# Philosophy of Logical Consequence

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Logical consequence is arguably the central concept of logic. The primary aim of logic is to tell us what follows logically from what. In order to simplify matters we take the logical consequence relation to hold for sentences rather than for abstract propositions, facts, state of affairs, etc. Correspondingly, logical consequence is a relation between a given class of sentences and the sentences that logically follow. One sentence is said to be a logical consequence of a set of sentences, if and only if, in virtue of logic alone, it is impossible for the sentences in the set to be all true without the other sentence being true as well. If sentence X is a logical consequence of a set of sentences K, then we may say that K implies or entails X, or that one may correctly infer the truth of X from the truth of the sentences in K. For example, Kelly is not at work is a logical consequence of Kelly is not both at home and at work and Kelly is at home. However, the sentence Kelly is not a football fan does not follow from All West High School students are football fans and Kelly is not a West High School student. The central question to be investigated here is: What conditions must be met in order for a sentence to be a logical consequence of others?

One popular answer derives from the work of Alfred Tarski, one of the preeminent logicians of the twentieth century, in his famous 1936 paper, “The Concept of Logical Consequence.” Here Tarski uses his observations of the salient features of what he calls the common concept of logical consequence to guide his theoretical development of it. Accordingly, we begin by
examining the common concept focusing on Tarski’s observations of the criteria by which we intuitively judge what follows from what and which Tarski thinks must be reflected in any theory of logical consequence. Then we examine two theoretic characterizations of logical consequence: the model theoretic and the deductive theoretic characterizations. They represent two major approaches to making the common concept of logical consequence more precise. Later in the article, criticisms of Tarski’s characterization of the common concept are entertained and challenges to the two theories of consequence are considered.

2. **The Concept of Logical Consequence**

2.1 *Tarski’s characterization of the common concept of logical consequence*

Tarski begins his article, “On the Concept of Logical Consequence,” by noting a challenge confronting the project of making precise the common concept of logical consequence.

The concept of *logical consequence* is one of those whose introduction into a field of strict formal investigation was not a matter of arbitrary decision on the part of this or that investigator; in defining this concept efforts were made to adhere to the common usage of the language of everyday life. But these efforts have been confronted with the difficulties which usually present themselves in such cases. With respect to the clarity of its content the common concept of consequence is in no way superior to other concepts of everyday language. Its extension is not sharply bounded and its usage fluctuates. Any attempt to bring into harmony all possible vague, sometimes contradictory, tendencies which are connected with the use of this concept, is certainly doomed to failure. We must reconcile ourselves from the start to the fact that every precise definition of this concept will show arbitrary features to a greater or less degree (1933) p. 409.

Not every feature of the technical account will be reflected in the ordinary concept, and we should not expect any clarification of the concept to reflect each and every deployment of it in everyday language and life. Nevertheless, despite its vagueness, Tarski believes that there are identifiable, essential features of the common concept of logical consequence.

…consider any class K of sentences and a sentence X which follows from this class. From an intuitive standpoint, it can never happen that both the class K consists of only true sentences and the sentence X is false. Moreover, since we are concerned here with the concept of logical, i.e., formal consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects to which the sentence X or the sentences of class K refer. The consequence relation cannot be affected by replacing designations of the objects referred to in these sentences by the designations of any other objects. (1936, pp.414-415)
According to Tarski, the logical consequence relation as it is employed by typical reasoners is (1) necessary, (2) formal, and (3) not influenced by empirical knowledge. I now elaborate on (1)-(3) in order to shape two preliminary characterizations of logical consequence.

2.1.1 The logical consequence relation has a modal element

Tarski countenances an implicit modal notion in the common concept of logical consequence. If X is a logical consequence of K, then not only is it the case that not all of the elements of K are true and X is false, but also this is necessarily the case. That is, X follows from K only if it is not possible for all of the sentences in K to be true with X false. For example, the supposition that All West High School students are football fans and that Kelly is not a West High School student does not rule out the possibility that Kelly is a football fan. Hence, the sentences All West High School students are football fans and Kelly is not a West High School student do not entail Kelly is not a football fan, even if she, in fact, isn’t a football fan. Also, Most of Kelly’s male classmates are football fans does not entail Most of Kelly’s classmates are football fans. What if the majority of Kelly’s class is composed of females who are not fond of football?

We said above that Kelly is not both at home and at work and Kelly is at home jointly imply Kelly is not at work. Note that it doesn’t seem possible for the first two sentences to be true and Kelly is not at work false. But it is hard to see what this comes to without further clarification of the relevant notion of possibility. For example, consider the following pairs of sentences.

- Kelly kissed her sister at 2:00pm
- 2:00pm is not a time during which Kelly and her sister were 100 miles apart
- There is a chimp in Paige’s house
- There is a primate in Paige’s house
- There is a female
- Kelly is a female
- Kelly is not the US President
- Kelly is not at work
- 10 is a prime number
- 10 is greater than 9

For each pair of sentences, there is a sense in which it is not possible for the first to be true and the second false. At the very least an account of logical consequence must distinguish logical possibility from other types of possibility. Should truths about physical laws, US political history, zoology, and mathematics constrain what we take to be possible in determining whether
or not the first sentence of each pair could logically be true with the second sentence false? If not, then this seems to mystify logical possibility (e.g., how could 10 be a prime number?). To paraphrase questions asked by G.E. Moore (1959, pp. 231-238), given that I know that George W. Bush is US President and that he is not a female named Kelly, isn’t it inconsistent for me to grant the logical possibility of the truth of *Kelly is a female* and the falsity of *Kelly is not the US President*? Or should I ignore my present state of knowledge in considering what is logically possible? Tarski does not derive a clear notion of logical possibility from the common concept of logical consequence. Perhaps there is none to be had, and we should seek the help of a proper theoretical development in clarifying the notion of logical possibility. Towards this end, let’s turn to the other features of logical consequence highlighted by Tarski, starting with the formality criterion of logical consequence.

2.1.2 *The logical consequence relation is formal*

Tarski observes that logical consequence is a formal consequence relation. And he tells us that a formal consequence relation is a consequence relation that is uniquely determined by the form of the sentences between which it holds. Consider the following pair of sentences

(1) Some children are both lawyers and peacemakers
(2) Some children are peacemakers

Intuitively, (2) is a logical consequence of (1). It appears that this fact does not turn on the subject matter of the sentences. Replace ‘children’, ‘lawyers’, and ‘peacemakers’ in (1) and (2) with the variables $S$, $M$, and $P$ to get the following.

(1') Some $S$ are both $M$ and $P$
(2') Some $S$ are $P$

(1') and (2') are forms of (1) and (2), respectively. Note that there is no interpretation of $S$, $M$, and $P$ according to which the sentence that results from (1') is true and the resulting instance of (2') is false. Hence, (2) is a formal consequence of (1) and on each interpretation of $S$, $M$, and $P$ the resulting (2') is a formal consequence of the sentence that results from (1') (e.g., *Some clowns are sad* is a formal consequence of *Some clowns are both lonely and sad*). Tarski’s observation is
that for any sentence $X$ and set $K$ of sentences, $X$ is a logical consequence of $K$ only if $X$ is a formal consequence of $K$. The formality criterion of logical consequence can work in explaining why one sentence doesn’t entail another in cases where it seems impossible for the first to be true and the second false. For example, (3) is false and (4) is true.

(3) Ten is a prime number
(4) Ten is greater than nine

Does (4) follow from (3)? One might think that (4) does not follow from (3) because being a prime number does not necessitate being greater than nine. However, this does not require one to think that ten could be a prime number and less than or equal to nine, which is probably a good thing since it is hard to see how this is possible. Rather, we take

(3') $a$ is a P
(4') $a$ is R than $b$

to be the forms of (5) and (6) and note that there are interpretations of ‘a’, ‘b’, ‘P’, and ‘R’ according to which the first is true and the second false (e.g. let ‘a’ and ‘b’ name the numbers two and ten, respectively, and let ‘P’ mean prime number, and ‘R’ greater). Note that the claim here is not that formality is sufficient for a consequence relation to qualify as logical but only that it is a necessary condition. I now elaborate on this last point by saying a little more about forms of sentences (i.e., sentential forms) and formal consequence.

