RISKS AND RATES OF RETURN

Returns

Current Income \((C_t)\) — CF received during period \(t\). usually at end of period

Capital gains \((P_t - P_{t-1})\) — Change in asset value during period \(t-1\) to \(t\)

Total Return = current income + capital gains

\[
TR_t = C_t + (P_t - P_{t-1})
\]

Rate of Return = Current Yield + Capital Gains Yield

\[
r_t = \frac{TR_t}{P_{t-1}} = \frac{C_t}{P_{t-1}} + \frac{(P_t - P_t)}{P_{t-1}}
\]
Example

Suppose you purchased 100 acres of land for $850/acre. You lease the land to a tenant for $75/acre and at the end of the year the land’s value has increased to $900/acre.

What is your return?

\[ TR_t = C_t + (P_t - P_{t-1}) = 75 + (900 - 850) = $125/acre \]

\[ r_t = \frac{C_t}{P_{t-1}} + \frac{(P_t - P_{t-1})}{P_{t-1}} = \frac{75}{850} + \frac{(900 - 850)}{850} = .08824 + 0.0588 \]

\[ = .147 \]

Important to include both current returns and capital gains in the return measures (even if the capital gains aren’t "realized")

— We haven’t worried about taxes
Measures of Location — describe "most likely" outcomes

Expected value (average, mean)

\[ E(\bar{x}) = \bar{x} = \sum_{i=1}^{n} p_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} p_i = 1 \]

\sim \quad \text{denotes randomness}

\[ P_i = \text{probability of random event } x_i \text{ occurring} \]

\[ n = \text{total number of possible events} \]

<table>
<thead>
<tr>
<th>State of Economy</th>
<th>Prob.</th>
<th>Inv. A</th>
<th>Inv. B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recession</td>
<td>.20</td>
<td>-20%</td>
<td>-40%</td>
</tr>
<tr>
<td>Normal</td>
<td>.60</td>
<td>20%</td>
<td>20%</td>
</tr>
<tr>
<td>Boom</td>
<td>.20</td>
<td>40%</td>
<td>60%</td>
</tr>
</tbody>
</table>

\[ E(\bar{r}_A) = (.2)(-.20) + (.60)(.2) + (.20)(.40) = .16 \]

\[ E(\bar{r}_B) = (.2)(-.40) + (.60)(.20) + (.20)(.20) + (.20)(.40) = .16 \]
Two useful properties of the expected value operator

\[ E(a + \bar{x}) = a + E(\bar{x}) \]

\[ a \text{ is constant} \]

\[ E(a\bar{x}) = aE(\bar{x}) \]

Example: Suppose \( P_0 = $30 \) is the price per unit of commodity XYZ. You've collected the following information

<table>
<thead>
<tr>
<th>( P_i )</th>
<th>Price</th>
<th>End-of-period Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>.3</td>
<td>25</td>
<td>-.16667</td>
</tr>
<tr>
<td>.4</td>
<td>35</td>
<td>.16667</td>
</tr>
<tr>
<td>.3</td>
<td>40</td>
<td>.3333</td>
</tr>
</tbody>
</table>

Calculate \( E(P_1) \) and expected return \( E(r) \)

\[ E(\bar{P}_1) = (.3)(25) + (.4)(35) + (.3)(40) = $33.50 \]

\[ E(\bar{r}_1) = (.3)(-0.16667) + (.4)(0.16667) + (.3)(0.3333) = .1167 \text{ or } 11.67\% \]

or

\[ \bar{r} = \frac{\bar{P}_1 - P_0}{P_0} \]

\[ E[\bar{r}] = E\left[ \frac{\bar{P}_1 - P_0}{P_0} \right] = \frac{E[P_1] - P_0}{P_0} = \frac{33.50 - 30}{30} = .1167 \text{ or } 11.67\% \]
Expected value is the most commonly used statistical measure to describe location

Problems

- May never get average
- May be poor measure with non symmetrical distribution
- Doesn’t describe dispersion or variability

Other measures

- Median — outcome (realization) in the middle (50\textsuperscript{th} percentile)

May be better measure of location then the mean if the distribution is skewed.

Consider the following set of numbers:

\{17, 0, 7, 10, 13, 3, 15, -4, 6, 0, -1, 17, 13, 13, 25, 13, 50, -1, 6, -8, 2, 54, 32, 202, 16, 13, 21, 120, 24, 29, 37\}
Mode — most frequent outcome

not very useful with return data which takes on fractional values
Expected values with sample data (historical series)

\[ \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{x_1 + x_2 + \ldots + x_n}{n} \]

n = number of observations

Example

You’ve collected the following data on common stock returns. From this information, what return would you expect from investing in stocks?

<table>
<thead>
<tr>
<th>Year</th>
<th>Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1988</td>
<td>16.81%</td>
</tr>
<tr>
<td>1989</td>
<td>31.49</td>
</tr>
<tr>
<td>1990</td>
<td>-3.17</td>
</tr>
<tr>
<td>1991</td>
<td>30.85</td>
</tr>
</tbody>
</table>

\[ \bar{r} = \frac{.1618 + 3149 + (-.0317) + 3055}{4} = .1892 \text{ or } 18.92\% \]
Measures of Dispersion — attempt to measure "risk"

- Range — difference between highest and lowest outcome
  
  Problem — not a consistent estimator becomes larger as sample size increases

Remember "set of numbers"

Range = 202 - (-8) = 210

- Semi interquartile range — difference between observation at the 75th quartile and observation at the 25th quartile

\[
\text{SIQ range} = \frac{x_{.75} - x_{.25}}{2}
\]

Go back to the "set of numbers"

\[
\text{SIQ} = \frac{(27 - 4.5)}{2} = 11.25
\]
- Provides consistent estimator (doesn’t increase as n→∞)
- Often used when variance doesn’t exist

\[ \text{Variance} — \text{most frequently used measure of dispersion.} \]

