CONSUMPTION, AND INVESTMENT

- Assumptions

  2–period model (decisions made in period $t = 0$)
  No transactions costs or taxes
  Income endowments \{\(y_0, y_1\)\}
  Utility: \(U\{C_0, C_1\}\)

  where \(C_t = \sum_{i=1}^{n} P_{it} q_{it}\) \(t = 0, 1\)

  \[
  U_0 = \frac{\partial U}{\partial C_0} > 0 \quad U_1 = \frac{\partial U}{\partial C_1} > 0
  \]

  \[
  U_{00} = \frac{\partial^2 U}{\partial C_0^2} < 0 \quad U_1 = \frac{\partial U}{\partial C_1^2} < 0
  \]

  □ More on Preferences
Slope of indifference curves:

Curves of utility function $U(C_0, C_1)$ defined at $U(C_0, C_1) = U^a$

Define $C_1 = C_1(C_0, U^a)$

Differentiate $U(C_0, C_1(C_0, U^a)) \equiv U^a$

$$U_0 dC_0 + U_1 \frac{\partial C_1}{\partial C_0} dC_0 \equiv 0$$

$$U_0 + U_1 \frac{\partial C_1}{\partial C_0} \equiv 0$$
\[ MRS_{c_1}^{c_0} = \frac{\partial c_1}{\partial c_0} = -\frac{U_0}{U_1} \]

\[ \begin{align*}
U_0 &> 0 \\
U_1 &> 0
\end{align*} \Rightarrow \text{negative slope} \]

\[ MRS_{c_1}^{c_0} = \text{avg. rate at substitution of } c_1 \text{ for } c_0 \]

- How much of \( c_1 \) do I need to receive to give up a unit of \( c_0 \).
- Subjective rate of time preference required return to forego current consumption.

\[ MRS(Y) > MRS(X) \]

At \( Y \) the rate of time preference is greater, which means you will require more future consumption to give up a unit of current consumption.

This makes sense because you are starting with less current consumption and more future consumption at point \( Y \) relative to \( X \).

- Utility of initial endowment: \( U = U(y_0, y_1) \)
- Capital Markets

- Facilitate transfer of funds between borrowers and lenders

- Market where people trade dollars today for dollars in the future

- Funds lent today return interest and principle in future period(s)

\[ x_1 = x_0 + x_0 k = x_0 (1 + k) \text{ or } x_0 = \frac{x_1}{1 + k} \]

\[
\begin{align*}
x_1 &= \text{Future value of borrowed funds} \\
x_0 &= \text{Present value of borrowed funds} \\
k &= \text{Interest rate}
\end{align*}
\]

Suppose cash flow today = \(y_0\)  
cash flow in 1 year = \(y_1\)  
borrowing and lending rate = \(k\)

If you don’t consume today you can lend \(y_0\) at rate \(k\) and consume \(c_1^* = y_1 + y_0 (1 + k) = w_1\).

If you don’t want to consume anything in period 1 you can borrow against \(y_1\) and consume \(c_0^* = y_0 + \frac{y_1}{1 + K} = w_0\).

\(W_1 = \text{future value at consumption bundle } \{y_0, y_1\}\)

\(W_0 = \text{present value of consumption bundle } \{y_0, y_1\}\)
By trading in the capital market we can move our consumption bundle anywhere along the capital market line.

Suppose we borrow \( \frac{y_1}{1 + k} \), then we can consume

\[
W_0 = y_0 + \frac{y_1}{1 + k} \text{ today.}
\]

Alternatively we can lend any amount of \( W_0 \) and adjust our consumption bundle to be \( \{c_o^*, c_1^* = (w_0 - C_0^*)(1 + k)\} \).
The present value of the consumption bundle is

\[
PV = c_0^* + \frac{c_1^*}{(1 + k)} = c_0^* + \frac{(w_0 - c_0^*)(1 + k)}{(1 + k)} = w_0
\]

So \( w_0 = c_0^* + \frac{c_1^*}{(1 + k)} \Rightarrow C_1^* = w_0(1 + k) - (1 + k)c_0^*

\[
= w_1 - (1 + k)c_0^*
\]

This last expression describes the Capital Market Line.

Remember the utility of our initial endowment is \( U\{y_0, y_1\} \) in the absence of capital markets.

- In the presence of capital markets, the utility maximizing consumption bundle is found solving

\[
\max_{c_0, c_1} U(c_0, c_1) \quad \text{s.t.} \quad c_1 = w_1 - (1 + k)C_0
\]

Substituting the constraint into the utility function our problem becomes
FOC

\[ \frac{\partial U}{\partial C_0} = U_0 + U_1 \frac{\partial c_1}{\partial c_0} = 0 \]

\[ = U_0 - U_1 (1 + k) = 0 \]

So the FOC implies

\[ - \frac{U_0}{U_1} = -(1 + k) \]

\[ MRS_{c_0}^{c_1} = \text{Slope of capital market line} \]

If \[ MRS_{c_1}^{c_0} < -(1 + k) \], then borrow (decreases \( U_0 \), increases \( U_1 \))

\[ MRS_{c_1}^{c_0} > -(1 + k) \], then lend (increases \( U_0 \), decreases \( U_1 \))
Borrowing

\[ U(y_0, y_1) < U(c_1, c_0) \]

Amount you are willing to forego in future consumption at \((y_0, y_1)\) is more than the market is requiring you to give up and so borrow.

Borrow until amount you are required to give up equals amount you are willing to give up.
The return you require to forego current consumption is less than what market will pay so lend at \((y_0, y_1)\).

