
Show And Explain Your Work! Write Neatly And In An Organized Fashion!

(I) (15 pts) Find \( \vec{T}'(t) \) and \( \vec{N}(t) \) for the curve \( \vec{r}(t) = e^t (\sin t + \cos t) \hat{i} + e^t (\sin t - \cos t) \hat{j} + e^t \hat{k} \).

[Helpful Comments: It is easiest to compute \( \vec{N} \) as \( \vec{T}' \) divided by \( \| \vec{T}' \| \).
Also, your answers will be simpler than the curve’s equation and have no exponentials!]

\[
\begin{align*}
\vec{r}'(t) &= e^t [\cos t \sin t + \sin t \cos t] \hat{i} + e^t [\cos t \sin t + \sin t \cos t] \hat{j} + e^t \hat{k} \\
&= 2e^t \cos t \hat{i} + 2e^t \sin t \hat{j} + e^t \hat{k}.
\end{align*}
\]

\( \| \vec{r}'(t) \| = \sqrt{4e^{2t} \cos^2 t + 4e^{2t} \sin^2 t + e^{2t}} = \sqrt{4e^{2t} + e^{2t}} = \sqrt{5}e^t = \sqrt{5}e^t. \)

\( \vec{T}(t) = \frac{\vec{r}'(t)}{\| \vec{r}'(t) \|} = \frac{2}{\sqrt{5}} \cos t \hat{i} + \frac{2}{\sqrt{5}} \sin t \hat{j} + \frac{1}{\sqrt{5}} \hat{k}. \)

\( \vec{T}'(t) = -\frac{2}{\sqrt{5}} \sin t \hat{i} + \frac{2}{\sqrt{5}} \cos t \hat{j}. \)

\( \| \vec{T}'(t) \| = \sqrt{\frac{4}{5} \sin^2 t + \frac{4}{5} \cos^2 t} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}. \)

\( \vec{N}(t) = \frac{\vec{T}'(t)}{\| \vec{T}'(t) \|} = -\sin t \hat{i} + \cos t \hat{j}. \)

(II) (15 pts) What is the length of one arch of the cycloid \( \vec{r}(t) = (t - \sin t) \hat{i} + (1 - \cos t) \hat{j} \)?

[Helpful Comments: Thus you will be integrating over the interval \( 0 \leq t \leq 2\pi \).
Also, when integrating, the Half-Angle Formula \( 1 - \cos t = 2 \sin^2 (t/2) \) should come in handy!]

\( \vec{r}'(t) = (1 - \cos t) \hat{i} + \sin t \hat{j}. \)

\( dl = \| \vec{r}'(t) \| \, dt = \sqrt{(1 - \cos t)^2 + \sin^2 t} \, dt \)

\( = \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} \, dt = \sqrt{2 - 2 \cos t} \, dt \)

\( = \sqrt{2} (1 - \cos t) \, dt \overset{\text{H} \text{a} \text{l} \text{f} \text{e} \text{a} \text{l} \text{e} \text{e} \text{N} \text{a} \text{t} \text{i} \text{e} \text{r} \text{a} \text{y} \text{e} \text{r} \text{e} \text{y}}{=} \sqrt{2} \sin (t/2) \, dt \)

\( = \sqrt{2} \sin (t/2) \, dt \text{ for } 0 \leq t \leq 2\pi. \)

\( l = \int dl = \int_0^{2\pi} \sqrt{2} \sin (t/2) \, dt \)

\( = -\sqrt{2} \cos (t/2) \bigg|_0^{2\pi} = -4 \cos (\pi) - \cos 0 = -4 (-1) - 1 = 8. \)
(III) (10 pts) Carefully and precisely describe in words the domain of the function \( f(x, y) = \sqrt{144 - 16x^2 - 9y^2} \).

\[
\text{Domain: } 144 - 16x^2 - 9y^2 \geq 0.
\]

\[
: 144 \geq 16x^2 + 9y^2.
\]

\[
: 16x^2 + 9y^2 \leq 144.
\]

\[
: \frac{x^2}{9} + \frac{y^2}{16} \leq 1.
\]

The domain consists of those points in the xy-plane lying on or in the ellipse \( \frac{x^2}{9} + \frac{y^2}{16} = 1 \).

(IV) (10 pts) The radius \( r \) and height \( h \) of a right circular cylinder are measured with possible relative errors of at most 4% and 2% respectively. Use a total differential to estimate the maximum possible relative error in the calculated volume. (Note: For a right circular cylinder, \( V = \pi r^2 h \).)

Given \( \frac{\Delta r}{r} \leq 4\% \) and \( \frac{\Delta h}{h} \leq 2\% \),

\[
\text{want } \frac{\Delta V}{V} \approx \left| \frac{\Delta V}{V} \right| = \frac{\left| \frac{\partial V}{\partial r} \Delta r + \frac{\partial V}{\partial h} \Delta h \right|}{V}.
\]

\[
= \frac{\left| 2\pi rh \Delta r + \pi r^2 \Delta h \right|}{\pi r^2 h} = \frac{\left| 2 \frac{\Delta r}{r} + \frac{\Delta h}{h} \right|}{r}.
\]

\[
\leq 2 \left( \frac{\Delta r}{r} + \frac{\Delta h}{h} \right) \leq 2 (4\%) + 2\% = 10\%.
\]

(V) (10 pts) If North points along the +y-axis and East along the +x-axis, then what is the slope of the surface \( f(x, y) = x^2 - y^2 \) at the point \((4, 3, 7)\) in the South-East direction?

