Confirmatory Factor Analysis

Psych 818
DeShon

Purpose

- Takes factor analysis a few steps further.
- Impose theoretically interesting constraints on the model and examine the resulting fit of the model with the observed data.
- Used to evaluate theoretical measurement structures.
- Provides tests and indices to evaluate fit.

CFA Model & Notation

\[ x = \Lambda \xi + \delta \]

Where:

- \( x \) = \((q \times 1)\) vector of indicator/manifest variable.
- \( \Lambda \) = \((q \times n)\) matrix of factor loadings.
- \( \xi \) = \((n \times 1)\) vector of latent constructs (factors).
- \( \delta \) = \((q \times 1)\) vector of errors of measurement.

CFA Example

- Measures for positive emotions, \( \xi_1 \):
  - \( x_1 \) = Happiness, \( x_2 \) = Pride
- Measures for negative emotions, \( \xi_2 \):
  - \( x_3 \) = Sadness, \( x_4 \) = Fear
- Model:
  \[
  x_1 = \lambda_{11} \xi_1 + \delta_1 \\
  x_2 = \lambda_{12} \xi_1 + \delta_2 \\
  x_3 = \lambda_{21} \xi_2 + \delta_3 \\
  x_4 = \lambda_{22} \xi_2 + \delta_4 
  \]

More on the 1.0s later.
CFA Model Matrices

\[ x_1 = \lambda_1 \xi + \delta_1 \]
\[ x_2 = \lambda_2 \xi + \delta_2 \]
\[ x_3 = \lambda_3 \xi + \delta_3 \]
\[ x_4 = \lambda_4 \xi + \delta_4 \]

\[ x = \Lambda \xi + \delta \]

Model Fitting

- The specified model results in an implied variance-covariance matrix, \( \Sigma_{\theta} \)

\[ \Sigma_{\theta} = \Lambda \Phi \Lambda + \Theta \]

\[ \Lambda \Phi \Lambda + \Theta = \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\
\lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\
\lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44}
\end{bmatrix} + \begin{bmatrix}
\theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\
\theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \\
\theta_{31} & \theta_{32} & \theta_{33} & \theta_{34} \\
\theta_{41} & \theta_{42} & \theta_{43} & \theta_{44}
\end{bmatrix} \]

Parameter Estimation

- Maximum Likelihood Estimation
  - Assumes multivariate normality
  - Iterative procedure
  - Requires starting values

\[ F_{ML} = Tr(SC^{-1}) - n + \ln(\det(C)) - \ln(\det(S)) \]
  - \( S \) is the sample variance-covariance matrix
  - \( C \) is the implied variance-covariance matrix

1 Factor, 2 indicators

- Not Identified!

- Population covariance matrix

\[ \Sigma = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{bmatrix} = \begin{bmatrix}
10 & 5 \\
5 & 10
\end{bmatrix} \]

- Implied Covariance Matrix

\[ \Sigma_{\theta} = \begin{bmatrix}
\lambda_{11}^2 & \lambda_{11} \lambda_{12} + \theta_{11} \\
\lambda_{21} \lambda_{11} & \lambda_{22} \lambda_{11} + \theta_{12} \\
\lambda_{31} \lambda_{11} & \lambda_{32} \lambda_{11} + \theta_{13} \\
\lambda_{41} \lambda_{11} & \lambda_{42} \lambda_{11} + \theta_{14}
\end{bmatrix} \]

- Solutions

<table>
<thead>
<tr>
<th>( \delta_{11} )</th>
<th>( \delta_{12} )</th>
<th>( \delta_{13} )</th>
<th>( \delta_{14} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>2.5</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>1.0</td>
<td>2.0</td>
<td>2.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
1 Factor, 3 Indicators

