Confirmatory Factor Analysis

Purpose

- Takes factor analysis a few steps further.
- Imposes theoretically interesting constraints on the model and examine the resulting fit of the model with the observed data
- Used to evaluate theoretical measurement structures
- Provides tests and indices to evaluate fit

CFA Model & Notation

\[ x = \Lambda \xi + \delta \]

Where:
- \( x \) = (q x 1) vector of indicator/manifest variable
- \( \Lambda \) = (q x n) matrix of factor loadings
- \( \xi \) = (n x 1) vector of latent constructs (factors)
- \( \delta \) = (q x 1) vector of errors of measurement

CFA Example

- Measures for positive emotions, \( \xi_1 \):
  - \( x_1 = \text{Happiness} \), \( x_2 = \text{Pride} \)
- Measures for negative emotions \( \xi_2 \):
  - \( x_3 = \text{Sadness} \), \( x_4 = \text{Fear} \)
- Model
  \[
  \begin{align*}
  x_1 &= \lambda_{11} \xi_1 + \delta_1 \\
  x_2 &= \lambda_{12} \xi_1 + \delta_2 \\
  x_3 &= \lambda_{21} \xi_2 + \delta_3 \\
  x_4 &= \lambda_{22} \xi_2 + \delta_4 
  \end{align*}
  \]
CFA Model Matrices

\[
x_1 = \lambda_1 \xi_1 + \delta_1 \\
x_2 = \lambda_2 \xi_1 + \delta_2 \\
x_3 = 0 + \lambda_3 \xi_2 + \delta_3 \\
x_4 = 0 + \lambda_4 \xi_3 + \delta_4
\]

More Matrices

\[
x_1 = \lambda_1 \xi_1 + \delta_1 \\
x_2 = \lambda_2 \xi_1 + \delta_2 \\
x_3 = 0 + \lambda_3 \xi_2 + \delta_3 \\
x_4 = 0 + \lambda_4 \xi_3 + \delta_4
\]

Model Fitting

- The specified model results in an implied variance-covariance matrix, \( \Sigma_n \)

\[
\Sigma_n = \Lambda \phi \Lambda + \Theta
\]

Parameter Estimation

- Maximum Likelihood Estimation
  - Assumes multivariate normality
  - Iterative procedure
  - Requires starting values

\[
F_{ML} = \text{Tr}(SC^{-1}) - n + \ln(\det(C)) - \ln(\det(S))
\]

- \( S \) is the sample variance-covariance matrix
- \( C \) is the implied variance-covariance matrix

Model Identification

- Definition:
  - The set of parameters \( \theta \) is not identified if there exists \( \theta_1 \neq \theta_2 \) such that \( \Sigma(\theta_1) = \Sigma(\theta_2) \).

1 Factor, 2 indicators

- Not Identified!

Implied Covariance Matrix

\[
\Sigma = \begin{bmatrix}
\sigma_{xx} & \sigma_{x1} \\
\sigma_{x1} & \sigma_{11}
\end{bmatrix} = \begin{bmatrix}
10 & 5 \\
5 & 10
\end{bmatrix}
\]

Solutions

\[
\Lambda^2 \phi_{11} + \theta_{11} = \begin{bmatrix}
\phi_{11} \\
\phi_{21}
\end{bmatrix} \quad \Lambda^2 \phi_{11} + \theta_{22} = \begin{bmatrix}
\phi_{11} \\
\phi_{21}
\end{bmatrix}
\]

\[
\phi_{11} = 3.0 \\
\phi_{21} = 2.5 \\
\theta_{11} = 5.0 \\
\theta_{22} = 7.5 \\
\theta_{33} = 0.0
\]
1 Factor, 3 Indicators