Distinguishing between a term of a sentence replaced with a variable and one held constant determines a form of the sentence. In Some children are both lawyers and peacemakers we may replace ‘Some’ with a variable and treat all the other terms as constant. Then

(1'') D children are both lawyers and peacemakers

is a form of (1), and each sentence generated by assigning a meaning to $D$ shares this form with (1). For example, the following three sentences are instances of (1''), produced by interpreting $D$ as ‘No’, ‘Many’, and ‘Few’.

No children are both lawyers and peacemakers
Many children are both lawyers and peacemakers
Few children are both lawyers and peacemakers

Whether X is a formal consequence of K then turns on a prior selection of terms as constant and others replaced with variables. Relative to such a determination, X is a formal consequence of K if and only if (iff) there is no interpretation of the variables according to which each of the K are true and X is false. So, taking all the terms, except for ‘Some’, in (1) *Some children are both philosophers and peacemakers* and in (2) *Some children are peacemakers* as constants makes the following forms of (1) and (2).

(1") D children are both lawyers and peacemakers
(2") D children are peacemakers

Relative to this selection, (2) is not a formal consequence of (1) because replacing ‘D’ with ‘No’ yields a true instance of (1") and a false instance of (2").

Consider the following pair.

(5) Kelly is female
(6) Kelly is not US President

(6) is a formal consequence of (5) relative to replacing ‘Kelly’ with a variable. Given current U.S. political history, there is no individual whose name yields a true (5) and a false (6) when it replaces ‘Kelly’. This is not, however, sufficient reason for seeing (6) as a logical consequence of (5). There are two ways of thinking about why, a metaphysical consideration and an epistemological one. First the metaphysical consideration. It seems possible for (5) to be true and (6) false. The course of U.S. political history could have turned out differently. One might think that the current US President could—logically—have been a female named, say, ‘Sally’. Using ‘Sally’ as a replacement for ‘Kelly’ would yield in that situation a true (5) and a false (6). Also, it seems possible that in the future there will be a female US President. In order for a formal consequence relation from K to X to qualify as logical it has to be the case that it is necessary that there is no interpretation of the variables in K and X according to which the K-sentences are true and X is false.
The epistemological consideration is that one might think that knowledge that X follows logically from K should not essentially depend on being justified by experience of extra-linguistic states of affairs. Clearly, the determination that (6) follows formally from (5) essentially turns on empirical knowledge, specifically knowledge about the current political situation in the US. This leads to the final highlight of Tarski’s rendition of the intuitive concept of logical consequence: that logical consequence cannot be influenced by empirical knowledge.

2.1.3 The logical consequence relation is a priori

Tarski says that by virtue of being formal, knowledge that X follows logically from K cannot be affected by knowledge of the objects that X and the sentences of K are about. Hence, our knowledge that X is a logical consequence of K cannot be influenced by empirical knowledge. However, as noted above, formality by itself does not insure that the extension of a consequence relation is not influenced by empirical knowledge. So, let’s view this alleged feature of logical consequence as independent of formality. We characterize empirical knowledge in two steps as follows. First, a priori knowledge is knowledge “whose truth, given an understanding of the terms involved, is ascertainable by a procedure which makes no reference to experience” (Hamlyn (1967), p. 141). Empirical, or a posteriori, knowledge is knowledge that is not a priori, i.e., knowledge whose validation necessitates a procedure that does make reference to experience.

Knowledge that

\[ 2+2=4, \]
\[ \text{If I am conscious, then I exist, and} \]
\[ \text{The shortest distance between two points in a plane is a straight line,} \]

is a priori. While knowledge that

George W. Bush is a Republican,
If Kelly is female, then she is not the US President, and
If that man is the Pope, then that man is Polish,

is empirical knowledge. We can safely read Tarski as saying that a consequence relation is logical only if knowledge that something falls in its extension is a priori, i.e., only if the relation
is a priori. Knowledge of physical laws, a determinant in people’s observed sizes, is not a priori and such knowledge is required to know that there is no interpretation of $k$, $h$, and $t$ according to which (7) is true and (8) false.

(7) $k$ kissed $h$ at time $t$
(8) $t$ is not a time during which $k$ and $h$ were 100 miles apart

So (8) cannot be a logical consequence of (7). However, my knowledge that *Kelly is not Paige’s only friend* follows from *Kelly is taller than Paige’s only friend* is a priori since I know a priori that *nobody is taller than herself*.

Let’s summarize and tie things together. We began by asking, for a given language $L$, what conditions must be met in order for a sentence $X$ of $L$ to be a logical consequence of a class $K$ of $L$-sentences? Tarski thinks that an adequate response must reflect the common concept of logical consequence, i.e., the concept as it is ordinarily employed. By the lights of this concept, an adequate account of logical consequence must reflect the formality and necessity of logical consequence, and must also reflect the fact that knowledge of what follows logically from what is a priori. Tying the criteria together, in order to fix what follows logically from what in a given language $L$, we must select a class of constants that determines a formal consequence relation that is both necessary and known, if at all, a priori. Such constants are called logical constants, and we say that the logical form of a sentence is a function of the logical constants that occur in the sentence and the pattern of the remaining expressions. As was illustrated above, the notion of formality does not presuppose a criterion of logical constancy. A consequence relation based on any division between constants and terms replaced with variables will automatically be formal with respect to the latter.

2.2 Logical and Non-logical terms

Tarski’s basic move from his rendition of the common concept of logical consequence is to distinguish between logical constants and non-logical terms and then say that $X$ is a logical consequence of $K$ only if there is no possible interpretation of the non-logical terms of the
language L that makes all of the sentences in K true and X false. The choice of the right terms as logical will reflect the modal element in the concept of logical consequence, i.e. will insure that there is no ‘possible’ interpretation of the variable, non-logical terms of the language L that makes all of the K true and X false, and will insure that this is known a priori. Of course, we have yet to spell out the modal notion in the concept of logical consequence. Tarski pretty much left this underdeveloped in his (1936). Lacking such an explanation hampers our ability to clarify the rationale for a selection of terms to serve as the logical ones.

Traditionally, logicians have regarded sentential connectives such as and, not, or, if…then, the quantifiers all and some, and the identity predicate = as logical terms. Remarking on the boundary between logical and non-terms, Tarski writes the following.

Underlying this characterization of logical consequence is the division of all terms of the language discussed into logical and extra-logical. This division is not quite arbitrary. If, for example, we were to include among the extra-logical signs the implication sign, or the universal quantifier, then our definition of the concept of consequence would lead to results which obviously contradict ordinary usage. On the other hand, no objective grounds are known to me which permit us to draw a sharp boundary between the two groups of terms. It seems to be possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stands in sharp contrast to ordinary usage. (1936, p. 419)

Tarski seems right to think that the logical consequence relation turns on the work that the logical terminology does in the relevant sentences. It seems odd to say that Kelly is happy does not logically follow from All are happy because the second is true and the first false when All is replaced with Few. However, by Tarski’s version of the ordinary concept of logical consequence there is no reason not to treat say taller than as a logical term along with not and, therefore no reason not to take Kelly is not taller than Paige as following logically from Paige is taller than Kelly. Also, it seems plausible to say that I know a priori that there is no possible interpretation of Kelly and is mortal according to which it is necessary that Kelly is mortal is true and Kelly is mortal is false. This makes Kelly is mortal a logical consequence of it is necessary that Kelly is mortal. Given that taller than and it is necessary that, along with other terms, were not generally regarded as logical terms by logicians of Tarski’s day, the fact that they seem to be logical terms
by the common concept of logical consequence, as observed by Tarski, highlights the question of what it takes to be a logical term. Tarski says that future research will either justify the traditional boundary between the logical and the non-logical or conclude that there is no such boundary and the concept of logical consequence is a relative concept whose extension is always relative to some selection of terms as logical (p.420). We shall further discuss Tarski’s views on logical terminology below at the end of section four and relate it to contemporary views.