Defined as expectation of the squared deviations from the mean.

\[
\text{var}(\bar{x}) = \sigma_x^2 = \mathbb{E}[(\bar{x}_i - \bar{x})^2] = \sum_{i=1}^{n} p_i (x_i - \bar{x})^2
\]

Remember commodity XYZ example

<table>
<thead>
<tr>
<th>( p_i )</th>
<th>( P_{i-} )</th>
<th>Realized Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>.3</td>
<td>$25</td>
<td>-1.667</td>
</tr>
<tr>
<td>.4</td>
<td>35</td>
<td>.1667</td>
</tr>
<tr>
<td>.3</td>
<td>40</td>
<td>.3333</td>
</tr>
</tbody>
</table>

\[
\text{Var}(\bar{P}_i) = (.3)(25 - 33.50)^2 + (.4)(35-33.50)^2 + (.3)(40 - 33.50)^2
\]

\[= 35.25 \text{ dollars}^2 \]

Standard deviation = \( \sigma_x = \sqrt{\text{Var}(\bar{x})} = \sqrt{35.25} = $5.94 \)
Two useful properties of the variance operator

\[ \text{var}(a + \bar{x}) = \text{var}(\bar{x}) \quad \text{when } a = \text{constant} \]

\[ \text{var}(a\bar{x}) = a^2\text{var}(\bar{x}) \]

Find the variance of the return for commodity XYZ.

\[ \bar{r} = \frac{\bar{P}_1 - P_0}{P_0} \]

\[ \text{var}(\bar{r}) = E[(\bar{r} - \bar{r})^2] \]

\[ = E\left[\left(\frac{P_1 - P_0}{P_0} - \frac{E(P_1) - P_0}{P_0}\right)^2\right] \]

\[ = \frac{1}{P_0^2} E[(\bar{P}_1 - P_0 - (\bar{P}_1 - P_0))^2] \]

\[ = \frac{1}{P_0^2} E[(\bar{P}_1 - \bar{P}_1)^2] \]

\[ = \frac{1}{P_0^2} \text{var}(\bar{P}_1) \]
\[ \sigma^2 = \text{var}(\bar{r}) = \frac{1}{p_0^2} \text{var}(\tilde{P}_1) = \frac{35.25}{(30)^2} = 0.39167 \]

\[ \sigma = \sqrt{0.39167} = .1979 \text{ or } 19.79\% \]

Check using returns directly

\[ \text{var}(\bar{r}) = (.3)(-.1667 - .1167)^2 + (.4)(.1667 - .1167)^2 \]
\[ + (.3)(.3333 - .1167)^2 \]
\[ = .0391169 \]

Summarize XYZ

\( \text{E}(\tilde{P}_1) = $33.50 \quad \text{E}(r_1) = 11.67\% \)

\( \sigma_{p_1} = $5.94 \quad \sigma_{r_1} = 19.79\% \)
Sample data

\[ \text{var}(\bar{x}) = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n - 1} \]

Example — Remember Common Stock

<table>
<thead>
<tr>
<th>Year</th>
<th>Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1988</td>
<td>16.81%</td>
</tr>
<tr>
<td>1989</td>
<td>31.49</td>
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<tr>
<td>1990</td>
<td>-3.17</td>
</tr>
<tr>
<td>1991</td>
<td>30.55</td>
</tr>
</tbody>
</table>

\[
\sigma^2(k_s) = [(0.1681 - 0.1892)^2 + (0.3149 - 0.1892)^2 \\
+ (-0.0317 - 0.1892)^2 + (0.3055 - 0.1872)^2] / (4-1) \\
= 0.2619
\]

\[
\sigma(k_s) = \sqrt{0.2619} = 0.1618 \text{ or } 16.18\%
\]

\[
\tilde{k}_s = 18.92\% \quad \text{and} \quad \sigma_{ks} = 16.18\%
\]
Two other measures of risk used in practice

- **Semi-variance** — expectation of squared deviation below mean
  - focus is on "downside" risk
  - variance gives equal weight to deviations above and below mean

- Let \( y_i = x_i - \bar{x} \) if \( x_i \leq \bar{x} \)
  \[ y_i = 0 \quad \text{if} \quad x_i > \bar{x} \]

- Semi-variance = \( E[(y_i)^2] \)

**Mean Absolute Deviation (MAD)**

- Expectation of absolute deviation from mean
- Variance gives more weight to observation far away from mean

\[
\text{MAD} = E[|x_i - \bar{x}|]
\]

- Most of the time we'll use mean and standard deviation (variance)
- Some situations, other measures useful
- Also highlights features of mean and variance
<table>
<thead>
<tr>
<th>Asset</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Risk Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land (1910-89)</td>
<td>14.78%</td>
<td>10.99%</td>
<td>10.9%</td>
</tr>
<tr>
<td>Common Stock</td>
<td>12.4</td>
<td>20.8</td>
<td>8.5</td>
</tr>
<tr>
<td>Small Company Stocks</td>
<td>17.5</td>
<td>35.3</td>
<td>14.6</td>
</tr>
<tr>
<td>Long-Term Corp. Bonds</td>
<td>5.7</td>
<td>8.5</td>
<td>1.8</td>
</tr>
<tr>
<td>Long-Term Govt. Bonds</td>
<td>5.1</td>
<td>8.6</td>
<td>1.2</td>
</tr>
<tr>
<td>Treasury Bills</td>
<td>3.9</td>
<td>3.4</td>
<td></td>
</tr>
<tr>
<td>Inflation</td>
<td>3.2</td>
<td>4.7</td>
<td></td>
</tr>
</tbody>
</table>

**Risk Premium:** rate of return over and above "risk-free" return (RP)

**T-bill:** risk-free return on short term asset

\[ \text{RP}_{CS} = \text{0.124} - \text{0.039} = 0.85 \text{ or 8.5%} \]
Some notes and thoughts

- Government borrows money which creates debt that is essentially risk-free, particularly in the short run.