* Just-Identified; always fits perfectly

\[
\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{pmatrix}
\]

\[
\Sigma_{\psi} = \begin{pmatrix}
\lambda_{11} \phi_{11} & \lambda_{12} \phi_{11} + \theta_{22} \\
\lambda_{21} \phi_{11} & \lambda_{21}^2 \phi_{11} + \theta_{11}
\end{pmatrix}
\]

Identification Rules

* Number of free parameters: \( t \leq \frac{1}{2} q (q+1) \)

* Three-Indicator Rule
  - Exactly 1 non zero element per row of \( \Lambda \)
  - 3 or more indicators per factor
  - \( \Theta \) Diagonal - uncorrelated errors

* Two-Indicator Rule
  - \( \phi_{ij} \neq 0 \) for at least one pair \( i, j, i \neq j \)
  - Exactly 1 non-zero element per row of \( \Lambda \)
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Scaling the Latent Variables

* The scale/metric of the latent variable is not determinant
  - Factor loadings and variances can take on any value unless the metric is specified

* Must impose a model constraint to yield a meaningful scale

* Two Constraints are possible:
  - Fix a loading to 1.0 from each factor to one of its indicators
    - Latent scale takes on the metric of the constrained indicator
  - Fix the latent variance to 1.0
    - Yields a standardized latent variable
### Covariance or Correlation matrix
- Cudeck shows that the covariance matrix is preferred for CFA
- Use of a correlation matrix can result in many problems:
  - Incorrect parameter estimates
  - Incorrect standard errors
  - Incorrect test statistics

### Parameter Evaluation – Local Fit
- **Parameter Estimate/SE = Z-statistic**
- Standard interpretation:
  - if $|Z| > 2$, then “significant”
- Consider both statistical and scientific value of including a variable in the model
- Significance testing in CFA:
  - Not usually interesting in determining if loadings are equal to zero
  - Might be interested in testing whether or not covariance between factors is zero.

### Goodness of Fit – Global Fit
- **Absolute indices**
  - derived from the fit of the obtained and implied covariance matrices and the ML minimization function.
  - Chi-square & functions of it
- **Relative fit indices**
  - Relative to baseline “worst-fitting” model
- **Adjusted fit indices**
  - Relative to number of parameters in model
  - Parsimony

### Chi Square: $\chi^2$
- $F_{crit}(n-1)$
- Sensitive to sample size
  - The larger the sample size, the more likely the rejection of the model and the more likely a Type II error (rejecting something true). In very large samples, even tiny differences between the observed model and the perfect-fit model may be found significant.
  - Informative when sample sizes are relatively small (100-200)
- Chi-square fit index is also very sensitive to violations of the assumption of multivariate normality

### Relative Fit Indices
- Fit indices that provide information relative to a baseline model
  - a model in which all of the correlations or covariances are zero
  - Very poor fit
  - Termed Null or Independence model
- Permits evaluation of the adequacy of the target model

### Goodness of Fit Index (GFI)
- $GFI = 1 - \frac{\chi^2_{model}}{\chi^2_{null}}$
- % of observed covariances explained by the covariances implied by the model
  - Ranges from 0-1
- Biased
  - Biased up by large samples
  - Biased downward when degrees of freedom are large relative to sample size
- GFI is often higher than other fit indices
  - Not commonly used any longer
**Normed Fit Index**  
(Bentler & Bonnet, 1980)

- Compare to a **Null or Independence** model — a model in which all of the correlations or covariances are zero
  \[ NFI = \frac{(X_{\text{null}}^2 - X_{\text{model}}^2)}{X_{\text{null}}^2} \]
- % of total covariance among observed variables explained by target model when using the null (independence) model as baseline - Hu & Bentler, 1995
- Not penalized for a lack of parsimony
- Not commonly used anymore

**Non-Normed Fit Index**  
-Tucker & Lewis, 1973

\[ \text{NNFI} = \frac{(X_{\text{null}}^2/df_{\text{null}} - X_{\text{model}}^2/df_{\text{model}})}{X_{\text{model}}^2/df_{\text{model}}} \]
- Penalizes for fitting too many parameters
- May be greater than 1.0
  - If so, set to 1.0