How, exactly, does the terminology usually regarded by logicians as logical work in making it the case that one sentence follows from others? In the next two sections two distinct approaches to understanding the nature of logical terms are sketched. Each approach leads to a unique way of characterizing logical consequence and thus yields a unique response to the above question.

2.2.1 The nature of logical constants explained in terms of their semantic properties

Consider the following metaphor, borrowed from Bencivenga, (1999).

The locked room metaphor

Suppose that you are locked in a dark windowless room and you know everything about your language but nothing about the world outside. A sentence X and a class K of sentences are presented to you. If you can determine that X is true if all the sentences in K are, X is a logical consequence of K.

Ignorant of US politics, I couldn’t determine the truth of Kelly is not US President solely on the basis of Kelly is a female. However, behind such a veil of ignorance I would be able to tell that Kelly is not US President is true if Kelly is female and Kelly is not US President is true. How? Short answer: based on my linguistic competence; longer answer: based on my understanding of the semantic contribution of and to the determination of the truth conditions of a sentence of the form P and Q. For any sentences P and Q, I know that P and Q is true just in case P is true and Q is true. So, I know, a priori, if P and Q is true, then Q is true. As noted by one philosopher, “This really is remarkable since, after all, it’s what they mean, together with the facts about the non-linguistic world, that decide whether P or Q are true” (Fodor 2000, p.12).
Taking *not* and *and* to be the only logical constants in (9) *Kelly is not both at home and at work*, (10) *Kelly is at home*, and (11) *Kelly is not at work*, we formalize the sentences as follows, letting *k* mean *Kelly*, *H* mean *is at home*, and *W* mean *is at work*.

(9') not-(Hk and Wk)  
(10') Hk  
(11') not-Wk

There is no interpretation of *k*, *H*, and *W* according to which (9') and (10') are true and (11') is false. The reason why turns on the semantic properties of *and* and *not*, which are knowable *a priori*. Suppose (9') and (10') are true on some interpretation of the variable terms. Then the meaning of *not* in (9') makes it the case that *Hk and Wk* is false, which, by the meaning of *and* requires that *Hk* is false or *Wk* is false. Given (10'), it must be that *Wk* is false, i.e., *not-Wk* is true. So, there can’t be an interpretation of the variable terms according to which (9’) and (10’) are true and (11’') is false, and, as the above reasoning illustrates, this is due exclusively to the semantic properties of *not* and *and*. So the reason that it is impossible that an interpretation of *k*, *H*, and *W* make (9’) and (10’) true and (11’) false is that the supposition otherwise is inconsistent with the semantic functioning of *not* and *and*. Compare: the supposition that there is an interpretation of *k* according to which *k is a female* is true and *k is not US President* is false does not seem to violate the semantic properties of the constant terms. If we identify the meanings of the predicates with their extensions in all possible worlds, then the supposition that there is a female U.S. President does not violate the meanings of *female* and *US President* for surely it is possible that there be a female US President. But, supposing that (9’) and (10’) could be true with (11’’) false on some interpretation of *k*, *H*, and *W*, violates the semantic properties of either *and* or *not*.

In sum, our first-step characterization of logical consequence is the following. For a given language *L*,

*X* is a logical consequence of *K* if and only if there is no possible interpretation of the non-logical terminology of *L* according to which all the sentence in *K* are true and *X* is false.
A possible interpretation of the non-logical terminology of the language L according to which sentences are true or false is a reading of the non-logical terms according to which the sentences receive a truth-value (i.e., is either true or false) in a situation that is not ruled out by the semantic properties of the logical constants. The philosophical locus of the technical development of ‘possible interpretation’ in terms of models is Tarski (1936). A model for a language L is the theoretical development of a possible interpretation of non-logical terminology of L according to which the sentences of L receive a truth-value. Models have become standard tools for characterizing the logical consequence relation, and the characterization of logical consequence in terms of models is called the Tarskian or model-theoretic characterization of logical consequence.

We say that X is a model-theoretic consequence of K if and only if all models of K are models of X. This relation may be represented as $K \models X$. If model-theoretic consequence is adequate as a representation of logical consequence, then it must reflect the salient features of the common concept, which, according to Tarski means that it must be necessary, formal and *a priori*. Below in section four we shall give a model-theoretic characterization of logical consequence for a sample language, and evaluate it by the lights of common concept of logical consequence.

2.2.2 *The nature of logical constants explained in terms of their inferential properties*

We now turn to a second approach to understanding logical constants. Instead of understanding the nature of logical constants in terms of their semantic properties as is done on the model-theoretic approach, on the second approach we appeal to their inferential properties conceived of in terms of principles of inference, i.e. principles justifying steps in deductions. We begin with a remark made by Aristotle. In his study of logical consequence, Aristotle comments that

A syllogism is discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so. I mean by the last phrase that they produce the consequence, and by this, that no further term is required from without in order to make the consequence necessary. *Prior Analytics* 24b
Adapting this to our X and K, we may say that X is a logical consequence of K when the sentences of K are sufficient to produce X. How are we to think of a sentence being produced by others? One way of developing this is to appeal to a notion of an actual or possible deduction. X is a deductive consequence of K if and only if there is a deduction of X from K. In such a case, we say that X may be correctly inferred from K or that it would be correct to conclude X from K. A deduction is associated with a pair <K, X>; the set K of sentences is the basis of the deduction, and X is the conclusion. A deduction from K to X is a finite sequence S of sentences ending with X such that each sentence in S (i.e., each intermediate conclusion) is derived from a sentence (or more) in K or from previous sentences in S in accordance with a correct principle of inference.

For example, intuitively, the following inference seems correct.

Kelly is not both at home and at work
Kelly is at home
(therefore)         Kelly is not at work

The set K of sentences above the line is the basis of the inference and the sentence X below is the conclusion. We represent their logical forms, again, as follows.

(9') not-(Hk and Wk)
(10') Hk
(therefore)       (11') not-Wk

Consider the following deduction of (11') from (10') and (9').

Deduction: Assume that (12') Wk. Then from (10') and (12') we may deduce that (13') Hk and Wk. (13') contradicts (9') and so (12'), our initial assumption, must be false. We have deduced not-Wk from not-(Hk and Wk) and Hk.

Since the deduction of not-Wk from not-(Hk and Wk) and Hk did not depend on the interpretation of k, W, and H, the deductive relation is formal. Furthermore, my knowledge of this is a priori because my knowledge of the underlying principles of inference in the above deduction is not empirical. For example, letting P and Q be any sentences, we know a priori that P and Q may be inferred from the set K={P, Q} of basis sentences. This principle grounds the move from (10') and (12') to (13'). Also, the deduction appeals to the principle that if we deduce a contradiction from an assumption, then we may infer that the assumption is false. The
correctness of this principle seems to be an *a priori* matter. Let’s look at another example of a deduction.

(1) Some children are both lawyers and peacemakers  
(therefore)   (2) Some children are peacemakers

The logical forms are, again, the following.

(1’) Some S are both M and P  
(therefore )  (2’) Some S are P

Again, intuitively, (2’) is deducible from (1’).

Deduction: The basis tells us that at least one S—let’s call this S ‘a’—is an M and a P. Clearly, *a is a P* may be deduced from *a is an M and a P*. Since we’ve assumed that *a is an S*, what we derive with respect to *a* we derive with respect to some *S*. So our derivation of *a is a P* is a derivation of *Some S is a P*, which is our desired conclusion.

Since the deduction is formal, we have shown not merely that (2) can be correctly inferred from (1), but we have shown that for any interpretation of *S, M, and P* it is correct to infer (2’) from (1’).

