CALC III (SPR 01): Third Exam. NAME (Print):

Show And Explain Your Work! Write Neatly And In An Organized Fashion!

(I) (15 pts) Let \( R \) be the region in the \( xy \)-plane bounded by the curves \( x = 1, \ y = 0, \) and \( y = \sqrt{x} \). Find the volume of the solid consisting of those points in space lying below the surface \( z = ye^x \) and above \( R \).

\[
V = \int_0^1 \int_0^{\sqrt{x}} ye^x \, dy \, dx \\
= \int_0^1 \left[ \frac{1}{2} e^x \right]_{y=0}^{y='\sqrt{x}} \, dx \\
= \int_0^1 \frac{1}{2} x e^x \, dx \\
= \frac{1}{2} \left[ xe^x - \int e^x \, dx \right]_0^1 \\
= \frac{1}{2} \left[ e - e^0 - \frac{e^2}{2} \right] = \frac{1}{2}.
\]

(II) (15 pts) Find the mass and center of mass of the triangular region \( R \) in the \( xy \)-plane bounded by the lines \( y = x, \ y = -x, \) and \( x = 2 \) if the density at a point \((x, y)\) of \( R \) is just \( \delta(x, y) = x \).

\[
M = \int \int_R \delta \, dA = \int_0^2 \int_{-x}^{x} x \, dy \, dx \\
= \int_0^2 2x^2 \, dx \\
= \frac{2}{3} x^3 \bigg|_0^2 = \frac{16}{3}.
\]

\[
\bar{x} = \frac{1}{M} \int \int_R x \delta \, dA = \frac{3}{16} \int_0^2 \int_{-x}^{x} x^2 \, dy \, dx \\
= \frac{3}{16} \int_0^2 2x^3 \, dx \\
= \frac{3}{4} \left( \frac{2x^4}{4} \right) \bigg|_0^2 = \frac{3}{2}.
\]

\[
\bar{y} = \frac{1}{M} \int \int_R y \delta \, dA = 0 \text{ by symmetry.}
\]
(III)(15 pts) Compute the surface area of the top part of the sphere \( x^2 + y^2 + z^2 = 1 \) that lies within the circular cylinder \( x^2 + y^2 = z \). (Note: Set the integral up in Cartesian coordinates but then switch it to polar coordinates for evaluation.)

\[
x^2 + y^2 + z^2 = 1 \Rightarrow z = \sqrt{1 - (x^2 + y^2)}.
\]

\[
A = \int_{xy} \sqrt{1 - (x^2 + y^2)} \, dx = \int_{xy} \frac{x^2}{\sqrt{1 - (x^2 + y^2)}} \, dy.
\]

\[
\Rightarrow \frac{dz}{dx} = \frac{-Xy}{x\sqrt{1 - (x^2 + y^2)}} \quad \Rightarrow \frac{dz}{dx} = \frac{-x^2y}{x\sqrt{1 - (x^2 + y^2)}}.
\]

\[
= 1 + \frac{\frac{x^2}{1 - (x^2 + y^2)}}{\sqrt{1 - (x^2 + y^2)}} = \frac{1}{\sqrt{1 - (x^2 + y^2)}}.
\]

\[
\therefore dS = \sqrt{1 + \left( \frac{x}{\sqrt{1 - (x^2 + y^2)}} \right)^2 + \left( \frac{-Xy}{x\sqrt{1 - (x^2 + y^2)}} \right)^2} \, dA = \frac{1}{\sqrt{1 - (x^2 + y^2)}} \, dA = \frac{1}{\sqrt{1 - r^2}} \, (rdr d\theta).
\]

\[
S = \int_0^{2\pi} \int_0^1 \rho r^2 \cos \theta \, r \, dr \, d\theta = 2 \int_0^{2\pi} \int_0^1 \rho r^2 \cos \theta \, dr \, d\theta = 2 \int_0^{2\pi} \left[ \frac{1}{3} \rho \right]_{r=0}^{r=1} \, d\theta = \frac{4}{3} \pi \rho \cos \theta.
\]

(IV)(30 pts) The solid bounded by the sphere with equation \( x^2 + y^2 + (z-1)^2 = 1 \) has density at a point equal to the point's distance to the origin. Set up (BUT DO NOT EVALUATE) triple integrals giving the mass of the solid in...

(a)(10 pts) Cartesian coordinates.

\[
M = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1-\sqrt{x^2+y^2}}^{1-\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2+z^2}} \, dz \, dy \, dx.
\]

(b)(10 pts) Cylindrical coordinates.

\[
M = \int_0^{2\pi} \int_0^1 \int_{1-\rho^2}^{1+\rho^2} r^2 dz \, dr \, d\theta.
\]

(c)(10 pts) Spherical coordinates.

\[
M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\rho_0} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.
\]
(V) (15 pts) The vector field displayed below is conservative. Find a potential function for it.

\[ \vec{F}(x, y, z) = 2xy^3 \mathbf{i} + (3x^2y^2 + e^z \cos y + 1/y) \mathbf{j} + (e^z \sin y + \sec^2 z) \mathbf{k}. \]

\[ f_x = P \Rightarrow f_x = 2xy^3 \quad \Rightarrow \quad f(x, y, z) = x^2 y^3 + g(y, z). \]

\[ f_y = Q \Rightarrow 3x^2y^2 + g_y(y, z) = \frac{3x^2y^2}{y} + e^z \cos y + \frac{1}{y} \]
\[ \Rightarrow g_y(y, z) = e^z \sin y + \ln y + h(z). \]

\[ f_z = R \Rightarrow e^z \sin y + h'(x) = e^z \sin y + \sec^2 z \]
\[ \Rightarrow h(z) = \tan z + c. \]

\[ \therefore B_j(\omega), (\beta), b(x), f(x, y, z) = x^2 y^3 + e^z \sin y + \ln y + \tan z + c \quad \text{will do}. \]

(VI) (10 pts) What is the work done by the force \( \vec{F}(x, y) = -2y \mathbf{i} - 3xy \mathbf{j} \) on an object moving along the path \( C: \vec{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, \ 0 \leq t \leq \pi \)?

\[ W = \int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \left\{ -2 \sin t \mathbf{i} - 3 \cos t \sin t \mathbf{j} \right\} \cdot \left\{ -\sin t \mathbf{i} + \cos t \mathbf{j} \right\} dt \]
\[ = \int_0^\pi 2 \sin^2 t - 3 \cos^2 t \sin t dt \]
\[ W = \int_0^\pi 1 - \cos 2t - 3 \cos^2 t \sin t dt \]
\[ = \left. t - \frac{1}{2} \sin 2t + \cos^3 t \right|_0^\pi \]
\[ = (\pi - 0 - 0) - (0 - 0 + 1) = \pi - 1. \]

Extra Credit (10 pts): Carefully sketch the solid over which the following triple integral is taking place and then switch the order of integration to \( dx \, dy \, dz \) instead of \( dz \, dy \, dx \):

\[ \int_0^\sqrt{7} \int_0^{7-x^2} \int_0^y f(x, y, z) \, dz \, dy \, dx. \]

\[ \therefore \int_0^7 \int_0^{\sqrt{7-x^2}} \int_0^y f(x, y, z) \, dx \, dy \, dz. \]