**Comparative Fit Index**  
-Bentler, 1989; 1990

\[ CFI = \frac{(X_{\text{null}}^2 - df_{\text{null}}) - (X_{\text{model}}^2 - df_{\text{model}})}{(X_{\text{null}}^2 - df_{\text{null}})} \]
- Based on noncentrality parameter for chi-square distribution
- Indicates reduction in model misfit of a target model relative to a baseline (independence) model
- Can be greater than 1.0 or less than 0.0
  - If so, set to 1.0 or 0.0

**Root Mean Square Error of Approximation (RMSEA)**

\[ \text{RMSEA} = \sqrt{\frac{X_{\text{null}}^2/(df_{\text{null}} - 1)}{N-1}} \]
- Discrepancy per degree of freedom
- \( \text{RMSEA} \leq 0.05 \Rightarrow \text{Close fit} \)
- \( 0.05 < \text{RMSEA} \leq 0.08 \Rightarrow \text{Reasonable fit} \)
- \( \text{RMSEA} > 0.1 \Rightarrow \text{Poor fit} \)

**Standardized Root Mean Square Residual (SRMR)**

- Standardized difference between the observed covariance and predicted covariance
- A value of zero indicates perfect fit
- This measure tends to be smaller as sample size increases and as the number of parameters in the model increases.

**General fit standards**

- NFI
  - \( \geq 0.95 \Rightarrow \text{acceptable}; \text{above } 0.95 \text{ is good} \)
  - NFI positively correlated with sample size
- NNFI
  - \( \geq 0.95 \Rightarrow \text{acceptable}; \text{above } 0.95 \text{ is good} \)
  - NFI and NNFI not recommended for small sample sizes
- CFI
  - \( \geq 0.95 \Rightarrow \text{acceptable}; \text{above } 0.95 \text{ is good} \)
  - No systematic bias with small sample size
General fit standards

- **RMSE**
  - Should be close to zero
  - 0.0 to 0.05 is good fit
  - Greater than 0.05 is moderate fit

- **SRMR**
  - Less than 0.08 is good fit

- **Hu & Bentler**
  - SRMR ≤ 0.08 AND (CFI ≥ .95 OR RMSEA ≤ .06)

Nested Model Comparisons

- Test between equivalent models except for a subset of parameters that are freely estimated in one and fixed in the other
- Difference in –2LL is distributed as chi-square variate
  - Each model has a χ² value based upon a certain degree of freedom
  - If models are nested (ie., identical but M₂ deletes one parameter found in M₁), significance of 'decrement' in fit = χ²₂ - χ²₁ with df = df₁ - df₂

Modification indexes

- If the model does not fit well, modification indices may be used to guide respecification (ie. how to improve the model)
- For CFA, the only sensible solutions are to add direct paths from construct to indicator to drop paths
- Rarely is it reasonable to let residuals covary as many times suggested by the output
- Respecify the model or drop factors or indicators

Reliability in CFA

\[ r_{xx} = \frac{\left( \sum \lambda_i^2 \right)^2}{\left( \sum \lambda_i^2 \right)^2 + \sum (1 - \lambda_i^2)} \]

- Standard reliability assumes tau-equivalence.
- Can estimate reliability in CFA with no restrictions
  - Congeneric measures

Equivalent models

- Equivalent models exist for almost all models
- Most quant people say you should evaluate alternative models
- Most people don’t do it
- For now, be aware that there are alternative models that will fit as well as your model

Heirarchical CFA

Quality of life for adolescents: Assessing measurement properties using structural equation modeling
Lynn B. Meuleners, Andy H. Lee, Colin W. Binns & Anthony Lower
Hierarchical CFA

Depression (CES-D)

Negative Affect

Positive Affect

Somatic Symptoms

Model fit statistics: N = 868, χ^2(26) = 68.690, p < .001, SRMR = .055, IFI = .976

All loadings significant at p < .001.