Typically, deductions leave out steps (perhaps because they are too obvious), and they usually do not justify each and every step made in moving towards the conclusion (again, obviousness begets brevity). The notion of a deduction is made precise by describing a mechanism for constructing deductions that are both transparent and rigorous (each step is explicitly justified and no steps are omitted). This mechanism is a deductive system (also known as a formal system or as a formal proof calculus). A deductive system D is a collection of rules that govern which sequences of sentences, associated with a given <K, X>, are allowed and which are not. Such a sequence is called a proof in D (or, equivalently, a deduction in D) of X from K. The rules must be such that whether or not a given sequence associated with <K, X> qualifies as a proof in D of X from K is decidable purely by inspection and calculation. That is, the rules provide a purely mechanical procedure for deciding whether a given object is a proof in D of X from K.
We say that a deductive system $D$ is correct when for any $K$ and $X$, proofs in $D$ of $X$ from $K$ correspond to intuitively valid deductions. For example, intuitively, there are no correct principles of inference according to which it is correct to conclude

Some animals are mammals and reptiles

on the basis of the following two sentences.

Some animals are mammals
Some animals are reptiles

Hence, a proof in a deductive system of the former sentence from the latter two is evidence that the deductive system is incorrect. The point here is that a proof in $D$ may fail to represent a deduction if $D$ is incorrect.

A rich variety of deductive systems have been developed for registering deductions. Each system has its advantages and disadvantages, which are assessed in the context of the more specific tasks the deductive system is designed to accomplish. Historically, the general purpose of the construction of deductive systems was to reduce reasoning to precise mechanical rules (Hodges 1983, p. 26). Some view a deductive system defined for a language $L$ as a mathematical model of actual or possible chains of correct reasoning in $L$. Sundholm (1983) offers a thorough survey of three main types of deductive systems. For a shorter, excellent introduction to the concept of a deductive system see Henkin (1967).

If there is a proof of $X$ from $K$ in $D$, then we may say that $X$ is a deductive consequence in $D$ of $K$, which is sometimes expressed as $K \vdash_D X$. Relative to a correct deductive system $D$, we characterize logical consequence in terms of deductive consequence as follows.

$X$ is a logical consequence of $K$ if and only if $X$ is a deductive consequence in $D$ of $K$, i.e. there is an actual or possible proof in $D$ of $X$ from $K$.

This is sometimes called the proof-theoretic characterization of logical consequence. In section five, we shall give a deductive system for a sample language and consider why logical consequence might not be equivalent with deducibility in a deductive system.
2.3 The Model-theoretic and Deductive-theoretic approaches to logic

Let’s step back and summarize where we are at this point in our discussion of the logical consequence relation. We began with Tarski’s observations of the common or ordinary concept of logical consequence that we employ in daily life. According to Tarski, if X is a logical consequence of a set of sentences, K, then, in virtue of the logical forms of the sentences involved, if all of the members of K are true, then X must be true, and furthermore, we know this a priori. The formality criterion makes logical constants the essential determinant of the logical consequence relation. The logical consequence relation is fixed exclusively in terms of the logical terminology. We highlighted two different approaches to the nature of a logical constant: (1) in terms of its semantic contribution to sentences in which it occurs and (2) in terms of its inferential properties. Each yields a distinct conception of the notion of necessity inherent in the common concept of logical consequence, and lead to the following characterizations of logical consequence.

1. X is a logical consequence of K if and only if there is no possible interpretation of the non-logical terminology of the language according to which all the sentence in K are true and X is false.

2. X is a logical consequence of K if and only if X is deducible from K.

We make the notions of possible interpretation in (1) and deducibility in (2) precise by appealing to the technical notions of model and deductive system. This leads to the following theoretical characterizations of logical consequence.

1. The model-theoretic characterization of logical consequence: X is a logical consequence of K iff all models of K are models of X.

2. The deductive- (or proof-) theoretic characterization of logical consequence: X is a logical consequence of K iff there is a deduction in a correct deductive system of X from K.

We said that the primary aim of logic is to tell us what follows logically from what. These two characterizations of logical consequence lead to two different orientations or conceptions of logic (see Tharp (1975), p.5).
Model-theoretic approach: Logic is a theory of possible interpretations. For a given language the class of situations that can—logically—be described by that language.

Deductive-theoretic approach: Logic is a theory of formal deductive inference.

Following Shapiro ((1991) p.3) define a logic to be a language L plus either a model-theoretic or a proof-theoretic account of logical consequence. A language with both characterizations is a full logic just in case both characterizations coincide. In what follows, we offer a sample of a full logic: we first define a language L and then characterize the logical consequence relation in L model-theoretically and proof theoretically.

3. Linguistic Preliminaries: the Language M

Here we define a simple language M, a language about the McKeon family, by first sketching what strings qualify as well-formed formulas (wffs) in M. Next we define sentences from formula, and then give an account of truth in M, i.e. we describe the conditions in which M-sentences are true.

3.1 Syntax of M

Building blocks of formulas

Terms


Predicates

1-place predicates—‘Female’, ‘Male’

Formulas

An atomic wff is any of the above n-place predicates followed by n terms which are enclosed in parentheses and separated by commas. The notion of a well-formed formula (wff) is defined recursively as follows.

Sentential Connectives
(1) All atomic wffs are wffs.
(2) If $\alpha$ is a wff, so is $\neg \alpha$.
(3) If $\alpha$ and $\beta$ are wffs, so is $(\alpha \& \beta)$.  
(4) If $\alpha$ and $\beta$ are wffs, so is $(\alpha \vee \beta)$.  
(5) If $\alpha$ and $\beta$ are wffs, so is $(\alpha \rightarrow \beta)$.  

Quantifiers

(6) If $\Psi$ is a wff and $v$ is a variable, then $\neg \exists v \Psi$ is a wff.  
(7) If $\Psi$ is a wff and $v$ is a variable, then $\neg \forall v \Psi$ is a wff.  

Finally, no string of symbols is a formula of $\mathcal{M}$ unless the string can be derived from (1)-(7).  

It will prove convenient to have available in $\mathcal{M}$ an infinite number of individual names as well as variables. The strings ‘Parent(beth, paige)’ and ‘Male(x)’ are examples of atomic wffs. We allow the identity symbol in an atomic formula to occur in between two terms, e.g., instead of ‘=(evan, evan)’ we allow ‘(evan = evan)’. The symbols ‘$\neg$’, ‘$\&$’, ‘$\vee$’, and ‘$\rightarrow$’ correspond to the English words ‘not’, ‘and’, ‘or’ and ‘if…then’, respectively. ‘$\exists$’ is our symbol for an existential quantifier and ‘$\forall$’ represents the universal quantifier. $\exists v \Psi$ and $\forall v \Psi$ correspond to for some $v$, $\Psi$ and for all $v$, $\Psi$, respectively. For every quantifier, its scope is the smallest part of the wff in which it is contained that is itself a wff. An occurrence of a variable $v$ is a bound occurrence iff it is in the scope of some quantifier of the form ‘$\exists v$’ or the form ‘$\forall v$’, and is free otherwise. For example, the occurrence of $x$ is free in ‘Male(x)’ and in ‘$\exists y \exists x \text{Married}(y,x)$’. The occurrence of $y$ in the second formula is bound because it is in the scope of the existential quantifier. A wff with at least one free variable is an open wff, and a closed formula is one with no free variables. A sentence is a closed wff. For example, ‘Female(kelly)’ and ‘$\exists y \exists x \text{Married}(y,x)$’ are sentences but ‘OlderThan(kelly,y)’ and ‘($\exists x \text{Male}(x) \& \text{Female}(z)$)’ are not. So, not all of the wff of $\mathcal{M}$ are sentences. As noted below, this will affect our definition of truth for $\mathcal{M}$.  

3.2 Semantics for $\mathcal{M}$
We now provide a semantics for M. This is done in two steps. First, we specify a domain of discourse, i.e. the chunk of the world that our language M is about, and interpret M’s predicates and names in terms of the elements composing the domain. Then we state the conditions under which each type of M-sentence is true. To each of the above syntactic rules (1-7) there corresponds a semantic rule that stipulates the conditions in which the sentence constructed using the syntactic rule is true. The principle of bivalence is assumed and so ‘not true’ and ‘false’ are used interchangeably. In effect, the interpretation of M determines a truth-value (true, false) for each and every sentence of M.

**Domain D**—The McKeons: Matt, Beth, Shannon, Kelly, Paige, and Evan.

Here are the referents and extensions of the names and predicates of M.

