PRESENT VALUE ANALYSIS

Time value of money — “equal” dollar amounts have different values at different points in time.

Present value analysis — tool to convert CFs at different points in time to comparable values at a given point in time.

Time Line

\[
\begin{array}{cccccccc}
CF_1 & CF_{t+1} & CF_{t+2} & CF_{t+3} & CF_{t+4} & CF_{t+5} & CF_{t+6} & \text{(cash flows)} \\
t & t+1 & t+2 & t+3 & t+4 & t+5 & t+6 & \text{(time periods)} \\
\hline
$100 & $100 & $100 & \hline
0 & 1 & 2 & 3 \\
\end{array}
\]
**FUTURE VALUE OF A LUMP SUM**

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lump sum</td>
<td>Single CF at a specific point in time.</td>
</tr>
<tr>
<td>Present value (PV)</td>
<td>Lump sum value at a point in time of CFs to be received in the future.</td>
</tr>
<tr>
<td>Future value (FV)</td>
<td>Lump sum value at point in time of CFs received in the past.</td>
</tr>
<tr>
<td>Rate of return (k)</td>
<td>The return you can earn on the best available investment option. (We’ll talk about risk adjustments later.)</td>
</tr>
<tr>
<td>Number of periods (n)</td>
<td>Number of time periods of a given length that you earn rate k.</td>
</tr>
</tbody>
</table>
Future value of a lump sum —

If you invest a single lump sum amount at a particular point in time, how much will you have at a specific point in time in the future?

Suppose you invest $1250 in the bank and earn 8% for one year. How much will you have a the end of the year?

\[
\begin{align*}
PV & = $1250 \\
k & = .08 \text{ (8\%)} \\
n & = 1 \\
FV & = ? \\
FV_1 & = 1250 + 1250 \times 0.08 \\
& = 1250 \times (1 + .08) \\
& = 1350
\end{align*}
\]

Now suppose you leave the money in the bank for 5 years. How much would you have at the end of 5 years?

\[
\begin{align*}
PV & = $1250 \\
k & = 0.08 \\
n & = 5 \\
FV_5 & = ?
\end{align*}
\]
Time Line

<table>
<thead>
<tr>
<th></th>
<th>FV₁</th>
<th>FV₂</th>
<th>FV₃</th>
<th>FV₄</th>
<th>FV₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1250</td>
<td>1350</td>
<td>1458</td>
<td>1250 (1 + .08)²</td>
<td>1250 (1 + .08)⁵</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

\[
FV₁ = PV (1 + k) = 1250 (1 + .08) = $1350
\]

\[
FV₂ = FV₁ (1 + k) = 1350 (1 + .08) = $1458
\]

\[
= PV (1 + k)(1 + k)
\]
\[
= PV (1 + k)²
\]
\[
= 1250 (1 + .08)²
\]
\[
= $1458
\]

\[
FV₅ = FV₄ (1 + k)
\]
\[
= FV₃ (1 + k)(1 + k)
\]
\[
= FV₂ (1 + k)(1 + k)(1 + k)
\]
\[
= FV₁ (1 + k)(1 + k)(1 + k)(1 + k)
\]
\[
= PV (1 + k)(1 + k)(1 + k)(1 + k)(1 + k)
\]
\[
= PV (1 + k)⁵
\]
\[
= 1250 (1 + .08)⁵
\]
\[
= $1836.66
\]
General Result:

\[ FV_n = PV \times (1 + k)^n \]

Future value interest factor

\[ FVIF_{k,n} = (1 + k)^n \]

\[ FV_5 = PV \times (1 + k)^n \]
\[ = PV \times (FVIF_{k,n}) \]
\[ = 1250 \times (FVIF_{8,5}) \]
\[ = 1250 \times (1.4693) \]
\[ = $1836.66 \]

Relationship with k and n

Increase in k causes increase in FV

Increase in n causes increase in FV
PRESENT VALUE OF A LUMP SUM

How much must you invest at a particular point in time to obtain a given amount of money on a specific date in the future?

General Formula (Inverse of FV):

\[ PV = FV_n \left( \frac{1}{(1 + k)^n} \right) \]

Suppose you will deposit some money in an account that pays 8% interest per year. How much do you need to deposit to have $1350 in the account after 1 year?

- \( FV_1 = $1350 \)
- \( n = 1 \)
- \( k = 0.08 \)
- \( PV = ? \)

\[ PV = 1350 \left( \frac{1}{(1 + .08)^1} \right) \]

\[ = $1250 \]
Suppose, instead, that you need to have $1,836.66 in five years in order to take a trip you’ve been planning. How much should you put in your bank account which pays 8% per year.

\[
\begin{align*}
FV_5 &= $1836.66 \\
n &= 5 \\
k &= 0.08 \\
PV &= ?
\end{align*}
\]

\[
PV = FV_5 \left(\frac{1}{(1 + .08)^5}\right) = 1836.66 \cdot .6806 = $1250
\]

Present value interest factor for a lump sum

\[
PVIF_{k,n} = \frac{1}{(1 + k)^n}
\]

\[
PV = FV_n \cdot (PVIF_{k,n}) = 1836.66 \cdot (PVIF_{8.5}) = 1836.66 \cdot .6806 = $1250
\]

Relationship with \(k\) and \(n\)

Increase in \(k\) causes decrease in \(PV\)

Increase in \(n\) causes decrease in \(PV\)
COMPOUND VS. SIMPLE INTEREST

Simple Interest — interest is only paid on principal amount each period

Example — Deposit $100 for 2 years at 10% simple interest. How much do you have at the end of 2 years?