- T-bills are short-run government bonds that have a duration of less than 1 year. Thus T-bill rates are a good measure of a "risk-free rate of return".

- Excess return on a risk asset (investment) over the risk-free return in termed the "risk premium."

\[ \text{RP}_{cs} = .124 - .039 = .085 \text{ or } 8.5\% \]

- Investors in common stock have historically been rewarded with an excess return of 8.5% relative to the return investors received from T-bills.

- Why then, do people invest in T-bills? Answer is forthcoming, but looking at standard deviations we might guess the excess return is related to risk.
Expected risk on the market = risk-free rate plus market risk premium

Risk-free rate = 1 year T-bill rate (easy to observe)

Risk premium is more difficult

Historical risk premium on market is easy to estimate

e.g. 1926-91 \( \text{RP}_{CS} = 12.4 - 3.9 = 8.5\% \)

If risk on market portfolio is the same over time and peoples distaste for risk is constant, then risk premium is constant over time.
Suppose T-bill rate is 6%, then

\[
\text{Expected return} = \text{current risk-free rate} + \text{risk premium on market}
\]

\[
14.5\% \quad 6\% \quad + \quad 8.5\%
\]

Suppose an asset's risk differs from market risk. If we assume required returns are a linear function of risk then using our figure

Project risk ≥ market risk suggest \( k_p \geq k_M \)

Question: How to measure risk?

Standard deviation is obvious starting point but may not be appropriate . . . . .
MEASURING PORTFOLIO RISK AND RETURNS

Financial theory often assumes decision makers maximize expected utility looking at the impacts of these choices on the mean and variance of returns

Assumption: \( E[u(r)] = V(\mu_r, \sigma_r^2) \)

Implications

- Utility function is quadratic; or
- Returns follow a normal distribution

Normal Distribution

\[
\tilde{\tilde{r}} \rightarrow N(\mu_r, \sigma_r^2) = \frac{1}{\sigma_r \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\tilde{\tilde{r}} - \mu_r}{\sigma_r} \right)^2}
\]

We often standardize the normal distribution to the unit normal distribution

\[
\tilde{Z} = \frac{\tilde{\tilde{r}} - \mu_r}{\sigma_r}
\]

\[
\tilde{Z} \overset{D}{\rightarrow} N(0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (z)^2}
\]
Standard normal can easily be rescaled and is often easier to work with.

Implications of normal distribution

- Standard deviation

\[
\begin{align*}
\text{Prob } & \left[ \mu_r - \sigma_r < \bar{r} < \mu_r + \sigma_r \right] = .6826 \\
\text{Prob } & \left[ \mu_r - 2\sigma_r < \bar{r} < \mu_r + 2\sigma_r \right] = .9544 \\
\text{Prob } & \left[ \mu_r - 3\sigma_r < \bar{r} < \mu_r + 3\sigma_r \right] = .9974
\end{align*}
\]

Probability of a return being a certain distance above or below the mean depends only on the standard deviation.

e.g. \( \bar{r} = 10\% \), \( \hat{\sigma}_r = 20\% \)

\[
\text{Prob } \left[ \bar{r} - \hat{\sigma}_r < \bar{r} < \bar{r} + \hat{\sigma}_r \right] = .6826
\]

\[
\text{Prob } \left[ -10\% < \bar{r} < 30\% \right] = .6826 \text{ or } 68.26\%
\]

\( \uparrow \sigma_r \rightarrow \text{increased dispersion} \)
Assuming normal distribution often done at least as an approximation because it simplified both conceptual and empirical work.

It is often difficult to reject normality because of lack of data (Meyer).

Even if distribution is non normal, we may proceed in some cases using mean-variance framework:

- Location-scale (Meyer)
- Elliptical Symmetry (Nelson)
- Approximation (Hanson and Ladd)
MEASURING RISK

If assets were held in isolation, then standard deviation would be a good measure of risk in many cases.

However, when assets are held in portfolios (two or more assets), the standard deviation may not be a good measure of the risk that each asset contributes to the portfolio.

Example 1

<table>
<thead>
<tr>
<th>Weather</th>
<th>Probability</th>
<th>Sunglasses</th>
<th>Umbrellas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rain</td>
<td>1/3</td>
<td>0%</td>
<td>20%</td>
</tr>
<tr>
<td>Normal</td>
<td>1/3</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Sunny</td>
<td>1/3</td>
<td>20</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\mu_s = \mu_u = 10\% \\
\sigma_s = \sigma_u = 8.16\%
\]
Now consider a portfolio in which 50% of your wealth is invested in the Sunglasses Company and 50% is invested in the Umbrella Company.

<table>
<thead>
<tr>
<th>Weather</th>
<th>Prob.</th>
<th>Realized Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rain</td>
<td>1/3</td>
<td>(.5)(0) + (.5)(.20) = .10</td>
</tr>
<tr>
<td>Normal</td>
<td>1/3</td>
<td>(.5)(.10) + (.5)(.10) = .10</td>
</tr>
<tr>
<td>Sunny</td>
<td>1/3</td>
<td>(.5)(.20) + (.5)(0) = .10</td>
</tr>
</tbody>
</table>

\[
\mu_p = 10\% \\
\sigma = 0\%
\]

Each asset in isolation is risky. However, when the assets are held in a portfolio the "risk" is reduced and in this case eliminated.
Calculating expected returns and variances in a portfolio

Let \( w_j \) designate proportion of wealth invested in asset \( j \). If \( n \) are available, then

\[
\sum_{j=1}^{n} w_j = 1
\]

Let \( \hat{r}_j \) be the risky return from asset \( j \).