Example...AMOS
Confirmatory Factor Analysis

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Purpose

• Takes factor analysis a few steps further.

• Impose theoretically interesting constraints on the model and examine the resulting fit of the model with the observed data

• Used to evaluate theoretical measurement structures

• Provides tests and indices to evaluate fit
Purpose

• CFA model is constructed in advance
  - specifies the number of (latent) factors
  - specifies the pattern of loadings on the factors
  - that specifies the pattern of unique variances specific to each observation
  - measurement errors may be correlated - Yuck!
  - factor loadings can be constrained to be zero (or any other value)
  - covariances among latent factors can be estimated or constrained
  - multiple group analysis is possible

• Can TEST if these constraints are consistent with the data
**CFA Model & Notation**

\[ x = \Lambda \xi + \delta \]

Where:

- \( x = (q \times 1) \) vector of indicator/manifest variable
- \( \Lambda = (q \times n) \) matrix of factor loadings
  - lambda
- \( \xi = (n \times 1) \) vector of latent constructs (factors)
  - ksi or xi
- \( \delta = (q \times 1) \) vector of errors of measurement
  - delta
CFA Example

• Measures for positive emotions, $\xi_1$:
  - $x_1 = \text{Happiness}$, $x_2 = \text{Pride}$

• Measures for negative emotions $\xi_2$:
  - $x_3 = \text{Sadness}$, $x_4 = \text{Fear}$

• Model

\[
\begin{align*}
x_1 &= \lambda_{11} \xi_1 + \delta_1 \\
x_2 &= \lambda_{12} \xi_1 + \delta_2 \\
x_3 &= \lambda_{21} \xi_2 + \delta_3 \\
x_4 &= \lambda_{22} \xi_2 + \delta_4
\end{align*}
\]
CFA Example

More on the 1.0s later
CFA Model Matrices

\[ x_1 = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ 0 & \lambda_{32} \\ 0 & \lambda_{42} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \]

\[ x = \Lambda \xi + \delta \]
More Matrices

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix} =
\begin{pmatrix}
  \lambda_{11} & 0 \\
  \lambda_{21} & 0 \\
  0 & \lambda_{32} \\
  0 & \lambda_{42}
\end{pmatrix}
\begin{pmatrix}
  \xi_1 \\
  \xi_2
\end{pmatrix} +
\begin{pmatrix}
  \delta_1 \\
  \delta_2 \\
  \delta_3 \\
  \delta_4
\end{pmatrix}
\]

Theta-delta

\[\Phi = \begin{pmatrix}
  \phi_{11} & \phi_{12} \\
  \phi_{21} & \phi_{22}
\end{pmatrix}\]

\[\Theta = \begin{pmatrix}
  \delta_{11} & 0 & 0 & 0 \\
  0 & \delta_{22} & 0 & 0 \\
  0 & 0 & \delta_{33} & 0 \\
  0 & 0 & 0 & \delta_{44}
\end{pmatrix}\]
Model Fitting

- The specified model results in an implied variance-covariance matrix, \( \Sigma_\theta \)

\[
\Sigma_\theta = \Lambda \Phi \Lambda + \Theta
\]

\[
\Lambda \Phi \Lambda + \Theta =
\begin{pmatrix}
\lambda_{11}^2 \phi_{11} + \theta_{11} & \lambda_{21} \lambda_{11} \phi_{11} & \lambda_{21}^2 \phi_{11} + \theta_{22} \\
\lambda_{32} \lambda_{11} \phi_{12} & \lambda_{32} \lambda_{21} \phi_{12} & \lambda_{32}^2 \phi_{22} + \theta_{33} \\
\lambda_{42} \lambda_{11} \phi_{12} & \lambda_{42} \lambda_{21} \phi_{12} & \lambda_{42} \lambda_{32} \phi_{22} + \theta_{44}
\end{pmatrix}
\]
Parameter Estimation