**Terms**: ‘matt’ refers to Matt, ‘beth’ refers to Beth, ‘shannon’ refers to Shannon, etc…

The meaning of a predicate is identified with its extension, i.e. the set (possibly empty) of elements from the domain D the predicate is true of. The extension of a one-place predicate is a set of elements from D, the extension of a two-place predicate is a set of ordered pairs of elements from D.

The extension of ‘Male’ is \{Matt, Evan\}.

The extension of ‘Female’ is \{Beth, Shannon, Kelly, Paige\}.

The extension of ‘Parent’ is \{<Matt, Shannon>, <Matt, Kelly>, <Matt, Paige>, <Matt, Evan>, <Beth, Shannon>, <Beth, Kelly>, <Beth, Paige>, <Beth, Evan>\}.

The extension of ‘Married’ is \{<Matt, Beth>, <Beth, Matt>\}.

The extension of ‘Sister’ is \{<Shannon, Kelly>, <Kelly, Shannon>, <Shannon, Paige>, <Paige, Shannon>, <Kelly, Paige>, <Paige, Kelly>, <Kelly, Evan>, <Paige, Evan>, <Shannon, Evan>\}.

The extension of ‘Brother’ is \{<Evan, Shannon>, <Evan, Kelly>, <Evan, Paige>\}.

The extension of ‘OlderThan’ is \{<Beth, Matt>, <Beth, Shannon>, <Beth, Kelly>, <Beth, Paige>, <Beth, Evan>, <Matt, Shannon>, <Matt, Kelly>, <Matt, Paige>, <Matt, Evan>, <Shannon, Kelly>, <Shannon, Paige>, <Shannon, Evan>, <Kelly, Paige>, <Kelly, Evan>, <Paige, Evan>\}.

The extension of ‘Admires’ is \{<Matt, Beth>, <Shannon, Matt>, <Shannon, Beth>, <Kelly, Beth>, <Kelly, Matt>, <Kelly, Shannon>, <Paige, Beth>, <Paige, Matt>, <Paige, Shannon>, <Paige, Kelly>, <Evan, Beth>, <Evan, Matt>, <Evan, Shannon>, <Evan, Kelly>, <Evan, Paige>\}.
The extension of ‘=’ is \{<Matt, Matt>, <Beth, Beth>, <Shannon, Shannon>, <Kelly, Kelly>, 
<Paige, Paige>, <Evan, Evan>\}.

(I) An atomic sentence with a one-place predicate is true iff the referent of the term is a
member of the extension of the predicate, and an atomic sentence with a two-place
predicate is true iff the ordered pair formed from the referents of the terms in order is a
member of the extension of the predicate.

The atomic sentence ‘Female(kelly)’ is true because, as indicated above, the referent of
‘kelly’ is in the extension of the property designated by ‘Female’. The atomic sentence
‘Married(shannon, kelly)’ is false because the ordered pair <Shannon, Kelly> is not in the
extension of the relation designated by ‘Married’.

Let \(\alpha\) and \(\beta\) be any M-sentences.

(II) \(\neg \alpha\) is true iff \(\alpha\) is false.

(III) \((\alpha \& \beta)\) is true when both \(\alpha\) and \(\beta\) are true; otherwise \((\alpha \& \beta)\) is false.

(IV) \((\alpha \lor \beta)\) is true when least one of \(\alpha\) and \(\beta\) is true; otherwise \((\alpha \lor \beta)\) is false.

(V) \((\alpha \rightarrow \beta)\) is true if and only if (iff) \(\alpha\) is false or \(\beta\) is true. So, \((\alpha \rightarrow \beta)\) is false just in
case \(\alpha\) is true and \(\beta\) is false.

The meanings for ‘\(~\)’ and ‘\(\&\)’ roughly correspond to the meanings of ‘not’ and ‘and’ as ordinarily
used. We call \(\neg \alpha\) and \((\alpha \& \beta)\) negation and conjunction formulas, respectively. The
formula \((\alpha \lor \beta)\) is called a disjunction and the meaning of ‘\(\lor\)’ corresponds to inclusive-or.

There are a variety of conditionals in English (e.g., causal, counterfactual, logical), each type
having a distinct meaning. The conditional defined by (V) is called the material conditional. One
way of following (V) is to see that the truth conditions for \((\alpha \rightarrow \beta)\) are the same for
\((\neg (\alpha \& \neg \beta))\).

By (II) ‘\(~Married(shannon, kelly)\)’ is true because, as noted above, ‘Married(shannon,kelly)’
is false. (II) also tells us that ‘\(~Female(kelly)\)’ is false since ‘Female(kelly)’ is true. According to
(III), ‘\((\neg Married(shannon,kelly) \& Female(kelly))\)’ is true because ‘\(~Married(shannon, kelly)\)’ is
ture and ‘Female(kelly)’ is true. And ‘\((Male(shannon) \& Female(shannon))\)’ is false because
‘Male(shannon)’ is false. (IV) confirms that ‘(Female(kelly) v Married(evan, evan))’ is true because, even though ‘Married(evan, evan)’ is false, ‘Female(kelly)’ is true. From (V) we know that the sentence ‘(¬(beth=beth) → Male(shannon))’ is true because ‘¬(beth=beth)’ is false. If α is false then ∀(α→β) is true regardless of whether or not β is true. The sentence ‘(Female(beth) → Male(shannon))’ is false because ‘Female(beth)’ is true and ‘Male(shannon)’ is false.

Before describing the truth conditions for quantified sentences we need to say something about the notion of satisfaction. We’ve defined truth only for the formula of M that are sentences. So, the notions of truth and falsity are not applicable to non-sentences such as ‘Male(x)’ and ‘((x=x) → Female(x))’ in which x occurs free. However, objects may satisfy wffs that are non-sentences. We introduce the notion of satisfaction with some examples. An object satisfies ‘Male(x)’ just in case that object is male. Matt satisfies ‘Male(x)’, Beth does not. This is the case because replacing ‘x’ in ‘Male(x)’ with ‘matt’ yields a truth while replacing the variable with ‘beth’ yields a falsehood. An object satisfies ‘((x=x) → Female(x))’ if and only if it is either not identical with itself or is a female. Beth satisfies this wff (we get a truth when ‘beth’ is substituted for the variable in all of its occurrences), Matt does not (putting ‘matt’ in for ‘x’ wherever it occurs results in a falsehood). As a first approximation, we say that an object with a name, say ‘a’, satisfies a wff ∀vΨ in which at most v occurs free if and only if the sentence that results by replacing v in all of its occurrences with ‘a’ is true. ‘Male(x)’ is neither true nor false because it is not a sentence, but it is either satisfiable or not by a given object. Now we define the truth conditions for quantifications, utilizing the notion of satisfaction. The notion of satisfaction will be revisited below when we formalize the semantics for M and give the model-theoretic characterization of logical consequence.

Let Ψ be any formula of M in which at most v occurs free.
(VI) $\exists v \Psi$ is true just in case there is at least one individual in the domain of quantification (e.g. at least one McKeon) that satisfies $\Psi$.

(VII) $\forall v \Psi$ is true just in case every individual in the domain of quantification (e.g. every McKeon) satisfies $\Psi$.

Here are some examples. ‘$\exists x (\text{Male}(x) \& \text{Married}(x, \text{beth}))$’ is true because Matt satisfies ‘$(\text{Male}(x) \& \text{Married}(x, \text{beth}))$’; replacing ‘$x$’ wherever it appears in the wff with ‘matt’ results in a true sentence. The sentence ‘$\exists x \text{OlderThan}(x,x)$’ is false because no McKeon satisfies ‘$\text{OlderThan}(x,x)$’, i.e. replacing ‘$x$’ in ‘$\text{OlderThan}(x,x)$’ with the name of a McKeon always yields a falsehood.