Expected return on a portfolio of \( n \) assets

\[
E(\bar{r}^P) = E\left[ \sum_{j=1}^{n} w_j \bar{r}_j \right] = E[w_1 \bar{r}_1 + w_2 \bar{r}_2 + \ldots + w_n \bar{r}_n]
\]

\[
= w_1 E(\bar{r}_1) + \ldots + w_n E(\bar{r}_n)
\]

\[
= w_1 \mu_1 + \ldots + w_n \mu_n
\]

\[
= \sum_{j=1}^{n} w_j \mu_j
\]

Sample data

\[
E(\bar{r}) = \sum_{j=1}^{n} w_j \bar{r}_j
\]
The portfolio mean return is simply the weighted average of expected returns on individual assets where the weights are the proportions invested in each asset.

Sunglasses example

\[
E(\tilde{r}_p) = w_s \mu_s + w_u \mu_u \\
= (5)(.10) + (.5)(.10) = .10 \text{ or } 10\
\]

Variance of a portfolio

Two asset case: \( \tilde{r}_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2 \)

\[
\text{var}(\tilde{r}_p) = E[(\tilde{r}_p - \mu_{r_p})^2] \\
= E[(w_1 \tilde{r}_1 - \mu_{r_1}) + w_2 (\tilde{r}_2 - \mu_{r_2})]^2] \\
= E[(w_1 (\tilde{r}_1 - \mu_{r_1}) + w_2 (\tilde{r}_2 - \mu_{r_2})]^2] \\
= E[w_1^2 (\tilde{r}_1 - \mu_{r_1})^2 + w_2^2 (\tilde{r}_2 - \mu_{r_2})^2 + 2w_1w_2(\tilde{r}_1 - \mu_{r_1})(\tilde{r}_2 - \mu_{r_2})] \\
= w_1^2 \text{var}(\tilde{r}_1) + w_2^2 \text{var}(\tilde{r}_2) + 2w_1w_2 \text{cov}(\tilde{r}_1, \tilde{r}_2)
where \( \text{cov}(\tilde{r}_1, \tilde{r}_2) = E[(\tilde{r}_1 - \mu_{r_1})(\tilde{r}_2 - \mu_{r_2})] \) is the covariance between \( r_1 \) and \( r_2 \), which measures how \( r_1 \) and \( r_2 \) move in relation to each other.

\[ \text{cov} > 0 \rightarrow \text{variables tend to move in same direction} \]
\[ \text{cov} < 0 \rightarrow \text{variables tend to move in opposite direction} \]

Covariance is extremely important because it measures the "risk" a single asset contributes to the portfolio.

Note: variance is really covariance of a random variable with itself

\[
\text{cov}(w_1 \tilde{r}_1, w_1 \tilde{r}_1) = E[w_1(\tilde{r}_1 - \mu_{r_1})w_1(\tilde{r}_1 - \mu_{r_1})] \\
= w_1^2 E[(\tilde{r}_1 - \mu_{r_1})^2] \\
= w_1^2 \text{var}(\tilde{r}_1)
\]
Sunglass-Umbrella Example

\[ \sigma_u = \sigma_s = .0816 \]

\[
\sigma_{u,s} = E[(\bar{r}_s - \mu_s)(\bar{r}_u - \mu_u)]
\]

\[
= \frac{1}{3}(.20 - .10)(0 - .10) + \frac{1}{3}(.10 - .10)(.10 - .10) + \frac{1}{3}(0 - .10)(.20 - .10)
\]

\[ = 0.006667 \]

\[
\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12}
\]

\[
= (.5)^2(.0816)^2 + (.5)^2(.0816)^2 + 2(.05)(.5)(-.006667)
\]

\[ = 0 \]

The negative covariance mean the assets returns tend to move in opposite directions.

This lets use "hedge" our risk by holding the assets in a portfolio — what we gain one, we tend to lose in the other asset.
Sample data

Two assets

\[ \hat{\sigma}_p^2 = w_1^2 \hat{\sigma}_1^2 + w_2^2 \hat{\sigma}_2^2 + 2 w_1 w_2 \hat{\sigma}_{1,2} \]

n assets

\[ \hat{\sigma}_p^2 = \frac{1}{n-1} \left[ (\bar{r}_1 - \bar{r})^2 + \ldots (\bar{r}_n - \bar{r})^2 \right] \]

\[ \hat{\sigma}_{r,s} = \frac{1}{n-1} \left[ (\bar{r}_1 - \bar{r})(\bar{s}_1 - \bar{s}) + \ldots (\bar{r}_n - \bar{r})(\bar{s}_n - \bar{s}) \right] \]
Example 2

<table>
<thead>
<tr>
<th>Prob</th>
<th>( r_x )</th>
<th>( r_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.2</td>
<td>11%</td>
<td>-3%</td>
</tr>
<tr>
<td>.2</td>
<td>9</td>
<td>15</td>
</tr>
<tr>
<td>.2</td>
<td>25</td>
<td>2</td>
</tr>
<tr>
<td>.2</td>
<td>7</td>
<td>20</td>
</tr>
<tr>
<td>.2</td>
<td>-2</td>
<td>6</td>
</tr>
</tbody>
</table>

\[
E(r_x) = (.2)(11\%) + (.2)(9\%) + (.2)(25\%) + (.2)(7\%) + (.2)(-2\%) = 1\%
\]

\[
E(r_y) = 8\%
\]

\[
\text{var}(r_x) = (.2)(.11-.10)^2 + (.2)(.09-.10)^2 + (.2)(.07-.10)^2 + (.2)(-.02-.10)^2
\]

\[
= .0076
\]