- Maximum Likelihood Estimation
  - Assumes multivariate normality
  - Iterative procedure
  - Requires starting values

\[ F_{ML} = \text{Tr}(SC-1) - n + \ln(\det(C)) - \ln(\det(S)) \]
- \( S \) is the sample variance-covariance matrix
- \( C \) is the implied variance-covariance matrix
**Model Identification**

- **Definition:**
  - The set of parameters $\theta=\{\Lambda, \Phi, \Theta\}$ is not identified if there exists $\theta_1 \neq \theta_2$ such that $\Sigma(\theta_1) = \Sigma(\theta_2)$. 
1 Factor, 2 indicators

• Not Identified!

• Population covariance matrix

\[
\Sigma = \begin{bmatrix}
\sigma_{yy} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{xx}
\end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}
\]

• Implied Covariance Matrix

• Solutions

\[
\Sigma_{\theta} = \begin{bmatrix}
\lambda_{11}^2 \phi_{11} + \theta_{11} \\
\lambda_{21} \lambda_{11} \phi_{11} & \lambda_{12}^2 \phi_{11} + \theta_{22}
\end{bmatrix}
\]

\[
\begin{align*}
\phi_{11} &= 5.0 & \phi_{11} &= 2.5 \\
\lambda_{21} &= 1.0 & \lambda_{21} &= 2.0 \\
\theta_{11} &= 5.0 & \theta_{11} &= 7.5 \\
\theta_{22} &= 5.0 & \theta_{22} &= 0.0
\end{align*}
\]
1 Factor, 3 Indicators

- Just-Identified; always fits perfectly

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{21} & \sigma_{22} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\phi_{11} + \theta_{11} \\
\lambda_{21} \phi_{11} & \lambda_{21}^2 \phi_{11} + \theta_{22} \\
\lambda_{31} \phi_{11} & \lambda_{31} \lambda_{21} \phi_{11} & \lambda_{31}^2 \phi_{11} + \theta_{33}
\end{pmatrix}
\]
1 Factor, 3 Indicators

- Just-Identified; always fits perfectly

\[ \lambda_{21} = \frac{\sigma_{32}}{\sigma_{21}} \]
\[ \lambda_{31} = \frac{\sigma_{32}}{\sigma_{21}} \]
\[ \phi_{11} = \frac{\sigma_{21} \sigma_{31}}{\sigma_{32}} \]
\[ \theta_{11} = \sigma_{11} - \phi_{11} \]
\[ \theta_{22} = \sigma_{22} - \lambda_{21}^2 \phi_{11} \]
\[ \theta_{33} = \sigma_{33} - \lambda_{31}^2 \phi_{11} \]
Identification Rules

• Number of free parameters: \( t \leq \frac{1}{2} q (q+1) \)

• Three-Indicator Rule
  - Exactly 1 non zero element per row of \( \Lambda \)
  - 3 or more indicators per factor
  - \( \Theta \) Diagonal – uncorrelated errors

• Two-Indicator Rule
  - \( \phi_{ij} \neq 0 \) for at least one pair \( i, j, i \neq j \)
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Scaling the latent variables

• The scale/metric of the latent variable is not determinant
  - Factor loadings and variances can take on any value unless the metric is specified

• Must impose a model constraint to yield a meaningful scale

• Two Constraints are possible:
  - Fix a loading to 1.0 from each factor to one of its indicators
    • Latent scale takes on the metric of the constrained indicator
  - Fix the latent variance to 1.0
    • Yields a standardized latent variable
Scaling the Latent Variables