<table>
<thead>
<tr>
<th>Principal</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest Year 1</td>
<td>100(.10) = $10</td>
</tr>
<tr>
<td>Interest Year 2</td>
<td>100(.10) = 10</td>
</tr>
<tr>
<td>Year 2 Balance</td>
<td>$120</td>
</tr>
</tbody>
</table>

In general,

\[ FV_n = PV(1 + nk) \]
\[ = $100 (1 + 2(.10)) \]
\[ = $120 \]
Compound Interest — Interest is paid on principal balance plus any previously earned interest.

Example — Deposit $100 for 2 years at 10% compounded interest. How much do you have at the end of year 2?

<table>
<thead>
<tr>
<th>Description</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Principal</td>
<td>$100</td>
</tr>
<tr>
<td>Interest Year 1</td>
<td>$10</td>
</tr>
<tr>
<td>Interest Year 2</td>
<td>$11</td>
</tr>
<tr>
<td>Balance Year 2</td>
<td>$121</td>
</tr>
</tbody>
</table>

In general,

\[ FV = PV(1 + k)^n \]
\[ = $100(1 + .10)^2 \]
\[ = $121 \]

Consider \( n = 2 \), \( k = .10 \), \( PV = $1 \)

Simple Interest \[ PV(1 + nk) = $1(1 + 2(.10)) = $1.20 \]

Compound Interest \[ PV(1 + k)^n = $1(1 + .10)^2 \]
\[ = $1(1 + 2(.10) + (.10)(.10)) \]
\[ = $1 + $.20 + $0.01 \]
\[ = $1.21 \]
POWER OF COMPOUNDING

■ PV = $1000  n = 30 years  k = 10%

Simple Interest

\[ FV_{30} = 1000(1 + 30(.10)) = 4000 \]

Compound Interest

\[ FV_{30} = 1000(1 + .10)^{30} = 17449.40 \]

■ FV = $100,000  n = 30 years  k = 10%

Simple Interest

\[ PV = \frac{100,000}{(1 + 30(.10))} = 25,000 \]

Compound Interest

\[ PV = \frac{100,000}{(1 + .10)^{30}} = 5,730.86 \]
PRESENT VALUE ANALYSIS

Lump Sums

- \( PV = FV \left( \frac{1}{(1 + k)^n} \right) \)

\[ PVIF_{k,n} = \frac{1}{(1 + k)^n} \]

- \( FV = PV(1 + k)^n \)

\[ FVIF_{k,n} (1 + k)^n \]

- Most problems can be handled with these formulas
RECOGNIZING PATTERNS IN CF STREAMS

Most problems can be solved using lump sum formulas. However calculations may become cumbersome

Calculations can be simplified when CF streams follow certain patterns

Perpetuity — constant CF stream that continues forever

\[
PV_0 = \frac{CF_1}{(1 + k)} + \frac{CF_2}{(1 + k)^2} + \frac{CF_3}{(1 + k^3)} + \ldots
\]

\[
= \frac{C}{(1 + k)} + \frac{C}{(1 + k)^2} + \frac{C}{(1 + k)^3} + \ldots
\]

PV of the perpetuity is the lump sum amount today that allows you to generate the same CF stream that is provided by the perpetuity.
Question: How much do we need today to recreate the same CF stream as the perpetuity generates?

Note: CF stream continues forever so CFs must be generated from interest without touching principal

So, \( C = (PV_t)k \quad \Rightarrow \quad PV_t = \frac{C}{k} \)

Formally,

\[
PV_t = \frac{C}{(1 + k)} + \frac{C}{(1 + k)^2} + \frac{C}{(1 + k)^3} + \ldots
\]

Let \( a = \frac{C}{(1 + k)} \) and \( x = \frac{1}{(1 + k)} \)

\[
PV_t = a (1 + x + x^2 + \ldots) \quad (1)
\]

multiply (1) by \( x \)

\[
x \ PV_t = ax + ax^2 + \ldots \quad (2)
\]
Subtract (2) from (1)

\[ PV_t - x PV_t = a \]

\[ PV_t (1 - x) = a \]

\[
 PV_t = \frac{a}{(1 - x)} = \frac{C}{(1 + k)} \frac{1}{1 - \frac{1}{(1 + k)}} = \frac{C}{k}
\]

\[ PV_t = \frac{C}{k} \]

Example — Suppose you receive $1,000 each year forever starting 1 year from today. If you can earn 10% elsewhere, what is the value of the perpetuity today?

\[ C = $1000 \quad k = 10\% \]

\[ PV_0 = \frac{1000}{.10} = $10,000 \]

Suppose k falls to 7%. What happens to the value of the perpetuity?

\[ PV_0 = \frac{1000}{.07} = $14,285.71 \]
Growing Perpetuity — CF stream that grows at a constant rate forever

\[ PV_t = \frac{CF_{t+1}}{(1 + k)} + \frac{CF_{t+1}(1 + g)}{(1 + k)^2} + \frac{CF_{t+1}(1 + g)^2}{(1 + k)^3} + \ldots \]

\[ CF_{t+1} = CF \text{ at } t + 1 \]
\[ g = \text{ constant growth rate of CF stream } (\%) \]
\[ k = \text{ rate of return elsewhere} \]