\[
\text{var}(r_y) = .00708
\]

\[
\text{cov}(r_x, r_y) = (.2)(.11-.10)(-.03-.08) + (.2)(.09-.10)(.15-.08)
\]
\[
+ (.2)(.25-.10)(.02-.08) + (.2)(.07-.10)(.20-.08)
\]
\[
+ (.2)(-.02-.10)(.06-.08)
\]

\[
= -0.0024
\]
Suppose $w_x = w_y = .5$

$$E(r_p) = (.5)(.10) + (.5)(.8) = 9\%$$

$$\text{var}(r_p) = (.5)^2(.0076) + (.5)^2(.00708) + 2(.5)(.5)(-.0024)$$

$$= .00247$$

$$\sigma_{r_p} = 4.97\%$$

Expected return is $\frac{1}{2}$ way between $E(r_x) + E(r_y)$ but variance is much lower than $\sigma^2_{rx}$ or $\sigma^2_{ry}$.

We could choose any combination of $x$ and $y$ s.t.

$$w_x + w_y = 1$$

Thus $E(r_p) = w_xE(r_x) + (1 - w_x)E(r_y)$

$$\frac{dE(r_p)}{dw_x} = E(r_x) - E(r_y)$$

$$= .10 - .08 = .02 \text{ or } 2\%$$
Portfolio returns are a linear function of $w_x$

Short sale: sell asset you don’t own (like borrowing)

e.g. $w_x = -.5$ \hspace{1cm} $w_y = 1.5$

$$E(R_p) = (.5)(.10) + (1.5)(.08) = 7\%$$
<table>
<thead>
<tr>
<th>$w_x$</th>
<th>$w_y$</th>
<th>$E(r_p)$</th>
<th>$\sigma_{r_p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0</td>
<td>10%</td>
<td>8.72%</td>
</tr>
<tr>
<td>0.75</td>
<td>0.25</td>
<td>9.5</td>
<td>6.18</td>
</tr>
<tr>
<td>0.50</td>
<td>0.50</td>
<td>9.0</td>
<td>4.97</td>
</tr>
<tr>
<td>0.25</td>
<td>0.75</td>
<td>8.5</td>
<td>5.96</td>
</tr>
<tr>
<td>0</td>
<td>1.00</td>
<td>8.0</td>
<td>8.41</td>
</tr>
</tbody>
</table>

$$w_x = -.5 \quad w_y = 1.5 \Rightarrow$$

$$\text{var}(r_p) = (-.5)^2(.0076) + (1.5)^2(.00708) + 2(-.5)(1.5)(-.0024)$$

$$= 0.2143$$

$$\sigma_{r_p} = 14.6\%$$
Trade off between mean and standard deviation

- each point represents $w_1$ and $w_2$ combination

Correlation:

- Independence
- Perfect Positive Correlation
- Perfect Negative Correlation
Perfectly Correlated Assets

Suppose $r_x = 1.703 + 1.037r_y$ (same $r_y$ as before)

<table>
<thead>
<tr>
<th>Prob.</th>
<th>$r_x$</th>
<th>$r_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.2</td>
<td>-1.408%</td>
<td>-3%</td>
</tr>
<tr>
<td>.2</td>
<td>17.258</td>
<td>14</td>
</tr>
<tr>
<td>.2</td>
<td>3.777</td>
<td>2</td>
</tr>
<tr>
<td>.2</td>
<td>22.443</td>
<td>20</td>
</tr>
<tr>
<td>.2</td>
<td>7.925</td>
<td>6</td>
</tr>
</tbody>
</table>

$E(r_x) = 10\%$  
$E(r_y) = 8\%$

$\sigma_x = 1.037\sigma_y$  
$= 8.72\%$

$\sigma_y = 8.41\%$

$\text{cov}(r_x, r_y) = .007334$

All points on a straight line
Correlation Coefficient

\[
\rho_{xy} = \frac{\text{cov}(r_x, r_y)}{\sigma_x \sigma_y}
\]

Suppose \( r_x \) is a linear function of \( r_y \)

\[
r_x = a + b r_y
\]

\[
E(r_x) = a + b E(r_y)
\]

\[
\text{var}(r_x) = b^2 \text{var}(r_y) \quad \Rightarrow \quad \sigma_x = b \sigma_y
\]

By def.

\[
\rho_{xy} = \frac{\text{cov}(r_x, r_y)}{\sigma_x \sigma_y} = \frac{E[(r_x - E(r_x))(r_y - E(r_y))]}{\sigma_x \sigma_y}
\]

\[
= \frac{E[(a + br_y - a - bE(r_y))(r_y - E(r_y))]}{b \sigma_y^2}
\]

\[
= \frac{E(b(r_y - E(r_y))^2]}{b \sigma_y^2} = \frac{b \sigma_y^2}{b \sigma_y^2} = 1
\]

Perfect positive correlation. Given \( r_y \) we know \( r_x \). Analogous result for \( \rho_{xy} = -1 \).
-1 < \rho_{xy} < 1

For our example

\[ \rho_{xy} = \frac{\text{cov}(r_x, r_y)}{\sigma_x \sigma_y} = \frac{-0.0024}{(0.0872)(0.0841)} = -0.33 \]

Another definition of covariance

\[ \text{cov}(r_x, r_y) = \sigma_{xy} = \rho_{xy} \sigma_x \sigma_y \]

Variance of 2 asset portfolio

\[ \sigma_p^2 = w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2 w_x w_y \rho_{xy} \sigma_x \sigma_y \]