Fix Variances

Fix Path
Scaling the Latent Variables

Fix variances:

\[ x_1 = \lambda_{11} \xi_1 + \delta_1 \]
\[ x_2 = \lambda_{21} \xi_1 + \delta_2 \]
\[ x_3 = \lambda_{31} \xi_1 + \delta_3 \]
\[ x_4 = \lambda_{42} \xi_2 + \delta_4 \]
\[ x_5 = \lambda_{52} \xi_2 + \delta_5 \]
\[ x_6 = \lambda_{62} \xi_2 + \delta_6 \]
\[ \text{cov}(\xi_1, \xi_2) = \varphi_{12} \]
\[ \text{var}(\xi_1) = 1 \]
\[ \text{var}(\xi_2) = 1 \]

Fix path:

\[ x_1 = \xi_1 + \delta_1 \]
\[ x_2 = \lambda_{21} \xi_1 + \delta_2 \]
\[ x_3 = \lambda_{31} \xi_1 + \delta_3 \]
\[ x_4 = \xi_2 + \delta_4 \]
\[ x_5 = \lambda_{52} \xi_2 + \delta_5 \]
\[ x_6 = \lambda_{62} \xi_2 + \delta_6 \]
\[ \text{cov}(\xi_1, \xi_2) = \varphi_{12} \]
\[ \text{var}(\xi_1) = \varphi_{11} \]
\[ \text{var}(\xi_2) = \varphi_{22} \]
Covariance or Correlation matrix

• Cudeck shows that the covariance matrix is prefered for CFA
• Use of a correlation matrix can result in many problems
  - Incorrect parameter estimates
  - Incorrect standard errors
  - Incorrect test statistics
Parameter Evaluation – Local Fit

- **Parameter Estimate/SE** = $Z$-statistic
- Standard interpretation:
  - if $|Z| > 2$, then “significant”
- Consider both statistical and scientific value of including a variable in the model
- Significance testing in CFA:
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• Absolute indices
  – derived from the fit of the obtained and implied covariance matrices and the ML minimization function.
  – Chi-square & functions of it

• Relative fit indices
  – Relative to baseline “worst-fitting” model

• Adjusted fit indices
  – Relative to number of parameters in model
  – Parsimony
Chi Square: $\chi^2$

- $F_{ML}^*(n-1)$
- Sensitive to sample size
  - The larger the sample size, the more likely the rejection of the model and the more likely a Type II error (rejecting something true). In very large samples, even tiny differences between the observed model and the perfect-fit model may be found significant.
  - Informative when sample sizes are relatively small (100-200)
- Chi-square fit index is also very sensitive to violations of the assumption of multivariate normality
Relative Fit Indices

- Fit indices that provide information relative to a baseline model
  - a model in which all of the correlations or covariances are zero
  - Very poor fit
  - Termed Null or Independence model

- Permits evaluation of the adequacy of the target model
Goodness of Fit Index (GFI)

\[ GFI = 1 - \frac{\chi^2_{\text{model}}}{\chi^2_{\text{null}}} \]

• % of observed covariances explained by the covariances implied by the model
  - Ranges from 0-1

• Biased
  - Biased up by large samples
  - Biased downward when degrees of freedom are large relative to sample size

• GFI is often higher than other fit indices
  - Not commonly used any longer
Normed Fit Index
(Bentler & Bonnet, 1980)

• Compare to a Null or Independence model
  – a model in which all of the correlations or covariances are zero

\[ NFI = \frac{\left( \chi^2_{null} - \chi^2_{model} \right)}{\chi^2_{model}} \]

• % of total covariance among observed variables explained by target model when using the null (independence) model as baseline - Hu & Bentler, 1995

• Not penalized for a lack of parsimony
• Not commonly used anymore
Non-Normed Fit Index
  - Tucker & Lewis, 1973

\[
NNFI = \frac{(\chi^2_{null} / df_{null} - \chi^2_{model} / df_{model})}{\chi^2_{model} / df_{model}}
\]