The universal quantification ‘$\forall x (\text{OlderThan}(x, \text{paige}) \rightarrow \text{Male}(x))$’ is false for there is a McKeon who doesn’t satisfy ‘$(\text{OlderThan}(x, \text{paige}) \rightarrow \text{Male}(x))$’. For example, Shannon does not satisfy ‘$(\text{OlderThan}(x, \text{paige}) \rightarrow \text{Male}(x))$’ because Shannon satisfies ‘$\text{OlderThan}(x, \text{paige})$’ but not ‘$\text{Male}(x)$’. The sentence ‘$\forall x (x=x)$’ is true because all McKeons satisfy ‘$x=x$’; replacing ‘$x$’ with the name of any McKeon results in a true sentence.

Note that in the explanation of satisfaction we suppose that an object satisfies a wff only if the object is named. But we don’t want to presuppose that all objects in the domain of discourse are named. For the purposes of an example, suppose that the McKeon’s adopt a baby boy, but haven’t named him yet. Then, ‘$\exists x \text{Brother}(x, \text{evan})$’ is true because the adopted child satisfies ‘$\text{Brother}(x, \text{evan})$’, even though we can’t replace ‘$x$’ with the child’s name to get a truth. To get around this is easy enough. We have added a list of names, ‘$w_1$’, ‘$w_2$’, ‘$w_3$’, etc. to $M$, and we may say that any unnamed object satisfies $\nabla \Psi_\forall$ iff the replacement of $v$ with a previously unused $w_i$ assigned as a name of this object results in a true sentence. In the above scenario, ‘$\exists x \text{Brother}(x, \text{evan})$’ is true because, ultimately, treating ‘$w_1$’ as a temporary name of the child, ‘$\text{Brother}(w_1, \text{evan})$’ is true.

We have characterized an interpreted language $M$ by defining what qualifies as a sentence of $M$ and by specifying the conditions under which any $M$ sentence is true. We complete our
specification of a full logic by defining model theoretic and deductive theoretic consequence relations for M. We shall regard just the sentential connectives, the quantifiers of M, and the identity predicate as logical constants. The language M is a first-order language. The full logic developed below may be viewed as a version of classical logic or a first-order theory.

4. Model-theoretic consequence

As highlighted above in section 2.3, one understanding of the necessity inherent in the common concept of logical consequence leads to the following semantic characterization.

X is a logical consequence of K if and only if there is no possible interpretation of the non-logical terminology of the language according to which all the sentences in K are true and X is false.

As promised, we now make the notion of possible interpretation precise by appealing to the technical notion of truth in a structure, in terms of which we give the model-theoretic characterization of logical consequence. We accomplish this by formalizing the semantics for M and making the notion of satisfaction precise. The technical machinery to follow is designed to clarify how it is that sentences receive truth-values owing to interpretations of them. We begin by introducing the notion of a structure. Then we revisit the notion of satisfaction, and link structures and satisfaction to model-theoretic consequence. We offer a modernized version of the model-theoretic characterization of logical consequence sketched by Tarski and so deviate from the details of Tarski’s presentation in his (1936).

4.1 Truth in a structure

Relative to our language M, a structure U is an ordered pair <D, I>.

1. D, a non-empty set of elements, is the domain of discourse. Two things to highlight here. First, the domain D of a structure for M may be any set of entities, e.g. the dogs living in Connecticut, the toothbrushes on Earth, the natural numbers, the twelve apostles, etc… Second, we require that D not be the empty set.

2. I is a function which assigns to each individual constant of M an element of D, and to each n-place predicate of M a subset of D^n (i.e., a set of n-tuples taken from D). In essence, I
interprets the individual constants and predicates of M, linking them to elements and sets of n-tuples of elements from of D. For individual constants $c$ and predicates $P$, the element $I_U(c)$ is the element of D designated by $c$ under $I_U$, and $I_U(P)$ is the set of entities assigned by $I_U$ as the extension of $P$.

By ‘structure’ we mean an L-structure for some first-order language L. The intended structure for a language L is the course-grained representation of the piece of the world that we intend L to be about. The intended domain D and its subsets represent the chunk of the world L is being used to talk about and quantify over. The intended interpretation of L’s constants and predicates assigns the actual denotations to L’s constants and the actual extensions to the predicates. The above semantics for our language M, may be viewed, in part, as an informal portrayal of the intended structure of M, which we refer to as $U^M$. That is we take M to be a tool for talking about the McKeon family with respect to gender, who is older than whom, who admires whom, etc. To make things formally prim and proper we should represent the interpretation of constants as $I_U^M(matt)=Matt$, $I_U^M(beth)=Beth$, and so on. And the interpretation of predicates can look like $I_U^M(Male)=\{Matt, Evan\}$, $I_U^M(Female)=\{Beth, Shannon, Kelly, Paige\}$, and so on. We assume that this has been done.

A structure $U$ for a language L (i.e., an L-structure) represents one way that a language can be used to talk about a state of affairs.Crudely, the domain D and the subsets recovered from D constitute a rudimentary representation of a state of affairs, and the interpretation of L’s predicates and individual constants makes the language L about the relevant state of affairs. Since a language can be assigned different structures, it can be used to talk about different states of affairs. The class of L-structures represents all the states of affairs that the language L can be used to talk about. For example, consider the following M-structure $U'$.

$$D=\text{the set of natural numbers}$$

$$I_{U'}(beth)=2\quad I_{U'}(Male)=\{d \in D \mid d \text{ is prime}\}$$

$$I_{U'}(matt)=3\quad I_{U'}(Female)=\{d \in D \mid d \text{ is even}\}$$

$$I_{U'}(shannon)=5\quad I_{U'}(Parent)=\emptyset$$
In specifying the domain D and the values of the interpretation function defined on M’s predicates we make use of brace notation, instead of the earlier list notation, to pick out sets. For example, we write

\[ \{d \in D | d \text{ is even}\} \]

to say “the set of all elements d of D such that d is even.” And

\[ \{<d, d'> \in D^2 | d > d'\} \]

reads: “The set of ordered pairs of elements d, d' of D such that d > d’.” Consider: the sentence

\[ \text{OlderThan(beth, matt)} \]

is true in the intended structure \( U^M \) for \(<I^M_U(\text{beth}), I^M_U(\text{matt})> \) is in \( I^M_U(\text{OlderThan}) \). But the sentence is false in \( U' \) for \(<I'_U(\text{beth}), I'_U(\text{matt})> \) is not in the extension of \( I'_U(\text{OlderThan}) \) (because 2 is not greater than 3). The sentence

\[ (\text{Female(beth)} & \text{Male(beth)}) \]

is not true in \( U^M \) but is true in \( U' \) for \( I'_U(\text{beth}) \) is in \( I'_U(\text{Female}) \) and in \( I'_U(\text{Male}) \) (because 2 is an even prime). In order to avoid confusion it is worth highlighting that when we say that the sentence ‘(Female(beth) & Male(beth))’ is true in one structure and false in another we are saying that one and the same wff with no free variables is true in one state of affairs on an interpretation and false in another state of affairs on another interpretation.

4.2 Satisfaction Revisited

Note the general strategy of giving the semantics of the sentential connectives: the truth of a compound sentence formed with any of them is determined by its component well-formed formulas (wffs), which are themselves (simpler) sentences. However, this strategy needs to be altered when it comes to quantificational sentences. For quantificational sentences are built out
of open wffs and, as noted above, these component wffs do not admit of truth and falsity. Therefore, we can’t think of the truth of, say,

$$\exists x (\text{Female}(x) \& \text{OlderThan}(x, \text{paige}))$$

in terms of the truth of ‘\text{Female}(x) \& \text{‘Older than}(x, \text{paige})’’ for some McKeon \(x\). What we need is a truth-relevant property of open formulas that we may appeal to in explaining the truth-value of the compound quantifications formed from them. Tarski is credited with the solution, first hinted at in the following.

The possibility suggests itself, however, of introducing a more general concept which is applicable to any sentential function [open or closed wff] can be recursively defined, and, when applied to sentences leads us directly to the concept of truth. These requirements are met by the notion of satisfaction of a given sentential function by given objects. (1933, p. 189)

The needed property is satisfaction. The truth of the above existential quantification will depend on there being an object that satisfies both ‘\text{Female}(x)’ and ‘\text{OlderThan}(x, \text{paige})’’. Earlier we introduced the concept of satisfaction by describing the conditions in which one element satisfies an open formula with one free variable. Now we want to develop a picture of what it means for objects to satisfy a wff with \(n\) free variables for any \(n \geq 0\). We begin by introducing the notion of a variable assignment.