- Just-Identified; always fits perfectly

\[
\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}
\]

\[
\Sigma_\theta = \begin{pmatrix}
\phi_{11} + \theta_{11} & \lambda_{11} \phi_{11} + \theta_{12} & \lambda_{11} \phi_{11} + \theta_{13} \\
\lambda_{11} \phi_{11} + \theta_{21} & \lambda_{11} \phi_{11} + \theta_{22} & \lambda_{11} \phi_{11} + \theta_{23} \\
\lambda_{11} \phi_{11} + \theta_{31} & \lambda_{11} \phi_{11} + \theta_{32} & \lambda_{11} \phi_{11} + \theta_{33}
\end{pmatrix}
\]

Identification Rules

- Number of free parameters: \( t \leq \frac{1}{2} q (q+1) \)
- Three-Indicator Rule
  - Exactly 1 non zero element per row of \( \Lambda \)
  - 3 or more indicators per factor
  - \( \Theta \) Diagonal - uncorrelated errors
- Two-Indicator Rule
  - \( \phi_{ij} \neq 0 \) for at least one pair \( i, j, i \neq j \)
  - Exactly 1 non-zero element per row of \( \Lambda \)
  - 2 or more indicators per factor
  - \( \Theta \) Diagonal - uncorrelated errors

Scaling the latent variables

- The scale/metric of the latent variable is not determinant
  - Factor loadings and variances can take on any value unless the metric is specified
- Must impose a model constraint to yield a meaningful scale
- Two Constraints are possible:
  - Fix a loading to 1.0 from each factor to one of its indicators
    * Latent scale takes on the metric of the constrained indicator
  - Fix the latent variance to 1.0
    * Yields a standardized latent variable
Covariance or Correlation matrix

- Cudeck shows that the covariance matrix is preferred for CFA
- Use of a correlation matrix can result in many problems:
  - Incorrect parameter estimates
  - Incorrect standard errors
  - Incorrect test statistics

Parameter Evaluation – Local Fit

- Parameter Estimate/SE = Z-statistic
- Standard interpretation:
  - if $|Z| > 2$, then “significant”
- Consider both statistical and scientific value of including a variable in the model
- Significance testing in CFA:
  - Not usually interesting in determining if loadings are equal to zero
  - Might be interested in testing whether or not covariance between factors is zero.

Goodness of Fit – Global Fit

- Absolute indices
  - derived from the fit of the obtained and implied covariance matrices and the ML minimization function
  - Chi-square & functions of it
- Relative fit indices
  - Relative to baseline “worst-fitting” model
- Adjusted fit indices
  - Relative to number of parameters in model
  - Parsimony

Chi Square: $\chi^2$

- $F_{ML}*(n-1)$
- Sensitive to sample size
  - The larger the sample size, the more likely the rejection of the model and the more likely a Type II error (rejecting something true). In very large samples, even tiny differences between the observed and the perfect-fit model may be found significant.
  - Informative when sample sizes are relatively small (100-200)
- Chi-square fit index is also very sensitive to violations of the assumption of multivariate normality

Relative Fit Indices

- Fit indices that provide information relative to a baseline model
  - a model in which all of the correlations or covariances are zero
  - Very poor fit
  - Termed Null or Independence model
- Permits evaluation of the adequacy of the target model

Goodness of Fit Index (GFI)

$$GFI = 1 - \frac{\chi^2_{model}}{\chi^2_{null}}$$

- % of observed covariances explained by the covariances implied by the model
  - Ranges from 0-1
- Biased
  - Biased up by large samples
  - Biased downward when degrees of freedom are large relative to sample size
- GFI is often higher than other fit indices
  - Not commonly used any longer
**Normed Fit Index**  
* (Bentler & Bonnet, 1980)

- Compare to a Null or Independence model  
  - a model in which all of the correlations or covariances are zero  
  \[ NFI = \frac{X^2_{\text{null}} - X^2_{\text{model}}}{X^2_{\text{model}}} \]
- % of total covariance among observed variables explained by target model when using the null (independence) model as baseline - Hu & Bentler, 1995
- Not penalized for a lack of parsimony
- Not commonly used anymore