Simplifies to

\[ PV_t = \frac{CF_{t+1}}{k - g} \]

Formally

\[ PV_t = \frac{CF_{t+1}}{(1 + k)} + \frac{CF_{t+1}(1 + g)}{(1 + k)^2} + \frac{CF_{t+1}(1 + g)^2}{(1 + k)^3} + \ldots \]

Let \( a = \frac{CF_{t+1}}{(1 + k)} \) and \( x = \frac{1 + g}{1 + k} \)
\[ PV_t = a(1 + x + x^2 + \ldots) \]  \hspace{1cm} (1)

\[ x \ PV_t = ax + ax^2 + ax^3 + \ldots \]  \hspace{1cm} (2)

Subtract (2) from (1)

\[ PV_t(1 - x) = a \]

\[ PV_t = \frac{a}{(1 - x)} \]

\[ PV_t = \frac{\frac{\text{CF}_{t+1}}{(1 + k)}}{1 - \frac{(1 + g)}{(1 + k)}} = \frac{\text{CF}_{t+1}}{(1 + k) - (1 + g)} = \frac{\text{CF}_{t+1}}{k - g} \]

Example — Suppose CF from your dairy operation next year is expected to be $60,000 and CFs are expected to grow at a constant rate of 4% each year indefinitely. If you can earn 12% elsewhere, what is the value of the dairy operation in today's dollars?

\[ \text{CF}_{t+1} = 60,000 \quad g = 4\% \quad k = 12\% \]

\[ PV_0 = \frac{60,000}{.12 - .04} = $750,000 \]
Important Points

- Numerator is CF in the next period from the point at which you wish to find the PV of the future CF stream.

To find the $PV_t$ you need $CF_{t+1}$ in the numerator.

Example — Find the $PV_1$ of dairy operation in the last example.

$$PV_1 = \frac{CF_2}{k - g} = \frac{60,000(1 + 0.04)}{0.12 - 0.04} = \frac{62,400}{0.08}$$

Constant growth formula always gives PV one period before CF in the numerator.

- Interest rate must be greater than growth rate $(k > g)$

As $k \rightarrow g$  $PV \rightarrow \infty$

If $k = g$  $PV$ is undefined.

- CFs assumed to occur at regular discrete intervals.
Suppose the ATCF from your dairy operation is expected to be $60,000 next year and you expect to CFs to grow at a 4% rate each year indefinitely. If you can earn 12% on any money you invest, what would you sell the operation for today?

\[
PV = \frac{\$60,000}{.12 - .04} = \$750,000
\]

What would you sell the operation for in 5 years?

\[
\begin{align*}
CF_1 &= \$60,000 \\
CF_2 &= \$62,400 \\
CF_3 &= 62,400 (1 + .04) = 64,896 \\
CF_4 &= 64,876 (1.04) = 67,491.84 \\
CF_5 &= 67,491.84 (1.04) = 70,191.51 \\
CF_6 &= 70,191.51 (1.04) = 72,999.17
\end{align*}
\]
Or 

\[ CF_6 = CF_1 (1 + g)^5 \]

\[ = 60,000 (1 + .04)^5 = 72,999.17 \]

Or 

\[ CF_6 = CF_1 (FVIF_{4,5}) = 72,999.17 \]

\[
PV_5 = \frac{CF_6}{k - g} = \frac{72,999.17}{.12 - .04} = \$912,489.63
\]

Why can’t we just take 

\[ $750,000 (FVIF_{12,5}) = \$1,321,756.26? \]
Annuity — stream of fixed CFs of the same size that occur at equal increment over a finite time horizon

Present Value of Annuity — how much do you need today to generate a stream identical to the annuity CF

Present value of a perpetuity starting today

\[ P_0 = \frac{C}{k} \]

Present value of a perpetuity starting at \( n \)

\[ P_0 = \frac{C}{k} \left( \frac{1}{(1 + k)^n} \right) \]

The value of the annuity will be the value of the perpetuity today, less the present value of the CFs that occur after the last annuity payment.
\[ PV_t = \frac{C}{k} - \frac{C}{k} \left[ \frac{1}{(1 + k)^{t+n}} \right] \]

\[ = C \left[ 1 - \frac{1}{(1 + k)^{t+n}} \right] / k \]

\[ PVIFA_{k,n} = \left[ 1 - \frac{1}{(1 + k)^n} \right] / k \]

Suppose \( C = 1000 \) \( n = 5 \) \( k = 12\% \)

\[ PV_0 = 1000(PVIFA_{12,5}) \]
\[ = 1000(3.6048) \]
\[ = 3604.80 \]

We can check the answer by using the lump sum formula on each annuity CF.
Present Value of an Annuity — Amount of money you need to deposit to generate given annuity stream of CFs.

Suppose you earn 12% on money you invest. How much do you need to invest today to be able to withdraw $1000 at end of each year for the next 5 years?