Continue to lend until the required return equals the market return.

- In the presence of the capital market:

\[
U(c_0^*, c_1^*) \geq U(y_0, y_1)
\]

Capital market allows consumption patterns to be adjusted across time, thus increasing \(U\).
Adding Production Opportunities

Suppose we can invest a portion of current consumption in productive assets.

Define investment $t = 0$ as $I_0 = w_0^y - d_0$

where $W_0^y = y_0 + \frac{y_1}{(1 + k)}$

The investment opportunity function can be defined as $d_1 = g(I_0)$ where $g' > 0$ and $g'' < 0$

In the absence of capital markets $c_0 = d_0, c_1 = d_1$
Thus \( c_1 = g(w_0^y - c_0) \)

\[
\frac{\partial c_1}{\partial c_0} = -g' = MRT \quad \text{(rate at which } c_0 \text{ can be transferred into } c_1)\]

The decision problem becomes

\[
\max_{c_0, c_1} U(c_0, c_1) \quad \text{s.t.} \quad c_1 = g(w_0^y - c_0)
\]

FOC

\[
\frac{\partial U}{\partial c_0} = U_0 - U_1 g' = 0
\]

\[
\frac{-U_0}{U_1} = -g'
\]

or

\[
\text{MRS}_{c_1}^{c_0} = \frac{MRT}{c_1}^{c_0}
\]

Rate at which \( c_0 \) will be given = Rate at which \( c_0 \) can be transformed up for \( c_1 \) into \( c_1 \)
\[ MRS = MRT \]
Productive opportunities and capital markets

If capital markets are available then the consumption and investment decisions may not be directly linked as before.

It may be optimal to generate one cash flow stream through productive investment and then adjust the consumption levels through the capital market.

Let’s start by asking how we would maximize the present value of wealth when both productive opportunities and capital markets are available.
Remember

\[ W_0 = d_0 + \frac{d_1}{1 + k} \]  

Present value of cash flows

where

\[ d_1 = g(w_0^y - d_0) \]

Maximize PV of cash flows

\[ \frac{\partial w_0}{\partial d_0} = 1 - \frac{g'}{1 + k} = 0 \Rightarrow -g' = -(1 + k) \]

\[ MRT_{c_1} = \text{Slope of capital market line} \]
Now consider the consumption problem when both investment opp. and capital markets are available.

\[
\begin{align*}
\max_{c_0, c_1, d_1, d_0} & \quad U(c_0, c_1) \quad \text{s.t.} \quad c_1 = w_1 - (1 + k)c_0 \\
& \quad w_1 = d_1 + (1 + k)d_0 \\
& \quad d_1 = g(w_0^y - d_0) \\
& \quad w_0^y = y_0 + \frac{y_1}{(1 + k)}
\end{align*}
\]

or

\[
\begin{align*}
\max_{c_0, d_0} & \quad U\left(c_0, g(w_0^y - d_0) + (1 + k)d_0 - (1 + k)C_0\right)
\end{align*}
\]

FOC

\[
(1) \quad \frac{\partial U}{\partial c_0} = U_0 - U_1(1 + k) = 0
\]

\[
(2) \quad \frac{\partial U}{\partial d_0} = U_1[-g' + (1 + k)] = 0
\]
Start with Equation (2)

\[- g' = -(1 + K)\]

\[MRT_{I_0}^I = \text{slope capital market line}\]

So invest to max PV of wealth from productive investments. Doesn’t depend on time preferences.

Equation (1) says adjust consumption so that

\[- \frac{U_0}{U_1} = -(1 + K)\]

\[MRS_{C_0}^{C_1} = \text{Slope at capital market line}\]

Same solution as before with capital market
Fisher Separation Theorem: Given perfect and complete capital markets, the production decision is determined solely be maximizing the present value of wealth without regard to individual preferences for consumption.
Major Implication — Businesses can be managed without concern for personal preferences of owners. Every investor wants some production decision regardless of preferences.
- Equation (1) says adjust consumption so that

\[- \frac{U_0}{U_1} = - (1 + K)\]

\[MRS_{c_0}^{c_1} = \text{Slope at capital market line}\]

Same solution as before with capital market.

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Major Implication — Businesses can be managed without concern for personal preferences of owners. Every investor wants the same production decision regardless of preferences.
Implications for NPV analysis

PV of future cash flow $d_1 = w_0 - d_0$

Investment $= w - d_0$

NPV = PVCF - Investment

$= (w_0 - d_0) - (w_0^y - d_0) = w_0 - w_0^y$

The investment rule $MRT_{d_0}^{d_1} = -(1 + K)$ is equivalent to maximizing NPV.

Alternatively, we can think of the problem in terms of our rate of return rule.
Invest until the rate of return (MRT) from the investment is equal to the return available on a alternative investment in the capital market.

- **Crucial Assumptions**

  - Borrowing and lending opportunities are identical

  1. No barriers preventing access to capital markets
  2. No individual can impact security prices
  3. Costless access to market (no frictions)
  4. No taxes
  5. Information about price and quality of each security is free and available to all participants.

    \[ \implies \text{markets are perfect and competitive.} \]

Approximation at best. How good?

- Over 50 million shareholders in US above
- Even largest investors control only fragment of traded securities
- Cost at trading in securities is small in absolute and relative sense
- On assets traded an exchanges, there are regulations put into place to ensure information is available to public or not al all.
Imperfect Capital Markets

Suppose there are transaction costs in the capital markets so that the lending rate is lower than the borrowing rate.

Now shareholder preference can impact investment decisions as investors have different discount rates.
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