CALC I (SPR 01): Third Exam. NAME (Print): Max Mirrilli
Show Your Work! Write Neatly And In An Organized Fashion!
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(1)(30 pts) Evaluate the following indeterminant limits:

(a) (10 pts) \( \lim_{x \to 0} \frac{\arcsin(2x)}{e^{5x} - 1} \)

\[
\lim_{x \to 0} \frac{1}{\sqrt{1 - (2x)^2}} \cdot \frac{2}{5e^x} = \frac{1}{5}.
\]

(b) (10 pts) \( \lim_{x \to 0} \frac{1}{\ln(x + 1)} - \frac{1}{x} \)

\[
\lim_{x \to 0} \frac{x - \ln(1+x)}{x \ln(1+x)} = \frac{1}{2}.
\]

(c) (10 pts) \( \lim_{\theta \to \pi/2} (\sin \theta - \cos \theta) \tan \theta \)

\[
\text{Call it } L. \text{ Then } L = \lim_{\theta \to \pi/2} \tan \theta \ln(\sin \theta - \cos \theta).
\]

\[
\frac{\cos \theta}{\cos \theta} = \lim_{\theta \to \pi/2} \frac{\ln(\sin \theta - \cos \theta)}{\cos \theta}
\]

\[
\lim_{\theta \to \pi/2} \frac{(\cos \theta + \sin \theta)/(\sin \theta - \cos \theta)}{-\csc^2 \theta} = -1.
\]

\[
\therefore \lim_{\theta \to \pi/2} (\sin \theta - \cos \theta) \tan \theta = L = e^{-1}.
\]
(II)(20 pts) For \( f(x) = \frac{4(x - 5)}{(x - 4)^2} \), we have \( f'(x) = \frac{4(6 - x)}{(x - 4)^3} \) and \( f''(x) = \frac{8(x - 7)}{(x - 4)^4} \). Figure out:

1. the \( x- \) and \( y- \) intercepts of \( f \),
2. vertical and horizontal asymptotes of \( f \),
3. all information about \( f \) gleaned from its first derivative, and
4. all information about \( f \) gleaned from its second derivative.

(For 1 and 2 say "none" where appropriate; for 3 and 4 display the information on number lines as in class.)

Then carefully graph the function \( f \) making sure your graph is consistent with all the information you’ve found.

1. \( x- \) int: \( x = 5 \), \( y- \) int: \( y = -\frac{5}{4} \).
2. VA: \( x = 4 \), HA: \( y = 0 \).

3. \[
\begin{array}{cccccc}
& -\infty & \rightarrow & 4 & \leftarrow & 6 & \rightarrow & \infty \\
f' & \text{DEC} & \text{VA} & \text{INC} & \text{MAX} & \text{DEC} \\
n & f(n) = 1.
\end{array}
\]

4. \[
\begin{array}{cccccc}
& -\infty & \rightarrow & 4 & \leftarrow & 7 & \rightarrow & 2 & \infty \\
f'' & \text{CCD} & \text{VA} & \text{CCD} & \text{IP} & \text{CCU} \\
n & f(7) = \frac{8}{7}.
\end{array}
\]
A box with no top is to have a base whose length is twice its width and a volume of 36 cubic feet. What dimensions should the box have to minimize the amount of material used in its construction?

Indicate the natural domain over which you are optimizing!
Don't forget to show that your purported answer is a minimum!
State your answer as a sentence and include proper units!

Must minimize surface area

\[ S = A_{\text{bottom}} + A_{\text{left\&right}} + A_{\text{front\&back}} \]
\[ = lw + 2wh + 2lh \]
\[ = w(2w) + 2wh + 2(3w)h \]
\[ = 2w^2 + 6wh \]

subject to the constraint \( V = 36 \), i.e., \( lwh = 36 \), i.e., \( (2w)wh = 36 \), i.e., \( h = \frac{36}{2w^2} = \frac{18}{w^2} \).

\[ \therefore \text{Must minimize} \quad S = 2w^2 + 6w \left( \frac{18}{w^2} \right) = 2w^2 + \frac{108}{w} \quad \text{for} \quad 0 < w < \infty. \]

\[ S' = 4w - \frac{108}{w^2} = \frac{4}{w^2} \left( w^3 - 27 \right). \quad \therefore \text{CP:} \quad w = 3. \]

\[ S'' = 4 + \frac{108}{w^3} > 0 \quad \text{for} \quad 0 < w < \infty. \quad \therefore \text{ABS MIN!} \]

\[ \therefore \text{Min occurs when} \quad w = 3, \quad l = 2w = 6, \quad h = \frac{18}{w^2} = 2. \]

The desired box has dimensions 6 ft by 3 ft by 2 ft

with 2 ft being the height.
(IV) (30 pts) Work out the following antiderivatives:

(a) (7 pts) \[
\int \frac{(x^4 + 1)^2}{x^3} \, dx = \int \frac{x^8 + 2x^4 + 1}{x^3} \, dx = \int x^5 + 2x + x^{-3} \, dx = \frac{1}{6} x^6 + \frac{1}{2} x^2 - \frac{1}{2} x^{-2} + C. \]

(b) (7 pts) \[
\int \tan^3(3\theta) \sec^2(3\theta) \, d\theta = \int u^5 \left(\frac{1}{3} \, du\right) = \frac{1}{18} u^6 + C.
\]

- \(u = \tan(3\theta)\).
- \(\frac{1}{3} \, du = 3\sec^2(3\theta) \, d\theta\).

(c) (8 pts) \[
\int \frac{x + 1}{x^2 + 1} \, dx = \int \frac{2x}{2x^2 + 2} \, dx + \int \frac{1}{2x^2 + 2} \, dx = \frac{1}{2} \ln(x^2 + 1) + \arctan x + C.
\]

(d) (8 pts) \[
\int \frac{1}{(x + 1)\sqrt{x^2 + 2x - 8}} \, dx = \int \frac{1}{(x+1)\sqrt{(x+1)^2 - 9}} \, dx = \frac{1}{3} \arcsin |u + 1| + C = \frac{1}{3} \arcsin \left|\frac{x+1}{3}\right| + C \text{ or } \frac{1}{3} \arccsc \left|\frac{x+1}{3}\right| + C.
\]

Extra Credit (10 pts): It takes 10 hours for a population of 500 tribbles to grow to 8,000 tribbles. Assuming the growth of tribbles is exponential, how long does it take for a population of tribbles to double? Write your answer as an exact decimal! You can do this without a calculator!

Formally Done: \(P(t) = P_0 e^{kt} \Rightarrow 8000 = 5000 e^{k \cdot 10} \Rightarrow e^{k \cdot 10} = \frac{8000}{5000} = 16 \Rightarrow k \cdot 10 = \ln 16 \Rightarrow k = \frac{\ln 16}{10} \).

\(\therefore\) Doubling Time \(D = \frac{\ln 2}{k} = \frac{10 \ln 2}{\ln 16} = \frac{10 \ln 2}{4 \ln 2} = \frac{10}{4} = 2.5\) hours.

Intuitively Done: \(500 \overset{D}{\rightarrow} 1000 \overset{D}{\rightarrow} 2000 \overset{D}{\rightarrow} 4000 \overset{D}{\rightarrow} 8000\).

\(\therefore 4D = 10\) hours, i.e., \(D = 2.5\) hours.