<table>
<thead>
<tr>
<th>Time</th>
<th>CF</th>
<th>PVIF</th>
<th>Present Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1000</td>
<td>$892.90</td>
<td>$892.90</td>
</tr>
<tr>
<td>2</td>
<td>1000</td>
<td>.7971</td>
<td>797.10</td>
</tr>
<tr>
<td>3</td>
<td>1000</td>
<td>.7118</td>
<td>711.80</td>
</tr>
<tr>
<td>4</td>
<td>1000</td>
<td>.6355</td>
<td>635.50</td>
</tr>
<tr>
<td>5</td>
<td>1000</td>
<td>567.40</td>
<td>567.40</td>
</tr>
</tbody>
</table>

PVA = $3,604.70
Present Value Interest Factor of Annuity

$$PVIFA_{k,n} = \frac{1}{k} \left( \frac{1}{(1 + k)^n} \right)$$

PVA = PMT (PVIFA _{k,n})
= $1000 (PVIFA_{12,5})
= $1000 (3.6048)
= $3,604.80

- PVIFA gives PV of annuity CF stream **one period before first PMT**
- n corresponds to the number of annuity payments
Suppose the PMTs were received at the beginning of each year?

\[
PVA = $1000 \times (PVIFA_{12,5}) \\
    = $3604.80
\]

\[
PV = $3604.80 \times (FVIF_{12,1}) \\
    = $4037.30
\]
Future Value of Annuity

How much will the stream of CFs be worth at the point of the last CF

<table>
<thead>
<tr>
<th>Year</th>
<th>C</th>
<th>FVIF&lt;sub&gt;K,n&lt;/sub&gt;</th>
<th>FV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1000</td>
<td>FVIF&lt;sub&gt;12,4&lt;/sub&gt; = 1.5375</td>
<td>1573.50</td>
</tr>
<tr>
<td>2</td>
<td>$1000</td>
<td>FVIF&lt;sub&gt;12,3&lt;/sub&gt; = 1.4049</td>
<td>1404.90</td>
</tr>
<tr>
<td>3</td>
<td>$1000</td>
<td>FVIF&lt;sub&gt;12,2&lt;/sub&gt; = 1.2544</td>
<td>1254.44</td>
</tr>
<tr>
<td>4</td>
<td>$1000</td>
<td>FVIF&lt;sub&gt;12,1&lt;/sub&gt; = 1.12</td>
<td>1120.00</td>
</tr>
<tr>
<td>5</td>
<td>$1000</td>
<td>FVIF&lt;sub&gt;12,0&lt;/sub&gt; = 1.00</td>
<td>1000.00</td>
</tr>
</tbody>
</table>

6352.80
Remember PV of an annuity is

\[ PV = C \left[ 1 - \frac{1}{(1 + k)^n} \right] / k \]

This is a lump sum value of the annuity one period before the first CF. We can find the value of annuity at the point of the last CF by moving the PV of the annuity forward as a lump sum.

\[ FV_n = PV (1 + k)^n \]

\[ = C \left[ \left[ 1 - \frac{1}{(1 + k)^n} \right] / k \right] (1 + k)^n \]

\[ = C[(1 + k)^n - 1] / k \]
Thus $FV_{AK,n} = \frac{[(1 + k)^n - 1]}{k}$

$FV = $1000 \left[ (1 + .12)^5 - 1 \right] / .12$

= $6352.84$

or

$FV = $1000 \left( FVIFA_{12,5} \right)$

= 1000 (6.3528) \quad \text{table value}$

= $6352.80$

- $FVIFA_{K,n}$ finds FV at point of last cash flow.

- $n$ corresponds to number of payments
Future Value of an Annuity Due

Suppose instead you received the PMTs as an annuity due.

How much would you have after 5 years.

\[
FVA_5 = 1000 \times (FVIFA_{12,5}) = 100 \times (6.3528)
\]

\[
= 6352.80 \times 1.12
\]

\[
= 7115.14 \text{ (Amount at end of year 5)}
\]
Example — Infrequent Annuities

Suppose you receive a $500 annuity every two years for the next 10 years. What will be the future value of the annuity at the point of the last payment if you can earn a return of 8% each year?

The payments occur every 2 years and so to use our formula we need the rate of return for each 2 year period.

2 year return \[= (1 + k)(1 + k) - 1\]
\[= (1 + .08)(1 + .08) - 1 = .1664\]

\[
\begin{array}{c}
\text{FV} = 500((1 + .1664)^5 - 1)/.1664 \\
= 3482.35
\end{array}
\]

Check:

\[\text{FV} = 500(\text{FVIF}_{8,8}) + 500(\text{FVIF}_{8,6}) + 500(\text{FVIF}_{8,4}) + 500(\text{PVIF}_{8,2}) + 500(\text{FVIF}_{8,0})\]
\[= 3482.35\]
Growing Annuity

Finite stream of CFs that grow at a constant rate

\[
PV = C \left[ \frac{1}{k-g} - \frac{1}{k-g} \left( \frac{1+g}{1+k} \right)^n \right]
\]

\[
= C \left[ 1 - \left( \frac{1+g}{1+k} \right)^n \right] / (k - g)
\]

Note: if \( g = 0 \) the formula reduces to the PV of an annuity

Proof:

Analogous to the derivation of the annuity formula, we can find the PV of the growing annuity by subtracting the PV of CFs after period \( n \) from the value of a growing perpetuity.
PV of a growing perpetuity

\[
PV = \frac{CF_1}{k - g}
\]  

(1)

PV of CFs after period n

\[
PV = \frac{CF_1(1 + g)^n}{k - g} \left( \frac{1}{1 + k^n} \right)
\]  

(2)

