Show Your Work! Write Neatly And In An Organized Fashion!

(I) (20 pts) A balloon rises at a rate of $25\pi/27$ feet per second from a point on the ground 30 feet from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 40 feet above the ground. State your answer in degrees per second. (Note: Recall that $\pi$ radians equals 180°.)

Given \( \frac{dh}{dt} = \frac{25\pi}{27} \text{ ft/sec} \), we want \( \frac{d\theta}{dt} \) when \( h = 40 \text{ ft} \).

\[
\tan \theta = \frac{h}{30} \implies \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{30} \frac{dh}{dt}
\]

\[
\therefore \frac{d\theta}{dt} = \frac{\cos^2 \theta \cdot \frac{dh}{dt}}{30} = \frac{(\frac{3}{5})^2 \cdot \frac{25\pi}{27}}{30} = \frac{\pi}{90} \text{ radians/sec}
\]

\[
= \frac{\pi \times \text{sec}}{90 \times \text{sec} \times \frac{180^\circ}{\pi \times \text{rad}}} = 2 \times \text{per sec.}
\]

(II) (15 pts) The measurement of the side of a square is found to be 12 inches, with a possible error of 1/64 inch. Use differentials to approximate the possible propagated error in computing the area of the square.

Want \( |\Delta A| \) when \( l = 12 \text{ inches} \) & \( |\Delta l| \leq \frac{1}{64} \text{ inches} \). Here \( A = l^2 \).

\[
|\Delta A| \approx |dA| = |A'| |\Delta l| = 2l |\Delta l|
\]

\[
\leq 2 (12 \text{ inches})(\frac{1}{64} \text{ inches}) = \frac{24}{64} \text{ inches}^2 = \frac{3}{8} \text{ inch}^2.
\]

(III) (15 pts) Find the absolute maximum and absolute minimum values of the function \( f(x) = x^3 - 9x^2 + 15x \) on the interval \([0, 3]\).

\[
\frac{df}{dx} = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x - 1)(x - 5).
\]

\[
\therefore \ CPs: \ x = 1 \ & \ x = 5 \quad \because \quad \begin{cases} f(0) = 0 \ & \text{Not In Interval.} \\ f(1) = 7. \leftarrow \text{Abs Max Value!} \\ f(3) = -9. \leftarrow \text{Abs Min Value!} \\
\end{cases}
\]

(\#) \ 17 - 8\# + 45 = 72 - 8\# = -9.
Suppose \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\).

Then \( \frac{f(b) - f(a)}{b - a} = f'(c) \) for some \( c \) in \((a, b)\).

(b)(5 pts) Consider the function \( f(x) = x^{2/3} \) and the interval \([a, b] = [-1, 8]\). If the MVT applies, find a point \( c \) as in its conclusion; if the MVT does not apply, state why not by noting which of its hypotheses is not satisfied.

\[
f'(x) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3} \frac{1}{\sqrt[3]{x}} \quad \text{DNE at } x = 0. \quad \text{But } x = 0 \text{ is in } (-1, 8).
\]

\[
\therefore f \text{ is not differentiable on } (-1, 8). \quad \therefore \text{The MVT does not apply here!}
\]

(V) Evaluate the following indeterminant limits:

(a)(10 pts) \( \lim_{x \to 0} \frac{e^x - 1 - 4x}{x^2} \)

\[
\begin{align*}
\text{LR} & \quad \lim_{x \to 0} \frac{4e^x - 4}{2x} \\
\text{LR} & \quad \lim_{x \to 0} \frac{16e^x}{2} = \frac{16}{2} = 8.
\end{align*}
\]

(b)(10 pts) \( \lim_{x \to 0^+} (e^{4x} + x)^{1/2} \)

\[
\ln L = \lim_{x \to 0^+} \ln (e^{4x} + x)^{1/2} = \lim_{x \to 0^+} \frac{\ln (e^x + x)}{x}
\]

\[
\text{LR} & \quad \lim_{x \to 0^+} \frac{4e^x + 1}{2e^x + x^2} = 5.
\]

\[
\therefore \lim_{x \to 0^+} (e^x + x)^{1/2} = 5.
\]

(c)(10 pts) \( \lim_{x \to \infty} \sqrt{x^2 + 4x - x} \)

\[
\text{LR} & \quad \lim_{x \to \infty} \frac{\{x^2 + 4x - x\}\{x^2 + 4x + x\}}{\sqrt{x^2 + 4x + x}}
\]

(\text{1.} \quad \sqrt{x^2 + 4x} = \sqrt{x^2 + 4x + 4 - 4} = \sqrt{(x + 2)^2} - 2 = x + 2 - 2 = x.
\]

\[
\therefore \lim_{x \to \infty} \frac{x}{x + 2} = \lim_{x \to \infty} \frac{4x}{x} = 4.
\]

\[
\text{Since } x \to \infty \Rightarrow x > 0
\]

\[
\Rightarrow |x| = x
\]

\[
\Rightarrow \frac{4}{1+1} = 2.
\]
(VI)(20 pts) For \( f(x) = \frac{x^3}{x^2 - 1} \), we have \( f'(x) = \frac{x^2(x^2 - 3)}{(x^2 - 1)^2} \) and \( f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3} \). Figure out...

1. the single intercept that \( f \) has: \((0, 0)\)

2. the two vertical asymptotes that \( f \) has: \( x = 1 \) & \( x = -1 \)

3. the single oblique asymptote that \( f \) has (it has no horizontal asymptotes): \( y = x \)

\[
\frac{x^3}{x^2 - 1} = \frac{x^3}{x^2 - 1} \cdot \frac{x - 1}{x - 1} = \frac{x^4 - x^3}{x^3 - x^2} = \frac{x^3}{x^2 - 1} \xrightarrow{x \to \pm \infty} \frac{\infty}{\infty}
\]

4. all information about \( f \) gleaned from its first derivative (display the information on a number line as in class):

\[
\begin{array}{cccccccc}
& f' \downarrow & + & 0 & - \text{DNE} & - & 0 & - \text{DNE} & + & \infty \\
-\infty & & & & & & & & & \\
f & \text{INC} & \xrightarrow{\frac{\sqrt{3}}{2}} & \text{DEC} & \text{VA!} & \text{DEC} & 0 & \text{DEC} & \text{VA!} & \text{DEC} & \frac{\sqrt{3}}{2} \\
& & \text{MAX!} & & & & & & & \text{MIN!} & \\
& & & & & & & & & \text{NOT AN EXT!} & \\
& & & & & & & & & \text{MIN!} &
\end{array}
\]

5. all information about \( f \) gleaned from its second derivative (display the information on a number line as in class):

\[
\begin{array}{ccccccc}
f'' & - & \text{DNE} & + & 0 & - \text{DNE} & + & \infty \\
-\infty & & & & & & & \\
& & \text{CCD} & \text{VA!} & \text{CCD} & \text{VA!} & \text{CCD} & \infty \\
& & & & & & & \\
& & & & & & & \text{IP!} & \\
& & & & & & & \end{array}
\]

Then carefully graph the function \( f \) making sure your graph is consistent with all the information you've found. (Note that \( \sqrt{3} \approx 1.75 \) and \( 3\sqrt{3}/2 \approx 2.6 \)).