(I)(40 pts) Work out the following antiderivatives:

(a)(10 pts) \( \int \frac{(x^3 + 1)^2}{x^4} \, dx = \int \frac{x^2 + 2x^3 + 1}{x^4} \, dx \)

\[ = \int x^2 + \frac{2}{x} + x^{-2} \, dx \]

\[ = \frac{1}{3} x^3 + 2 \ln|x| - \frac{1}{3} x^{-3} + C. \]

(b)(10 pts) \( \int \tan^7(\theta) \sec^2(\theta) \, d\theta = \int u^7 \left( \frac{1}{2} \, du \right) \)

\[ u = \tan(\theta), \]

\[ \frac{1}{2} \, du = \sec^2(\theta) \, d\theta. \]

\[ = \frac{1}{2} \left( \frac{1}{8} u^8 \right) + C \]

\[ = \frac{1}{16} \tan^8(\theta) + C. \]

(c)(10 pts) \( \int \frac{6z + 5}{z^2 + 1} \, dz = 3 \int \frac{2z}{z^2 + 1} \, dz + 5 \int \frac{1}{z^2 + 1} \, dz \)

\[ = 3 \ln(z^2 + 1) + 5 \arctan z + C. \]

(d)(10 pts) \( \int \frac{1}{(x + 3) \sqrt{x^2 + 6x + 5}} \, dx = \int \frac{1}{(x+3) \sqrt{(x+3)^2-4}} \, dx \)

\[ = \int \frac{1}{2u \sqrt{u^2 - 1}} (2 \, du) \]

\[ = \frac{1}{2} \arccsc |u| + C \]

\[ = \frac{1}{2} \arccsc \frac{|x+3|}{2} + C. \]
(II)(20 pts)(a)(10 pts) State the Fundamental Theorem of Calculus, both parts, precisely and completely.

\[
\text{Suppose } f \text{ is continuous on } [a, b].
\]

Then: (I) \( \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \) for all \( x \) in \([a, b]\)

&

(II) \( \int_a^b f(t) \, dt = \left[ f(t) \right]_a^b = f(b) - f(a) \).

(b)(10 pts) Figure out \( \frac{d}{dx} \int_0^{\sin x} (1 - t^2)^{2003} \, dt \). Simplify as much as possible!

\[
\frac{d}{dx} \int_0^{\sin x} (1 - t^2)^{2003} \, dt = \frac{d}{dx} \int_0^{\sin x} (1 - t^2)^{2003} \, dt \cdot \frac{du}{dx}
\]

\[
= (1 - \sin^2 x)^{2003} \cdot \cos x
\]

\[
= \cos^{2003} x \cdot \cos x = \cos^{4007} x
\]

(III)(30 pts) Consider the region bounded by the curves \( y = x^2 \) and \( y = 12 - 2x^2 \).

(a)(10 pts) Set up and evaluate a definite integral giving the area of this region.

\[
\int_{-2}^2 y \, dx = \int_{-2}^2 (12 - 2x^2) \, dx = 3x^2 \bigg|_{-2}^2 = 4 \Rightarrow x = \pm 2.
\]

\[
A = \int dA = \int_{-2}^2 \left[ (12 - 2x^2) - x^2 \right] \, dx
\]

\[
= 2 \left[ 12 - 3x^2 \right] \bigg|_{-2}^2 = 2 \left[ 12x - x^3 \right]_0^2
\]

\[
\]

(b)(10 pts) Set up (BUT DO NOT EVALUATE) a definite integral giving the volume of the solid obtained by revolving this region about the \( x \)-axis. Use whichever of washers or shells is more convenient.

\[
\text{Washers!}
\]

\[
\begin{align*}
\text{Thickness} & = dx. \\
\text{Outer} & = 12 - 2x^2. \\
\text{Inner} & = x^2.
\end{align*}
\]

\[
V = \int dV = \int_{-2}^2 \pi \left[ (12 - 2x^2)^2 - (x^2)^2 \right] \, dx.
\]
(c) (10 pts) Set up (BUT DO NOT EVALUATE) a definite integral giving the volume of the solid obtained by revolving this region about the vertical line \( x = -3 \). Use whichever of washers or shells is more convenient.

\[
\text{Shells!}\]
\[
\begin{align*}
\text{thickness} &= dx, \\
\text{radius} &= x + 3, \\
\text{height} &= (12-2x^2)-x^3.
\end{align*}
\]

\[ V = \int_{-2}^{2} 2\pi (x+3) [(12-2x^2)-x^3] \, dx. \]

(IV) (20 pts) A box with no top is to have a base whose length is twice its width and a volume of 36 cubic feet. What dimensions should the box have to minimize the amount of material used in its construction?

Indicate the natural domain over which you are optimizing!
Don't forget to show that your purported answer is a minimum!
State your answer as a sentence and include proper units!

\[
\begin{align*}
&\text{Must minimize } S = lw + 2wh + 2lh, \quad l=2w, \\
&= 2w^2 + 2wh + 4wh^2 \\
&= 2w^2 + 6wh
\end{align*}
\]

subject to the constraint 36 = V = lwh = 2wh.

\[
\therefore h = \frac{36}{2w^2} = \frac{18}{w^2}, \quad \text{so} \quad S = 2w^2 + 6\left(\frac{18}{w^2}\right)w = 2w^2 + \frac{108}{w}.
\]

\[
\therefore \text{Must minimize } S = 2w^2 + \frac{108}{w} \quad \text{for } 0 < w < \infty.
\]

\[ S' = 4w - \frac{108}{w^2} = \frac{4}{w^2}\left[w^3-27\right]. \]

\[
\therefore \text{CPEs: } w = 3 \quad \text{(corresponding to } l=2w=6 \text{ & } h = \frac{18}{w^2} = 2).\]

\[
\begin{align*}
S' &\overset{0}{\overset{3}{\text{DEC}}} &\overset{\infty}{\text{INC}} \quad \Rightarrow \quad S'' = 4 + \frac{216}{w^3} > 0 \text{ always on } (0, \infty).
\end{align*}
\]

\[
\therefore \text{CCV always: } \overset{\circ}{\text{ABS MIN!}}
\]

\[
\therefore \text{To minimize the amount of material used in the construction of the box make the length, width, & height 6ft, 3ft, & 2ft respectively.}
\]