Subtract (2) from (1) to get the PV of the growing annuity

\[
PV = \frac{CF_1}{k - g} - \frac{CF_1(1 + g)^n}{k - g} \left( \frac{1 + g}{1 + k} \right)
\]

\[
= CF_1 \left[ 1 - \left( \frac{1 + g}{1 + k} \right)^n \right] / k - g
\]

Example — Suppose you take a job that pays you $40,000/year after taxes. You also expect the after-tax salary to increase by 4% each year for the next 30 years. What is the value of your job if you can earn 12% elsewhere?

\[
PV = $40,000 \left[ \frac{1}{(.12 - .04)} - \frac{1}{(.12 - .04)} \left( \frac{1 + .04}{1.12} \right)^{30} \right] = $445,871.06
\]
Formula only works for $k \neq g$

If $k = g$

$$PV = \frac{CF_{t+1}(n)}{(1 + k)}$$

Example — Suppose you need to make tuition payments each of the next 4 years. The first payment will be $15,000 next year and the payments will increase 8% each year. If you can earn 8% elsewhere, what is the PV of the tuition payments today? (How much needs to be deposited today to make the payments?)

$$PV = \frac{15,000(4)}{(1 + .08)} = $55,555.56$$
Uneven Streams of CFs

Treat each CF as a lump sum or look for embedded patterns of CFs that we can simplify

Example —

\[
PV = 1000(PVIF_{10,1}) + 500(PVIF_{10,2}) + 500(PVIF_{10,3}) + 500(PVIF_{10,4}) + 0(PVIF_{10,5}) + 750(PVIF_{10,6})
\]

\[= 2462.83\]

or

\[
PV = 1000(PVIF_{10,1}) + 500(PVIFA_{10,3})(PVIF_{10,1}) + 750(PVIF_{10,6})
\]

\[= 2462.83\]
Finding Annuity CFs

We can find annuity streams given the number of payments, the interest rate, and PV (or FV) of the annuity.

\[ PV = C(PVIFA_{K,n}) \Rightarrow C = \frac{PV}{PVIFA_{K,n}} \]

\[ FV = C(FVIFA_{K,n}) \Rightarrow C = \frac{FV}{FVIFA_{K,n}} \]

Example — Suppose you borrow $10,000 today and agree to repay the loan in 5 equal sized annual payments beginning one year from today. If the interest rate is 10%, what is the size of each payment?

\[ PV = 10,000, \quad n = 5, \quad k = 10\% \]

\[ CF = \frac{10,000}{\left(1 - \frac{1}{(1 + .10)^5}\right)} \times .10 = \frac{10,000}{3.7908} \times .9091 = \$2637.97 \]

How large is each payment if you make the first payment at the end of year 2?

\[ $10,000 = X(PVIFA_{10,5})(PVIF_{10,1}) \]

\[ X = \frac{10,000}{3.4462} = \$2901.76 \]
EQUATING ANNUITIES

Suppose you will need $50,000 each year for tuition when your daughter starts school 15 years from today. If you can earn 9% on money you invest, how much will you need to invest each year (beginning today) in order have enough to pay for her education? You plan to stop making payments one year before she starts school and tuition payments are due at the beginning of each school year.

Amount needed at year 14 to make tuition payments

\[ PV = \$50,000 \times (PVIFA_{9,4}) = \$161,985 \]

Payments needed to have $161,985 in the bank at \( t = 14 \)

\[ \$161,985 = C \times (FVIFA_{9,15}) \]

\[ C = \frac{161,985}{29.3609} = \$5,517.03 \]
Suppose you start making payments one year from now and make your last payment on the day your daughter starts school. How large will each payment need to be?

Amount needed at $t = 15$

$$PV = 50,000(PVIFA_{9,4})(FVIF_{9,1})$$

$$= $176,563.65$$

Payments needed to achieve $176,563.65 at $t = 15$

$$29.309$$

$$C = $6,013.56$$
Finding Interest Rates

Lump Sums

Suppose you borrow $5000 today and pay back $7013 at the end of 5 years. What interest rate are you paying on the loan?

PV = $5000 \quad FV = $7013 \quad n = 5 \quad k = ?

\[ 5000 = 7013 \left( PVIF_{K,5} \right) \]

\[ PVIF_{K,5} = 0.713 \]

from table \( k = 7\% \)

Using formula directly

\[ 5000 = 7013 \left( \frac{1}{(1 + k)^5} \right) \]

\[ \ln(5000) = \ln(7013) - 5 \ln(1 + k) \]

\[ \ln(1 + k) = \frac{\ln(7013) - \ln(5000)}{5} = .067666 \]

\[ (1 + k) = e^{.067666} = 1.07 \]

\( k = .07 \) or 7\%
Annuities

Suppose you borrow $5000 and repay $1,285.45 at the end of each year for the next 5 years. Find k.