A variable assignment is a function \(g\) from a set of variables (its domain) to a set of objects (its range). We shall say that the variable assignment \(g\) is suitable for a well-formed formula (wff) \(\Psi\) of \(M\) if every free variable in \(\Psi\) is in the domain of \(g\). In order for a variable assignment to satisfy a wff it must be suitable for the formula. For a variable assignment \(g\) that is suitable for \(\Psi\), \(g\) satisfies \(\Psi\) in \(U\) iff the object(s) \(g\) assigns to the free variable(s) in \(\Psi\) satisfy \(\Psi\). Unlike the earlier first-step characterization of satisfaction, there is no appeal to names for the entities assigned to the variables. This has the advantage of not requiring that new names be added to a language that does not have names for everything in the domain. In specifying a variable assignment \(g\), we write \(\alpha/v, \beta/v', \chi/v''\), … to indicate that \(g(v)=\alpha, g(v')=\beta, g(v'')=\chi\), etc… We understand
to mean that $g$ satisfies $\Psi$ in $U$.

$$U^M \models \text{OlderThan}(x,y)[\text{Shannon}/x, \text{Paige}/y]$$

This is true: the variable assignment $g$, identified with $[\text{Shannon}/x, \text{Paige}/y]$, satisfies ‘OlderThan(x,y)’ because Shannon is older than Paige.

$$U^M \models \text{Admires}(x,y)[\text{Beth}/x, \text{Matt}/y]$$

This is false for this variable assignment does not satisfy the wff: Beth does not admire Matt. However, the following is true because Matt admires Beth.

$$U^M \models \text{Admires}(x,y)[\text{Matt}/x, \text{Beth}/y]$$

For any wff $\Psi$, a suitable variable assignment $g$ and structure $U$ together ensure that the terms in $\Psi$ designate elements in $D$. The structure $U$ insures that individual constants have referents, and the assignment $g$ insures that any free variables in $\Psi$ get denotations. For any individual constant $c$, $c[g]$ is the element $I_U(c)$. For each variable $v$, and assignment $g$ whose domain contains $v$, $v[g]$ is the element $g(v)$. In effect, the variable assignment treats the variable $v$ as a temporary name.

We define $t[g]$ as ‘the element designated by $t$ relative to the assignment $g$’.

4.3 Formalized definition of truth

We now give a definition of truth for the language $M$ via the detour through satisfaction. The goal is to define for each formula $\alpha$ of $M$ and each assignment $g$ to the free variables, if any, of $\alpha$ in $U$ what must obtain in order for $U \models \alpha[g]$.

(I) Where $R$ is an $n$-place predicate and $t_1, \ldots, t_n$ are terms, $U \models R(t_1, \ldots, t_n)[g]$ if and only if (iff) the $n$-tuple $< t_1[g], \ldots, t_n[g]>$ is in $I_U(R)$.

(II) $U \models \neg \alpha[g]$ iff it is not true that $U \models \alpha[g]$

(III) $U \models (\alpha \& \beta)[g]$ iff $U \models \alpha[g]$ and $U \models \beta[g]$

(IV) $U \models (\alpha \lor \beta)[g]$ iff $U \models \alpha[g]$ or $U \models \beta[g]$

(V) $U \models (\alpha \rightarrow \beta)[g]$ iff either it is not true that $U \models \alpha[g]$ or $U \models \beta[g]$
Before going on to the (VI) and (VII) clauses for quantificational sentences, it is worthwhile to introduce the notion of a variable assignment that comes from another. Consider

\[ \exists y(\text{Female}(x) \& \text{OlderThan}(x, y)) \].

We want to say that a variable assignment \( g \) satisfies this wff if and only if there is a variable assignment \( g' \) differing from \( g \) at most with regard to the object it assigns to the variable \( y \) such that \( g' \) satisfies ‘(\text{Female}(x) \& \text{OlderThan}(x, y))’. We say that a variable assignment \( g' \) comes from an assignment \( g \) when the domain of \( g' \) is that of \( g \) and a variable \( v \), and \( g' \) assigns the same values as \( g \) with the possible exception of the element \( g' \) assigns to \( v \). In general, we represent an extension \( g' \) of an assignment \( g \) as follows.

\[ [g, d/v] \]

This picks out a variable assignment \( g' \) which differs at most from \( g \) in that \( v \) is in its domain and \( g'(v) = d \), for some element \( d \) of the domain \( D \). So, it is true that

\[ U^M \models \exists y(\text{Female}(x) \& \text{OlderThan}(x, y)) [\text{Beth}/x] \]

since

\[ U^M \models (\text{Female}(x) \& \text{OlderThan}(x, y)) [\text{Beth}/x, \text{Paige}/y]. \]

What this says is that the variable assignment \( \text{Paige}/y \) which comes from \( \text{Beth}/x \) satisfies ‘(\text{Female}(x) \& \text{OlderThan}(x, y))’ in \( U^M \). This is true for Beth is a female who is older than Paige.

Now we give the satisfaction clauses for quantificational sentences. Let \( \Psi \) be any formula of \( M \).

(VI) \[ U \models \exists v \Psi[g] \iff \text{for at least one element } d \text{ of } D, \ U \models \Psi[g, d/v] \]

(VII) \[ U \models \forall v \Psi[g] \iff \text{for all elements } d \text{ of } D, \ U \models \Psi[g, d/v] \]

If \( \alpha \) is a sentence, then it has no free variables and we write \( U \models \alpha[g_{\emptyset}] \) which says that the empty variable assignment satisfies \( \alpha \) in \( U \). The empty variable assignment \( g_{\emptyset} \) does not assign objects to any variables. In short: the definition of truth for language \( L \) is

A sentence \( \alpha \) is true in \( U \) if and only if \( U \models \alpha[g_{\emptyset}] \), i.e. the empty variable assignment satisfies \( \alpha \) in \( U \).
The truth definition specifies the conditions in which a formula of M is true in a structure by explaining how the semantic properties of any formula of M are determined by its construction from semantically primitive expressions (e.g., predicates, individual constants, and variables) whose semantic properties are specified directly. If a set of sentences is true in a structure $U$ we say that $U$ is a model of the set. We now work through some examples. The reader will be aided by referring when needed back to 19.

It is true that $U^M \not\models \neg\text{Married}(\text{kelly}, \text{kelly})[g_\emptyset]$, i.e., by (II) it is not true that $U^M \models \text{Married}(\text{kelly}, \text{kelly})[g_\emptyset]$, because $< \text{kelly}[g_\emptyset], \text{kelly}[g_\emptyset]>$ is not in $I^M_U(\text{Married})$. Hence, by (IV)

$U^M \models (\text{Married}(\text{shannon}, \text{kelly}) \lor \neg\text{Married}(\text{kelly}, \text{kelly})[g_\emptyset]$.

Our truth definition should confirm that

$\exists x \exists y \text{Admires}(x, y)$

is true in $U^M$. Note that by (VI) $U^M \models \exists y \text{Admires}(x, y)[g_\emptyset, \text{Paige}/x]$ since $U^M \models \text{Admires}(x, y)[[g_\emptyset, \text{Paige}/x], \text{kelly}/y]$. Hence, by (VI)

$U^M \models \exists x \exists y \text{Admires}(x, y)[g_\emptyset]$.

The sentence, ‘$\forall x \exists y (\text{Older}(y, x) \rightarrow \text{Admires}(x, y))$’ is true in $U^M$. By (VII) we know that

$U^M \models \forall x \exists y (\text{Older}(y, x) \rightarrow \text{Admires}(x, y))[g_\emptyset]$ if and only if

for all elements $d$ of $D$, $U^M \models \exists y (\text{Older}(y, x) \rightarrow \text{Admires}(x, y))[g_\emptyset, d/x]$.