**Non-Normed Fit Index**  
* - Tucker & Lewis, 1973

\[ \text{NNFI} = \frac{X^2_{\text{null}}/df_{\text{null}} - X^2_{\text{model}}/df_{\text{model}}}{X^2_{\text{model}}/df_{\text{model}}} \]
- Penalizes for fitting too many parameters
- May be greater than 1.0
  - If so, set to 1.0

**Comparative Fit Index**  
* - Bentler, 1989; 1990

\[ CFI = \frac{(X^2_{\text{null}} - df_{\text{null}}) - (X^2_{\text{model}} - df_{\text{model}})}{(X^2_{\text{null}} - df_{\text{null}})} \]
- Based on noncentrality parameter for chi-square distribution
- Indicates reduction in model misfit of a target model relative to a baseline (independence) model
- Can be greater than 1.0 or less than 0.0
  - If so, set to 1.0 or 0.0

**Root Mean Square Error of Approximation (RMSEA)**

\[ \text{RMSEA} = \sqrt{\frac{X^2_{\text{model}}/(df_{\text{model}}-1)}{N-1}} \]
- Discrepancy per degree of freedom
- \( \text{RMSEA} \leq 0.05 \Rightarrow \text{Close fit} \)
- \( 0.05 < \text{RMSEA} \leq 0.08 \Rightarrow \text{Reasonable fit} \)
- \( \text{RMSEA} > 0.1 \Rightarrow \text{Poor fit} \)

**Standardized Root Mean Square Residual (SRMR)**

- Standardized difference between the observed covariance and predicted covariance
- A value of zero indicates perfect fit
- This measure tends to be smaller as sample size increases and as the number of parameters in the model increases.

**General fit standards**

- NFI
  - .90-.95 = acceptable; above .95 is good
  - NFI positively correlated with sample size
- NNFI
  - .90-.95 = acceptable; above .95 is good
  - NFI and NNFI not recommended for small sample sizes
- CFI
  - .90-.95 = acceptable; above .95 is good
  - No systematic bias with small sample size
General fit standards

- RMSE
  - Should be close to zero
  - 0.0 to 0.05 is good fit
  - 0.05 to 0.08 is moderate fit
  - Greater than .10 is poor fit
- SRMR
  - Less than .08 is good fit
- Hu & Bentler
  - SRMR ≤ .08 AND (CFI ≥ .95 OR RMSEA ≤ .06)

Nested Model Comparisons

- Test between equivalent models except for a subset of parameters that are freely estimated in one and fixed in the other
- Difference in –2LL is distributed as chi-square variate
  - Each model has a $\chi^2$ value based upon a certain degree of freedom
  - If models are nested (ie., identical but M2 deletes one parameter found in M1), significance of 'increment' or 'decrement' in fit = $\chi^2_{1} - \chi^2_{2}$ with df = df$_{1} -$ df$_{2}$

Modification indexes

- If the model does not fit well, modification indices may be used to guide respecification (ie. how to improve the model)
- For CFA, the only sensible solutions are to add direct paths from construct to indicator to drop paths
- Rarely is it reasonable to let residuals covary as many times suggested by the output
- Respecify the model or drop factors or indicators

Reliability in CFA

$$r = \frac{(\sum \lambda_i)^2}{(\sum \lambda_i)^2 + \sum (1 - \lambda_i^2)}$$

- Standard reliability assumes tau-equivalence.
- Can estimate reliability in CFA with no restrictions
  - Congeneric measures

Equivalent models

- Equivalent models exist for almost all models
- Most quant people say you should evaluate alternative models
- Most people don't do it
- For now, be aware that there are alternative models that will fit as well as your model
Heirarchical CFA

Depression (CES-D)

Positive Affect

Negative Affect

Somatic Symptoms

Happy, Enjoy, Bothered, Blues, Depressed, Sad, Mind, Effort, Sleep

Model Fit Statistics: N= 868, $\chi^2(26)= 68.690, p<.001$, SRMR=.055, IFI= .976

a Second-order loadings were set equal for empirical identification. All loadings significant at $p < .001$. 

Example...AMOS