$5000 = $1285.45 (PVIFA_{k,5})

PVIFA_{k,5} = \frac{5000}{1285.45} = 3.8927

K = 9%

Linear Interpolation

Suppose you loan $5000 to a friend and require equal installments of $1,150 at the end of each year for 5 years. What rate of return did you earn?

k = 4%  \quad PVIFA_{4,5} = 4.4518

PVIFA_{4,5} = 4.3478 \cdot 1.04 \cdot 1.223

k = 5%  \quad PVIFA_{5,5} = 4.3295

k = 4% + \frac{4.4518 - 4.3478}{4.4518 - 4.3295} (1\%) = 4.85%

.104

.1223
Finding n

How many years does it take your money to double if you can earn 10% on the funds you invest?

\[
PV = \$1000 \quad FV = 2000 \quad k = 10\% \quad n = ?
\]

\[
2000 = 1000(FVIF_{10,n})
\]

\[
FVIF_{10,n} = 2
\]

\[
n = 7 \quad FVIF = 1.9487
\]

\[
n = 8 \quad FVIF = 2.1438
\]

\[
n = 7 + \frac{2 - 1.9487}{2.1438 - 1.9487} \quad (1 \text{ year}) = 7.26 \text{ years}
\]

\[
\begin{align*}
\text{Rule of 72} \\
\text{Number of years to double} & \approx \frac{72}{(k)100} \\
& \approx \frac{72}{10}
\end{align*}
\]
Solving for $n$ exactly

\[ 2000 = 1000 \ (1 + .10)^n \]

\[
\ln \left( \frac{2000}{1000} \right) = n \ln(1 + .10)
\]

\[
n = \frac{0.693147}{0.09531} = 7.27254 \text{ years}
\]
Compounding More Frequently

Up until now → annual compounding

\( k_a = \) stated annual interest rate (nominal rate)

Compounding may occur at intervals that are less than 1 year in length

Example — Semiannual compounding mean 1/2 of the annual rate is paid after six months. During the second six months interest is compounded so you earn interest on the interest paid the first six months.

Invest $100 at \( k_a = 10\% \). Compounding is semiannual. How much will you have after 1 year?

After six months \( 100(1 + .05) = $105 \)
After 1 year \( 100(1 + .05)(1 + .05) = $110.25 \)

\[ \begin{array}{c}
105 \\
\text{Interest on 1st six months interest}
\end{array} \]
Previous formulas are still valid if you adjust \( k \) and \( n \)

\[ k = \frac{k_a}{M} \quad \text{(interest rate per compounding period)} \]

\[ n = n_a(M) \quad \text{(number of compounding periods)} \]

\( k_a \) = interest rate per period  
\( n_a \) = number of periods (usually years)  
\( M \) = number of compoundings per period

Example — lump sum formula for future value

\[ FV = PV\left(1 + \frac{k_a}{M}\right)^{(M)(n_a)} \]

Suppose you deposit $100 for 1 year at \( k_a = 8\% \)

<table>
<thead>
<tr>
<th>Compounding Frequency</th>
<th>PV</th>
<th>( \times )</th>
<th>( \text{FVIF} )</th>
<th>=</th>
<th>FV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual</td>
<td>100</td>
<td>(1+ ( \frac{0.08}{1} )) ( ^{1\times1}=1.08 )</td>
<td>108</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Semiannual</td>
<td>100</td>
<td>(1+ ( \frac{0.08}{2} )) ( ^{2\times1}=1.0816 )</td>
<td>108.16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quarterly</td>
<td>100</td>
<td>(1+ ( \frac{0.08}{4} )) ( ^{4\times1}=1.0824 )</td>
<td>108.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Monthly</td>
<td>100</td>
<td>(1+ ( \frac{0.08}{12} )) ( ^{12\times1}=1.083 )</td>
<td>108.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Daily</td>
<td>100</td>
<td>(1+ ( \frac{0.08}{365} )) ( ^{365\times1}=1.0833 )</td>
<td>108.33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Continuous</td>
<td>100</td>
<td>( e^{(0.08)(1)} = 1.0833 )</td>
<td>108.33</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using Formulas
Suppose you deposit $1000 in an account paying 8% compounded semiannually. How much would you have after 5 years in the account?

\[ k = \frac{k_a}{M} = \frac{0.8}{2} = 0.04 \]

\[ n = (n_a)(M) = (5)(2) = 10 \]

\[ FV_5 = 1000(FVIF_{4,10}) = 1480.20 \]

Suppose you purchase a new truck and your payments will be $350/month for 5 years. Your parent's want to give you a graduation present and offer to deposit enough in an account paying 12% compounded monthly so that you can make your payments. How much needs to be deposited today if your first payment is next month?

\[ k = \frac{12}{12} = 1 \]

\[ n = (12)(5) = 60 \]

\[ PV = 350(PVIFA_{1,50}) = 350 \times 144.955 \]

\[ = 51,734.26 \]
CONTINUOUS COMPOUNDING

- When compounding occurs M times per period then our FV interest factor for n periods is

\[
\left( 1 + \frac{k_a}{M} \right)^{(M)(n_a)}
\]

Example — 10% semiannually for 2 years (M=2, n=2, k_a=.10

\[
FVIF = \left( 1 + \frac{.10}{2} \right)^{(2)(2)} = 1.2155
\]

- The number e

By definition \( e = \lim_{M \to \infty} \left( 1 + \frac{1}{M} \right)^M \approx 2.71828 \)

FVIF if \( k = 100\% \) and continuous compounding

Invest $1 at 100% compound continuously gives you $1

\( e = \$1(2.71828) = \$2.71828 \) after 1 year

Two years: \( FV = \$1 \ (e)^2 = \$7.38906 \)

n years: \( FV = 1(e)^n \)
Continuous Compounding at $k_a$

\[ FV_n = \lim_{M \to \infty} \left( 1 + \frac{k_a}{M} \right)^{(M)(n)} \]

\[ = PV \lim_{M \to \infty} \left( 1 + \frac{k_a}{M} \right)^{\left( \frac{M}{k_a} \right)(k_a n)} \]