Minimum variance portfolio

Since \( w_x + w_y = 1 \)

\[ \text{var}(r_p) = w_x^2 \sigma_x^2 + (1 - w_x)^2 \sigma_y^2 + 2 w (1 - w_x) \rho_{xy} \sigma_x \sigma_y \]
Minimum Variance

\[
\frac{d\sigma_{r_p}^2}{dw_x} = 2w_x\sigma_x^2 + 2(1 - w_x)(-1)\sigma_y^2 + 2(1 - w_x)\rho_{xy}\sigma_x\sigma_y
\]

\[+ 2w_x(-1)\rho_{xy}\sigma_x\sigma_y = 0\]

\[
w_x^* = \frac{\sigma_y^2 - \rho_{xy}\sigma_x\sigma_y}{(\sigma_x^2 + \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y)}
\]

\[
= \frac{.00708 - (-.33)(.0872)(.0841)}{.0076 + .00708 - 2(-.33)(.0872)(.0841)} = .487
\]

\[
E(r_p) = (.487)(.10) + (.513)(.08) = 8.974\%
\]

\[
\sigma_{r_p}^2 = (.487)^2(.0076) + (.513)^2(.00708) + 2(.487)(.513)(-.33)(.0872)(.0841)
\]

\[
= .0024565
\]

\[\Rightarrow \sigma_r = 4.956\%
\]
Risk-return tradeoff is a straight line when assets are perfectly correlated.

\[ \rho_{xy} = \begin{cases} 1 & 0 \leq w_x \leq 1 \\ \rho_{xy} = 1 & 0 \leq w_y \leq 1 \end{cases} \]

\[ \rho_{xy} = 1 \Rightarrow E(r_p) = w_x E(r_x) + w_y E(r_y) \]

\[ \sigma_{r_p}^2 = w_x^2 \sigma_x^2 + (1 - w_x)^2 \sigma_y^2 + 2w_x(1 - w_x)\sigma_x\sigma_y \]

\[ = (w_x \sigma_x + (1 - w_x) \sigma_y)^2 \]

\[ \sigma_p = w_x \sigma_x + (1 - w_x) \sigma_y \]
Consider minimum variance portfolio

\[
\begin{align*}
w_x^* &= \frac{\sigma_y^2 - \rho_{xy} \sigma_x \sigma_y}{(\sigma_x^2 + \sigma_y^2 - 2 \rho_{xy} \sigma_x \sigma_y)} \\
\rho_{xy} &= -1 \\
w_x^* &= \frac{\sigma_y^2 + \sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2 + 2 \sigma_x \sigma_y} = \frac{\sigma_y}{\sigma_x + \sigma_y} = \frac{.0841}{.0872 + .0841} = 49.095\% \\
E(r_p) &= .4095 (.10) + (1 - .49095) (.08) = 8.982\% \\
\sigma_r^p &= (.49095)(.0872) - (1 - .49095)(.0841) = 0\%
\end{align*}
\]

Proof that AB curve is a straight line if \( \rho_{xy} = 1 \)
Show slope doesn’t change as $w_x$ and $w_y$ change

$$\frac{dE(r_p)}{\sigma_{r_p}} = \frac{dE(r_p)}{dw_x} = \frac{E(r_x) - E(r_y)}{\sigma_x - \sigma_y}$$

$$= \frac{.10 - .08}{.0872 - .0841} = 6.45$$

Suppose $x$ and $y$ are perfectly inversely correlated

— can construct perfect hedge (sunglass-umbrella example)

$$\rho_{xy} = -1 \Rightarrow E(r_p) = w_x E(r_x) + (1 - w_x)E(r_y)$$

$$\sigma_{r_p}^2 = w_x^2 \sigma_x^2 + (1 - w_x)^2 \sigma_y^2 - 2w_x (1 - w_x) \sigma_x \sigma_y$$

$$= (w_x \sigma_x - (1 - w_x) \sigma_y)^2$$

$$\sigma_p = \pm (w_x \sigma_x - (1 - w_x) \sigma_y)$$

(positive and negative root)
Slope of curves CA and CB

Two roots: \( \sigma_{rp} \) must be \( \geq 0 \)

If \( w_x \geq \frac{\sigma_y}{\sigma_x + \sigma_y} \) \( \Rightarrow \sigma_{rp} = w_x \sigma_x - (1 - w_x)\sigma_y \)

\( < \frac{\sigma_y}{\sigma_x + \sigma_y} \) \( \Rightarrow \sigma_{rp} = (1 - w_x)\sigma_y - w_x \sigma_x \)

For CA, \( w_x > \sigma_y/(\sigma_x + \sigma_y) \)

\[
\frac{dE(r_p)}{d\sigma_{rp}} = \frac{dE(r_p)}{dw_x} = \frac{E(r_x) - E(r_y)}{(\sigma_x + \sigma_y)} = .117 > 0
\]

(positive slope and linear)

For CB, \( w_x < \sigma_y/(\sigma_x + \sigma_y) \)

\[
\frac{dE(r_p)}{d\sigma_{rp}} = \frac{dE(r_p)/dw_x}{d\sigma_{rp}/dw_x} = \frac{E(r_x) - E(r_y)}{-(\sigma_x + \sigma_y)} = -.117 < 0
\]

(negative slope and linear)
Perfect correlation cases are extremes

General slope of opportunity set is solid line

Minimum variance opportunity set:

Locus of risk and return combinations offered by portfolios of risky assets that yield the minimum variance for a given rate of return.

