February 26th: Expectation
Expectation

Suppose that a doctor get only two kinds of patients, those with insurance type `a’ and one with insurance `b’. If a patient has insurance `a’ the doctor gets 40 dollars per visit, if the patient is from insurance `b’ he gets 55. Let INCOME be a random variable from the event space of insurance types to the space of dollars defined by \( f(A) = 40 \) and \( f(B) = 55 \). \( P(INCOME=a) \) is 1/3 and \( P(INCOME=b) \) is 2/3. How much money does the doctor get on average from every patient? The answer is

\[
\frac{1}{3} \times 40 + \frac{2}{3} \times 55 = 50
\]

This value is known as the **expected value** or expectation of INCOME.
Expectation Value

The expectation value of a function $f(x)$ in a variable $x$ is denoted $\langle f(x) \rangle$ or $E\{f(x)\}$. For a single discrete variable, it is defined by

$$\langle f(x) \rangle = \sum_x f(x) P(x),$$

(1)

where $P(x)$ is the probability function.

For a single continuous variable it is defined by,

$$\langle f(x) \rangle = \int f(x) P(x) \, dx.$$

(2)

The expectation value satisfies

$$\langle ax + by \rangle = a \langle x \rangle + b \langle y \rangle$$

(3)

$$\langle a \rangle = a$$

(4)

$$\langle \sum x \rangle = \sum \langle x \rangle.$$  

(5)
$E(LENGTH) = 3 \times 0.068 + 2 \times 0.035 + 3 \times 0.0284 + 2 \times 0.0257 + 1 \times 0.0229$
Linearity of Expectation

Linear Operator

An operator \( \tilde{L} \) is said to be linear if, for every pair of functions \( f \) and \( g \) and scalar \( t \),

\[
\tilde{L}(f + g) = \tilde{L} f + \tilde{L} g
\]

and

\[
\tilde{L}(t f) = t \tilde{L} f.
\]
SOME THEOREMS ON EXPECTATION

Theorem 3-1: If $c$ is any constant, then

$$E(cX) = cE(X)$$  \hspace{1cm} (8)

Theorem 3-2: If $X$ and $Y$ are any random variables, then

$$E(X + Y) = E(X) + E(Y)$$  \hspace{1cm} (9)

Theorem 3-3: If $X$ and $Y$ are independent random variables, then

$$E(XY) = E(X)E(Y)$$  \hspace{1cm} (10)
<table>
<thead>
<tr>
<th>First die</th>
<th>Second die</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>7 8 9 10 11 12</td>
</tr>
<tr>
<td>5</td>
<td>6 7 8 9 10 11</td>
</tr>
<tr>
<td>4</td>
<td>5 6 7 8 9 10</td>
</tr>
<tr>
<td>3</td>
<td>4 5 6 7 8 9</td>
</tr>
<tr>
<td>2</td>
<td>3 4 5 6 7</td>
</tr>
<tr>
<td>1</td>
<td>2 3 4 5 6 7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>2 3 4 5 6 7 8 9 10 11 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(X = x)$</td>
<td>$\frac{1}{36}$ $\frac{1}{18}$ $\frac{1}{12}$ $\frac{1}{9}$ $\frac{5}{36}$ $\frac{1}{6}$ $\frac{5}{36}$ $\frac{1}{9}$ $\frac{1}{12}$ $\frac{1}{18}$ $\frac{1}{36}$</td>
</tr>
</tbody>
</table>

**Figure 2.2** A random variable $X$ for the sum of two dice. Entries in the body of the table show the value of $X$ given the underlying basic outcomes, while the bottom two rows show the pmf $p(x)$. 
Variance, the expected squared deviation
Variance is the difference $\E(X^2) - (\E(X))^2$

(130) \[ V(X) = \E(X^2) - (\E(X))^2 \]

For a proof notice that

\[
\E(X - \E X)^2 = \E(X - \E X)(X - \E X)
\]

\[
= \E(X^2 - 2X \cdot \E X + (\E X)^2)
\]

(131) \[
= \E(X^2) - 2\E((\E X) \cdot X) + (\E X)^2
\]

\[
= \E(X^2) - 2(\E X)(\E X) + (\E X)^2
\]

\[
= \E(X^2) - (\E X)^2
\]
SOME THEOREMS ON VARIANCE

Theorem 3-4:  \[ \sigma^2 = E[(X - \mu)^2] = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2 \]  
where \( \mu = E(X) \).

Theorem 3-5: If \( c \) is any constant,  \[ \text{Var}(cX) = c^2 \text{Var}(X) \]  

Theorem 3-6: The quantity \( E[(X - a)^2] \) is a minimum when \( a = \mu = E(X) \).

Theorem 3-7: If \( X \) and \( Y \) are independent random variables,  \[ \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \quad \text{or} \quad \sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 \]  \[ \text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) \quad \text{or} \quad \sigma_{X-Y}^2 = \sigma_X^2 + \sigma_Y^2 \]