Let \( w = \frac{M}{k_a} \)

\[ FV_n = PV \lim_{M \to \infty} \left[ \left( 1 + \frac{1}{w} \right)^w \right]^{k_a n} \]

As \( M \to \infty, w \to \infty \Rightarrow \left( 1 + \frac{1}{w} \right)^w \to e \)

\[ FV_n = PV \cdot e^{k_a n} \]

Suppose you invest $1000 at an interest rate of 10% compounded continuously. How much do you have after 10 years?

\[ FV_{10} = 1000e^{\cdot10(10)} = 2718.28 \]

Note:

\[ PV = FV \left( \frac{1}{e^{k_n}} \right) \]
Continuous CFs — formulas still work
— need to use continuous return

Constant Payment Annuities

Assume asset pays constant amount per unit of time

\[ C_t = C \]

stream of payments starts now and continues for n time periods

Continuously compounded discount rate = \( k \)

\[
PV = \int_{0}^{n} C_t e^{-kt} dt = \int_{0}^{n} C e^{-kt} dt = C \int_{0}^{n} e^{-kt} dt
\]

Remember "Fundamental Theorem of Calculus"

\[
\int_{a}^{b} f(x) dx = F(b) - F(a)
\]

Find the indefinite integral of the integral, substitute the upper and lower bound, and find the difference.
\[ \text{PV} = C \int_{0}^{n} e^{-kt} \, dt = C \left[ -\frac{1}{k} e^{-kt} \right]^n_0 \\
= C \left[ -\frac{e^{-kn}}{k} - \frac{e^{-k0}}{k} \right] \\
= C \left[ 1 - \frac{e^{-kn}}{k} \right] \]

Analogous to discrete time annuity factor

\[ \text{PV} = C \left[ 1 - (1 + k)^{-n} \right] \text{ where } k = \text{ discrete rate/period} \]

Suppose \( n \to \infty \) (Perpetuity)

\[ \text{PV} = \frac{C}{k} \text{ where } k \text{ is continuously compounded return} \]
Example 1 — Suppose you are going to receive a perpetuity of $1000 each year paid continuously during the year. If you can earn 10% compounded continuously, what would you sell the perpetuity for today?

\[ PV = \frac{C}{k} = \frac{1000}{.10} = 10,000 \]

What is the value of the perpetuity if you receive the $1000 payments at the end of each year, but you can still earn 10% compounded continuously?

Interest earned during the year

\[ e^{(10)\left(1\right)} = 1.10517 - 1 = .10517 \text{ or } 10.517\% \]

\[ PV = \frac{1000}{.10517} = 9508.42 \]

Check: \( 9508.42 \times .10517 = 1000 \)
Example 2 — Suppose you want to deposit enough in an account today so that you will be able to withdraw $20,000 per year for 20 years. You want the withdrawals to be made in 4 equal installments each quarter. How much should you deposit today if the first payment is to be withdrawn 20 years and 1 quarter from today? Elsewhere you can get a return of 10% compounded quarterly.

\[
PV = \left( \frac{20,000}{4} \right) \left( \text{PVIFA}_{0.10, (20)(4)} \right) \left( \text{PVIF}_{0.10, (20)(4)} \right)
\]

\[
= 5000 \left( \frac{1}{0.025} - \frac{1}{0.025} \left( \frac{1}{(1 + 0.025)^{80}} \right) \right) \left( \frac{1}{(1 + 0.025)^{80}} \right)
\]

\[
= \$23,893.18
\]
Now suppose, instead, that you want the payments to be made continuously starting at the end of year 20. You can earn a return elsewhere of 10% compounded continuously.

What would you sell the stream of payments for today?

\[
PV = 20,000 \left( \frac{1}{.10} - \frac{1}{.10} \left( \frac{1}{e^{0.10(20)}} \right) \right) \left( \frac{1}{e^{0.10(20)}} \right)
\]

\[
= \$23,403.93
\]

- Effective Annual Percentage Rate (EAPR)

- annual return equivalent to the more frequently compounded return

\[
EAPR = \left( 1 + \frac{k_a}{M} \right)^M - 1
\]
Example 1 — Suppose we have

\[ k_a = 10\% \] compounded quarterly

\[ k_a = 9.8771\% \] compounded continuously

\[
\text{EAPR}_{10,\text{quarterly}} = \left(1 + \frac{.10}{4}\right)^4 - 1 = .103813 \text{ or } 10.38\%
\]

\[
\text{EAPR}_{9.871\%,\text{continuously}} = e^{.098771} - 1 = .103813 \text{ or } 10.38\%
\]

Example 2 — Credit card companies state APR = 18%

However, monthly rate is 1.5%

\[
\text{EAPR} = \left(1 + \frac{.18}{12}\right)^{12} - 1 = (1 + .015)^{12} - 1
\]

\[ = .1956 \text{ or } 19.56\% \]
Annuities with Growing Payments

\[ PV = \int_{0}^{n} c_t e^{-kt} \, dt \]

Let \( c_t = c e^{gt} \) where \( g \) is the continuous growth rate

\[ PV = C \int_{0}^{n} e^{-(k-g)t} \, dt \]