- Penalizes for fitting too many parameters
- May be greater than 1.0
  - If so, set to 1.0
Comparative Fit Index
-Bentler, 1989; 1990

\[
CFI = \frac{(\chi^2_{null} - df_{null}) - (\chi^2_{model} - df_{model})}{(\chi^2_{model} - df_{model})}
\]

- Based on noncentrality parameter for chi-square distribution
- Indicates reduction in model misfit of a target model relative to a baseline (independence) model
- Can be greater than 1.0 or less than 0.0
  - If so, set to 1.0 or 0.0
Root Mean Square Error of Approximation (RMSEA)

\[ RMSEA = \sqrt{\frac{\chi^2_{model} \cdot I(df_{model} - 1)}{N - 1}} \]

- Discrepancy per degree of freedom
- \( RMSEA \leq 0.05 \Rightarrow \text{Close fit} \)
- \( 0.05 < RMSEA \leq 0.08 \Rightarrow \text{Reasonable fit} \)
- \( RMSEA > 0.1 \Rightarrow \text{Poor fit} \)
Standardized Root Mean Square Residual (SRMR)

• Standardized difference between the observed covariance and predicted covariance
• A value of zero indicates perfect fit
• This measure tends to be smaller as sample size increases and as the number of parameters in the model increases.
General fit standards

- **NFI**
  - .90-.95 = acceptable; above .95 is good
  - NFI positively correlated with sample size

- **NNFI**
  - .90-.95 = acceptable; above .95 is good
  - NFI and NNFI not recommended for small sample sizes

- **CFI**
  - .90-.95 = acceptable; above .95 is good
  - No systematic bias with small sample size
General fit standards

• RMSE
  - Should be close to zero
  - 0.0 to 0.05 is good fit
  - 0.05 to 0.08 is moderate fit
  - Greater than .10 is poor fit

• SRMR
  - Less than .08 is good fit

• Hu & Bentler
  - SRMR ≤ .08 AND (CFI ≥ .95 OR RMSEA ≤ .06)
Nested Model Comparisons

• Test between equivalent models except for a subset of parameters that are freely estimated in one and fixed in the other

• Difference in $-2\text{LL}$ is distributed as chi-square variate
  – Each model has a $\chi^2$ value based upon a certain degree of freedom
  – If models are nested (i.e., identical but $M_2$ deletes one parameter found in $M_1$), significance of “increment” or “decrement” in fit = $\chi^2_1 - \chi^2_2$ with df = df$_1$ – df$_2$
Modification indexes

• If the model does not fit well, modification indices may be used to guide respecification (ie. how to improve the model)

• For CFA, the only sensible solutions are to add direct paths from construct to indicator to drop paths

• Rarely is it reasonable to let residuals covary as many times suggested by the output

• Respecify the model or drop factors or indicators
Reliability in CFA

\[ r_{xx} = \frac{(\sum \lambda_i)^2}{(\sum \lambda_i)^2 + \sum (1 - \lambda_i^2)} \]

- Standard reliability assumes tau-equivalence.
- Can estimate reliability in CFA with no restrictions
  - Congeneric measures
Equivalent models

• Equivalent models exist for almost all models
• Most quant people say you should evaluate alternative models
• Most people don't do it

• For now, be aware that there are alternative models that will fit as well as your model
Quality of life for adolescents: Assessing measurement properties using structural equation modelling
Lynn B. Meuleners, Andy H. Lee, Colin W. Binns & Anthony Lower
Heirarchical CFA

Depression (CES-D)

Positive Affect
  Happy
  Enjoy

Negative Affect
  Bothered
  Blues
  Depressed
  Sad

Somatic Symptoms
  Mind
  Effort
  Sleep

Model Fit Statistics: N=868, χ²(26)=68.690, p<.001, SRMR=.055, IFI=.976

*a Second-order loadings were set equal for empirical identification.
All loadings significant at p < .001.
Example...AMOS