Bounded by triangle ABC
Notes:

- Diversification occurs whenever $\rho_{AB} < 1$
- MV represent minimum variance portfolio
- "Feasible set" is curve AB
- Curve is backward bending
- No investor will hold portfolio with $E(r)$ below MV
  => "Efficient Set" = MV to A
- Can construct efficient sets where to individual assets are themselves portfolios
Portfolios with N Risky Assets

Mean: \[ E(r_p) = \sum_{j=1}^{n} w_j E(r_j) \]

Variance: Start with 3 asset case

\[ r_p = w_1 r_1 + w_2 r_2 + w_3 r_3 \]

\[ \sigma_{r_p}^2 = E \left\{ \left[ (w_1 r_1 + w_2 r_2 + w_3 r_3) - (w_1 E(r_1) + w_2 E(r_2) + w_3 E(r_3)) \right]^2 \right\} \]

\[ = E \left\{ \left[ w_1 (r_1 - E(r_1)) + w_2 (r_2 - E(r_2)) + w_3 (r_3 - E(r_3)) \right]^2 \right\} \]

\[ = E \left\{ w_1^2 (r_1 - E(r_1))^2 + w_2^2 (r_2 - E(r_2))^2 + w_3^2 (r_3 - E(r_3))^2 \right\} \]

\[ + 2 w_1 w_2 (r_1 - E(r_1)) (r_2 - E(r_2)) \]

\[ + 2 w_1 w_3 (r_1 - E(r_1)) (r_3 - E(r_3)) \]

\[ + 2 w_2 w_3 (r_2 - E(r_2)) (r_3 - E(r_3)) \}

\[ = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + 2 w_1 w_2 \sigma_{12} + 2 w_1 w_3 \sigma_{13} + 2 w_2 w_3 \sigma_{23} \]

\[ = \sum_{i=1}^{3} \sum_{j=1}^{3} w_i w_j \sigma_{ij} \]

where \( \sigma_{ij} = \text{covariance of } i \text{ with } j \) (Remember \( \sigma_{11} = \sigma_1^2 \))
Generalizing to N assets

\[ \text{var}(r_p) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \]

Suppose \( N = 2 \)

\[ \text{var}(r_p) = w_1 w_1 \sigma_{11} + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2 w_2 \sigma_{22} \]
\[ = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12} \]
Sometimes useful to use matrix rotation

\[ E(r_p) = R'w \]

\[ \text{var}(r_p) = w' \]

\[
R = \begin{bmatrix}
    r_1 \\
    \vdots \\
    r_n
\end{bmatrix}
\]

\[
w = \begin{bmatrix}
w_1 \\
\vdots \\
w_n
\end{bmatrix}
\]

\[
\Sigma = \begin{bmatrix}
    \sigma_{11} & \cdots & \sigma_{1n} \\
    \vdots & \ddots & \vdots \\
    \sigma_{N1} & \cdots & \sigma_{m}
\end{bmatrix}
\]
More on working with Matrix notation later, if needed.

Application of minimum-variance portfolio technique

\[ r_p = w_c \tilde{r}_c + w_H \tilde{r}_H + x \tilde{r}_f \]

- \( w_c \) = proportion of wealth invested in corn
- \( w_H \) = proportion of wealth invested in hogs
- \( \tilde{r}_c \) = return from corn
- \( \tilde{r}_H \) = return from hogs

\( x \) = proportion of wealth held in futures contract (soy corn)

\( \tilde{r}_f \) = return on futures contract

\( \tilde{r}_f = \frac{(\tilde{r}_1 - f_0)}{f_0} \)

\( x > 0 \) (buy futures)
\( x < 0 \) (sell futures)

\[ \text{var}(r_p) = w_c^2 \sigma_c^2 + w_H^2 \sigma_H^2 + x^2 \sigma_f^2 + 2 w_c w_H \sigma_{cH} + 2 w_c x \sigma_{cf} + 2 w_H x \sigma_{Hf} \]

\[ \frac{2 \text{var}(r_p)}{2x} = 2x \sigma_f^2 + 2 w_c \sigma_{cf} + 2 w_H \sigma_{Hf} = 0 \]
\[ x^* = \frac{-(w_c \sigma_{cf} + w_H \sigma_{Hf})}{\sigma_f^2} \]

Suppose

\begin{align*}
  w_c &= 0.6 \quad \text{and} \quad w_H = 0.4 \\
  \sigma_f &= 0.03 \\
  \sigma_{cf} &= 0.025 \\
  \sigma_{Hf} &= 0.006
\end{align*}

\[ x^* = \frac{-(w_c \sigma_{cf} + w_H \sigma_{Hf})}{\sigma_f^2} \]

Sell futures equal to 42% of wealth
OPPORTUNITY SET WITH N RISKY ASSETS

Needed:  
- expected returns and variances of ind. assets  
- covariance between each pair of assets  
- N assets ⇒ N variances and ½ N(N-1) covariance terms

Method:  Construct each possible portfolio and plot $E(r_p)$ and $\sigma_{r_p}$

- Same shape as 2 asset portfolio  
- Some portfolios fall inside shaded area (but not important)  
- Individuals want to be on upper edge (MV to A) which is the efficient set

Finding efficient set requires programming approach  
(Markowitz)

Min var  
Opp. set : $\min \sigma_{r_p}$ s.t. $E(r_p) = X \ \forall \ X$  
Eff. set : $\max E(r_p)$ s.t. $\sigma_{r_p} = X \ \forall \ X$
PORTFOLIO DIVERSIFICATION AND INDIVIDUAL ASSET RISK

\[ \text{var}(r_p) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \]

\text{variance terms} = N

\text{covariance terms} = N(N-1) \text{ or } \frac{1}{2} N(N-1) \text{ if you count } \sigma_{ij} + \sigma_{ji} = 2\sigma_{ij} \text{ as one term}

Premise: As number of asset increases, the portfolio variance decreases and approaches the average covariance.