Using rules of integral calculus, the solution to the integral is

\[ PV = C \left[ \frac{-e^{-(k-g)t}}{(k - g)} \right]_{0}^{n} \]
\[ = C \left[ \frac{-e^{-(k-g)n}}{(k - g)} - \frac{-e^{0}}{(k - g)} \right] \]
\[ = C \left[ \frac{1 - e^{-(k-g)n}}{k - g} \right] \]

Suppose you have growing perpetuity

\[ \lim_{n \to \infty} PV = C \left[ \frac{1 - e^{-(k-g)n}}{k - g} \right] = \frac{C}{k - g} \]
OPTIMAL LIFE WITHOUT REPLACEMENT, DISPOSAL, OR SALVAGE VALUE

\[ \text{NPV} = \int_{0}^{n} [R(t)e^{-kt}] dt - I_0 \]

where

- \( R(t) \) = return or cash flow at \( t \)
- \( I_0 \) = initial investment
- \( k \) = continuously compounded opportunity cost
- \( n \) = life of asset

Rule for differentiating an integral

\[ y = \int_{a_1(x)}^{a_2(x)} f(t, x) dt \]

\[ \frac{dy}{dx} = \int_{a_1(x)}^{a_2(x)} \frac{\partial f}{\partial x} dt + f(a_2(x), x) \frac{da_2}{dx} - f(a_1(x), x) \frac{da_1}{dx} \]

\[ \frac{d\text{NPV}}{dN} = 0 + R(n)e^{-kn}(1) - R(0)e^{-k0}(0) \]

So

FOC: \[ \frac{d\text{NPV}}{dn} = R(n)e^{-m} = 0 \]

SOC: \[ \frac{d^2\text{NPV}}{dn^2} = rR(n) + \frac{\partial R}{\partial n} < 0 \]
Suppose \( R(t) = 100 - 10t \)
\( k = 10\% \)
\( I = 750 \)

\[
\text{NPV} = \int_{0}^{n} (100 - 10t)e^{-10t} \, dt - 750
\]

\[
\frac{d\text{NPV}}{dn} = (100 - 10n)e^{-10n} = 0
\]

\( n^* = 10 \)  
Since \( \frac{d^2\text{NPV}}{dn^2} < 0 \)

\[
\text{NPV} = \int_{0}^{10} (100 - 10t)e^{-10t} \, dt - 750 = 250
\]

\[ F(a) - F(b) - 750 \]

\[
\frac{-(100 - 10(10)e^{-10(10)}}{0.10} - \frac{-(100 - 0)e^{-10(0)}}{0.10} - 750
\]

\[ 1000 - 750 = 250 \]
PRICES, RETURNS, AND COMPOUNDING

- Financial Economics Often Works with Returns
  - scale-free summary of investment opportunity
  - more attractive than prices for theoretical and empirical reasons
    
    e.g. general equilibrium models often yield non-stationary prices, but stationary returns (more later)

Definitions and Conventions

Simple net return

\[ r_t = \frac{P_t}{P_{t-1}} - 1 \]  
(gross return = 1 + r_t)
Multi-year returns often annualized to make comparison

Annualized \([r_t(n)] = \left[ \prod_{j=0}^{n-1} (1 + r_{t-j}) \right]^{\frac{1}{n}} - 1\)

(geometric average)

Approximate Annualized \([r_t(k)] = \frac{1}{n} \sum_{j=0}^{n-1} r_{t-j}\)

(arithmetic average)

Continuous Compounding

\[ r_t^c = \ln(1 + r_t) = \ln\left( \frac{P_t}{P_{t-1}} \right) = \ln(P_t) - \ln(P_{t-1}) = \rho_t - \rho_{t-1} \]

Remember:

\[ P_t = P_{t-1} e^{r_t^c} \]

\[ \ln(P_t) = \ln(P_{t-1}) + \ln(e^{r_t^c}) \]

\[ \rho_t - \rho_{t-1} = r_t^c \]
Gross returns over most recent k periods (t - k to t) defined as

\[ 1 + r_t(k) = (1 + r_t)(1 + r_{t-1}) \ldots (1 + r_{t-k+1}) \]

\[ = \left( \frac{P_t}{P_{t-1}} \right) \left( \frac{P_{t-1}}{P_{t-2}} \right) \ldots \left( \frac{P_{t-k+1}}{P_{t-k}} \right) = \frac{P_t}{P_{t-k}} \]

net return = \( r_t(k) \)

Multi-period return called compound return

Returns are scale-free but not unit less

always defined w.r.t. some time period 10% return is not a complete description . . . need length of time period

Convention: quote annual rates
Consider multi-period return

\[ r_t^c(K) = \ln(1 + r_t(K)) = \ln(1 + r_t + \ln(1 + r_{t-1}) \ldots + (1 + r_{t-k+1}) = r_t^c + r_{t-1}^c + \ldots + r_{t-k+1}^c \]

Simplifies multiplication properties to additive properties

More importantly: easier to derive time series properties of additive process

Disadvantage with portfolio analysis

portfolio returns \( r_{pt} = \sum_{i=1}^{n} w_i r_i \neq \sum_{i=1}^{n} w_i r_i^c \)

problem - log of sum not the same as sum of log

\( r_{pt} = \sum_{i=1}^{n} w_i r_i^c \) (can be used as approx.)

In general

- use continuous return in temporal analysis
- use simple returns in cross-sectional work

Dividends