What we want is \( \nabla f(4, 3) \) where \( \hat{u} \) is the unit vector pointing SE.

\[
\nabla f(4, 3) = \frac{\partial f}{\partial x}(4, 3) \hat{i} + \frac{\partial f}{\partial y}(4, 3) \hat{j} = 2x \hat{i} - 2y \hat{j} \bigg|_{(4, 3)} = 8 \hat{i} - 6 \hat{j}.
\]

\[
\Delta \hat{u} = \frac{\hat{i} - \hat{j}}{\sqrt{2}}.
\]

\[
: \nabla f(4, 3) \cdot \hat{u} = \frac{8 + 6}{\sqrt{2}} = \frac{14}{\sqrt{2}} = 7\sqrt{2}.
\]

(VI) (10 pts) Find an equation for the tangent plane to the surface \( f(x, y) = x^3 - 3xy + y^3 \) at the point \((1, 2, 3)\).

Rewrite the surface as \( F(x, y, z) = x^3 - 3xy + y^3 - z = 0 \).

Then \( \vec{N} = \nabla F(1, 2, 3) = \left( 3x^2 - 3y, -3x + 3y^2 - 1 \right) \bigg|_{(1, 2, 3)} = -3 \hat{i} + 9 \hat{j} - \hat{k} \).

\[
: \text{Equation Plane: } -3x + 9y - z = -3(1) + 9(2) - 3 = 12.
\]

\[
3x - 9y + z = -12.
\]
(VII) (15 pts) Find and classify the critical points of the function \( f(x, y) = 2x^2 + 2xy + y^2 + 2x \).

\[
\begin{align*}
 f_x &= 0 \Rightarrow 4x + 2y + 2 = 0 \quad \Rightarrow \quad 2x + 0 + 2 = 0 \quad \Rightarrow \quad x = -1 \quad \text{and} \quad y = 1. \\
 f_y &= 0 \Rightarrow 2x + 2y = 0 \quad \therefore \quad \text{There is only one critical point: } (-1, 1). \\
 f_{xx} &= 4, \\
 f_{yy} &= 2, \\
 f_{xy} &= 2. \\
 \therefore D = f_{xx}f_{yy} - f_{xy}^2 = 4 > 0 \quad \text{and} \quad f_{xx} = 4 > 0.
\end{align*}
\]

By the 2nd Partial Test, \((-1, 1)\) is where \( f \) has a local minimum.

(VIII) (15) An open-faced shelter for waiting bus passengers is to have a volume of 350 cubic feet. If the material for its three sides costs \$5 per square foot, while that for its top costs \$7 per square foot, then what dimensions will result in the cheapest shelter?

(You may solve this problem either by eliminating a variable or by using Lagrange multipliers.)

We wish to min \( C = 7xy + 10xz + 5yz \) subject to the constraint \( xyz = y = 350 \).

But then \( z = \frac{350}{xy} \) and so \( C = 7xy + 10x \cdot \frac{350}{xy} + 5y \cdot \frac{350}{xy} = 7xy + \frac{3500}{y} + \frac{1750}{x} \).

\[
\begin{align*}
 C_x &= 7y - \frac{1750}{x^2} = 0 \quad \Rightarrow \quad 7x^2y = 1750 \quad \Rightarrow \quad x = \frac{7x^2y}{1750} = \frac{7}{350} = \frac{1}{2} \quad \Rightarrow \quad y = 2x, \\
 C_y &= 7x - \frac{3500}{y^2} = 0 \quad \Rightarrow \quad 7x^2 = 3500 \quad \Rightarrow \quad x^2 = \frac{3500}{7} \quad \Rightarrow \quad x = \frac{10}{\sqrt{7}}, \quad y = 2x, \quad y = \frac{20}{\sqrt{7}}, \\
 & \therefore \quad 7x^2y = 1750 \quad \text{becomes} \quad 14x^3 = 750, \quad \text{i.e.,} \quad x = \frac{1750}{\frac{14}{9}} = 12.5, \quad \text{i.e.,} \quad x = 5. \\
 & \therefore \quad \text{The cheapest shelter has dimensions } 5' \times 10' \times 7'.
\end{align*}
\]

Extra Credit: (10 pts) Consider the function \( f(x, y) = (x^2 + y^2)^2 - 2(x^2 + y^2) \) on the closed disc of radius 2 centered at the origin, i.e., on the region \( R = \{(x, y) : x^2 + y^2 \leq 4\} \). Find the absolute maximum and minimum values of \( f \) on \( R \) as well as the points of \( R \) where these maximum and minimum values occur.

**Interior CPs:** \( f_{xx} = 2(x^2 + y^2) \cdot 2x - 4x = 4x \left[ x + y^2 - 1 \right] = 0 \) \( \Rightarrow \quad x = 0 \) and \( y = 0 \).

\[
\begin{align*}
 f_{yy} &= 2(x^2 + y^2) \cdot 2y - 4y = 4y \left[ x + y^2 - 1 \right] = 0 \\
 & \Rightarrow \quad x^2 + y^2 = 1.
\end{align*}
\]

**Boundary Pts:** At all these points \( x^2 + y^2 = 4 \).

\[
\begin{align*}
 \text{Candiates:} & \quad f(0, 0) = 0, \\
 & \quad f(x, y) = \left[ x^2 + y^2 \right] \cdot \left[ x^2 + y^2 - 1 \right] = 1 \quad \text{everywhere on the circle } x^2 + y^2 = 1, \quad \text{Abs Min!} \\
 & \quad f(x, y) = \left[ x^2 + y^2 \right] \cdot \left[ x^2 + y^2 - 1 \right] = 8 \quad \text{everywhere on the circle } x^2 + y^2 = 4, \quad \text{Abs Max!}
\end{align*}
\]