<table>
<thead>
<tr>
<th>Proportion</th>
<th>Variance Terms</th>
<th>Covariance Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 asset port.</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3 asset port.</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4 asset port.</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>N asset port.</td>
<td>N</td>
<td>N(N-1)</td>
</tr>
</tbody>
</table>

Suppose we have an equally weighted proportion

\[ w_i = w_j = \frac{1}{N} \]
\[
\text{var}(r_p) = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N} \frac{1}{N} \sigma_{ij} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}
\]

\[
= \frac{1}{N^2} \sum_{i=1}^{N} \sigma_{ii} + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}
\]

\[
\text{variance terms}
\]

\[
\text{covariance terms}
\]

Suppose largest asset variance = L.

\[
\text{variance term} \leq \frac{1}{N^2} \sum_{i=1}^{N} \text{L} = \frac{1}{N^2} (NL) = \frac{L}{N}
\]

\[
\lim_{N \to \infty} \left( \frac{L}{N} \right) = 0 \quad \text{(variance term approaches zero as N increases)}
\]

Let \( \bar{\sigma}_{ij} \) = average covariance. Since there are \( N(N - 1) \) covariance terms, the covariance term on R.H.S. can be written as

\[
\text{cov term} = \frac{1}{N^2} (N^2 - N) \bar{\sigma}_{ij} = \frac{N^2}{N^2} \bar{\sigma}_{ij} - \frac{1}{N} \bar{\sigma}_{ij}
\]

\[
\lim_{N \to \infty} \left( \frac{N^2}{N^2} \bar{\sigma}_{ij} - \frac{1}{N} \bar{\sigma}_{ij} \right) = \bar{\sigma}_{ij} \quad \text{(covariance term approaches average covariance as N increases)}
\]
Result: \[ \lim_{n \to \infty} \text{var}(r_p) = \bar{\sigma}_{ij} \]

Covariance terms become relatively more important as \( N \) goes up.

Total Risk = Market Risk + Asset Specific Risk
(systematic) (unsystematic)

\[
\left( \text{Variance of Asset i} \right) = \left( \text{weighted average covariance with assets to the market} \right) + (\text{var}_i - \text{cov})
\]

Not Appropriate measure can be important of risk diversified in portfolio
Portfolio Selection

Suppose: \[ \text{Eu}(r_p) = G(E(r_p), \sigma_{r_p}) \]

\[ \frac{\partial G}{\partial E(r_p)} > 0 \quad \quad \frac{\partial^2 G}{\partial E(r_p)^2} < 0 \]

\[ \frac{\partial G}{\partial \sigma_{r_p}} < 0 \quad \quad \frac{\partial^2 G}{\partial \sigma_{r_p}^2} > 0 \]

Slope of indifference curve (take total derivative)

\[ \frac{dE(r_p)}{d\sigma_{r_p}} = -\frac{\frac{\partial G}{\partial \sigma_{r_p}}}{\frac{\partial G}{2E(r_p)}} > 0 \]

\[ \frac{d^2E(r_p)}{d\sigma_{r_p}^2} = \frac{-\frac{\partial^2 G}{\partial \sigma_{r_p}^2}}{\frac{\partial^2 G}{\partial E(r_p)^2}} > 0 \]

\[ \left( d\text{Eu}(r_p) = \frac{\partial G}{\partial E(r_p)} dE(r_p) + \frac{\partial G}{\partial \sigma_{r_p}} d\sigma_{r_p} \right) \]
Slope at B > Slope at A
Eu₂ > Eu₁
Portfolio selection along efficient set

Opt. Port. depends on individual preferences
Opt. Port. always lies on efficient set
Opt. Port. at tangent point between highest indifference curve and efficient set
Requires estimation of preferences and efficient set
Efficient set with N risky assets and a risk-free asset

Start with 1 risky asset and 1 risk free asset

\[ r_p = w_1 r_1 + (1 - w_1) r_0 \]  \quad (r_0 = \text{risk-free return})

\[ E(r_p) = w_1 E(r_1) + (1 - w_1) r_0 \]

\[ \text{var}(r_p) = w_1^2 \text{var}(r_1) \text{ or } \sigma_{r_p} = w_1 \sigma_1 \]

Slope:

\[ \frac{dE(r_p)}{d\sigma(r_p)} = \frac{dE(r_p)}{dw_1} \cdot \frac{dw_1}{d\sigma(r_p)} = \frac{E(r_1) - r_0}{\sigma_1} \]

Opportunity Set

\(0 \leq w_1 \leq 1\)

Lending

Borrowing

\(w_1 > 1\)
• Assumes risk-free asset exists
• Borrow and lend at risk-free rate
• Frictionless world
  assets infinitely divisible
  no transaction cost

We use assumption of frictionless world to derive equilibrium pricing theories.

Assumption is unrealistic but,

• empirical evidence shows theories useful in describing reality
• possible to relax assumptions

Efficient set — long position in risky asset with borrowing and lending

(Short sales can only exist with differing expectations)

(i.e., $E(r_1) < 0$)
N risky assets and risk-free asset

- Can obtain any point on a straight line between $r_0$ and the risk/return for any asset or portfolio of assets.

However, line XY dominates. This is the efficient set, sometimes called the capital market line.

Implication: all an investor needs to know is combination of assets that make up portfolio M and the risk-free return.

This is true for any investor regardless of risk preference.
Capital Market Line
• All investor hold M.

• Equilibrium requires set of market clearing prices so that excess demand for any asset in M is zero.

• All assets will be held in proportion to market value weights.

\[
wick = \frac{V_i}{\sum_{i=1}^{N} V_i} \quad \text{(mrkt value of i)}
\]

\[
\sum_{i=1}^{N} V_i \quad \text{(total mrkt. value)}
\]
Two-Fund Separation

Each investor will have EU maximizing portfolio that is a combination of the risk-free asset and a portfolio of risky assets determined by a line drawn from risk-free return tangent to the investors efficient set of risky assets.

Implications

- Investment in risky assets independent of preferences.
- Borrow and lend to max EU.
- Homogeneous belief $\rightarrow$ same $M$. 