This is true. For each element $d$ and assignment $[g_\emptyset, d/x]$, $U^M \models (\text{Older}(y, x) \rightarrow \text{Admires}(x, y))[[g_\emptyset, d/x], d'/y]$ i.e. there is some element $d'$ and variable assignment $g$ differing from $[g_\emptyset, d/x]$ only in assigning $d'$ to $y$, such that $g$ satisfies ‘$(\text{Older}(y, x) \rightarrow \text{Admires}(x, y))$’ in $U^M$.

4.4 Model-theoretic consequence defined

For any set $K$ of M-sentences and M-sentence $X$, we write

$K \models X$
to mean that every M-structure that is a model of K is also a model of X, i.e., X is a model theoretic consequence of K.

(1) OlderThan(paige, matt)
(2) ∀x(Male(x) → OlderThan(paige, x))

Note that both (1) and (2) are false in the intended structure U^M. We show that (2) is not a model theoretic consequence of (1) by describing a structure which is a model of (1) but not (2). The above structure U' will do the trick. By (I) it is true that U' ⊨ OlderThan(paige, matt) [g∅] because <(paige) g∅, (matt) g∅> is in I_U'(OlderThan) (because 11 is greater than 3). But, by (VII), it is not the case that

U' ⊨ ∀x(Male(x) → OlderThan(paige, x)) [g∅]

since the variable assignment [g∅, 13/x] doesn’t satisfy ‘(Male(x) → OlderThan(paige, x))’ in U’ according to (V) for U' ⊨ Male(x) [g∅, 13/x] but not U' ⊨ OlderThan(paige, x)) [g∅, 13/x]. So, (2) is not a model-theoretic consequence of (1). Consider the following sentences.

(3) (Admires(evan, paige) → Admires(paige, kelly))
(4) (Admires(paige, kelly) → Admires(kelly, beth))
(5) (Admires(evan, paige) → Admires(kelly, beth))

(5) is a model-theoretic consequence of (3) and (4). For assume otherwise. That is assume, that there is a structure U'' such that

(i) U'' ⊨ (Admires(evan, paige) → Admires(paige, kelly)) [g∅]

and

(ii) U'' ⊨ (Admires(paige, kelly) → Admires(kelly, beth)) [g∅]

but not

(iii) U'' ⊨ (Admires(evan, paige) → Admires(kelly, beth)) [g∅].

By (V), from the assumption that U'' does not satisfy (iii) it follows that U'' ⊨ Admires(evan, paige) [g∅] and not U'' ⊨ Admires(kelly, beth) [g∅]. Given the former, in order for (i) to hold according to (V) it must be the case that U'' ⊨ Admires(paige, kelly)) [g∅]. But then it is true that
U'' \models \text{Admires(paige,kelly))}[g_\emptyset] \text{ and false that } U'' \models \text{Admires(kelly, beth)}[g_\emptyset], \text{ which, again appealing to (V), contradicts our assumption (ii). Hence, there is no such } U'', \text{ and so (5) is a model-theoretic consequence of (3) and (4).}

Here are some more examples of the model-theoretic consequence relation in action.

(6) \quad \exists x \text{Male}(x)
(7) \quad \exists x \text{Brother}(x, shannon)
(8) \quad \exists x (\text{Male}(x) \& \text{Brother}(x, shannon))

(8) is not a model-theoretic consequence of (6) and (7). Consider the following structure U'''.

D={1, 2, 3}

For all M-individual constants c, I_U(c)=1.
I_U(\text{Male})={2}, I_U(\text{Brother})={<3,1>}. For all other M-predicates P, I_U(P)=\emptyset.

Appealing to the satisfaction clauses (I), (III), and (VI), it is fairly straightforward to see that the structure U''' is a model of (1) and (2) but not of (3). For example, U''' is not a model of (3) for there is no element d of D and assignment [d/x] such that

U''' \models (\text{Male}(x) \& \text{Brother}(x, shannon))[g_\emptyset, d/x].

Consider the following two sentences

(1) \quad \text{Female}(shannon)
(2) \quad \exists x \text{Female}(x)

(2) is a model-theoretic consequence of (1). For an arbitrary M-structure U, if U \models \text{Female}(shannon) g_\emptyset, then by satisfaction clause (I), shannon[g_\emptyset] is in I_U(\text{Female}), and so there is at least one element of D, shannon[g_\emptyset], in I_U(\text{Female}). Consequently, by (VI), U \models \exists x \text{Female}(x)[g_\emptyset].

For a sentence X of M, we write

\models X.
to mean that X is a model-theoretic consequence of the empty set of sentences. This means that every M-structure is a model of X. Such sentences represent logical truths; it is not logically possible for them to be false. For example,

\[ \models (\forall x \text{Male}(x) \rightarrow \exists x \text{Male}(x)) \]

is true. Here’s one explanation why. Let U be an arbitrary M-structure. We now show that

\[ U \models (\forall x \text{Male}(x) \rightarrow \exists x \text{Male}(x)) [\emptyset]. \]

If \( U \models (\forall x \text{Male}(x)) [\emptyset] \) holds, then by (VII) for every element d of the domain D, \( U \models \text{Male}(x) [\emptyset, d/x] \). But we know that D is non-empty, by the requirements on structures (see the beginning of section 4.1), and so D has at least one element d. Hence for at least one element d of D, \( U \models \text{Male}(x) [\emptyset, d/x] \), i.e. by (VI), \( U \models \exists x \text{Male}(x)) [\emptyset] \). So, if \( U \models (\forall x \text{Male}(x)) [\emptyset] \) then \( U \models \exists x \text{Male}(x)) [\emptyset] \), and, therefore according to (V),

\[ U \models (\forall x \text{Male}(x) \rightarrow \exists x \text{Male}(x)) [\emptyset]. \]

Since U is arbitrary, this establishes

\[ \models (\forall x \text{Male}(x) \rightarrow \exists x \text{Male}(x)). \]

If we treat ‘=’ as a logical constant and require that for all M-structures U, \( I_U (=) = \{ <d, d'> \in D^2 | d=d'> \} \), then M-sentences asserting that identity is reflexive, symmetrical, and transitive are true in every M-structure, i.e. the following hold.

\[ \models \forall x(x=x) \]
\[ \models \forall x \forall y((x=y) \rightarrow (y=x)) \]
\[ \models \forall x \forall y \forall z((x=y) \& (y=z)) \rightarrow (x=z)) \]

Structures which assign \{ <d, d'> \in D^2 | d=d'> \} to the identity symbol are sometimes called normal models. Letting \( \Gamma \Psi(v) \vdash \) be any wff in which just variable v occurs free,

\[ \forall x \forall y ((x=y) \rightarrow (\Psi(x) \rightarrow \Psi(y))) \]

represents the claim that identicals are indiscernible—if x=y then whatever holds of x holds of y—and it is true in every M-structure U that is a normal model. Treating ‘=’ as a logical constant
(which is standard) requires that we restrict the class of M-structures appealed to in the above model-theoretic definition of logical consequence to those which are normal models.

4.5 The model-theoretic definition and the common concept of logical consequence

Our aim has been to define the logical consequence relation in M in terms of the model-theoretic consequence relation. In order to determine the success of the definition, we need to know what type of definition it is. Clearly it is not intended as a lexical definition. As Tarski’s opening passage in his (1933) (reproduced above, pp.1-2) makes clear, a theory of logical consequence need not yield a report of what ‘logical consequence’ means. On other hand, it is clear that Tarski doesn’t see himself as offering just a stipulative definition. Tarski is not merely stating how he proposes to use ‘logical consequence’ and ‘logical truth’ (but see Tarski (1986)) any more than Newton was just proposing how to use certain words when he defined force in terms of mass and acceleration. Newton was invoking a fundamental conceptual relationship in order to improve our understanding of the physical world. Similarly, Tarski’s definition of ‘logical consequence’ in terms of model-theoretic consequence is supposed to help us formulate a theory of logical consequence that deepens our understanding of the common concept. As Tarski says, “The concept of logical consequence is one of those whose introduction into a field of strict formal investigation was not a matter of arbitrary decision on the part of this or that investigator; in defining this concept efforts were made to adhere to the common usage of the language of everyday life.”

Let’s follow this approach to Tarski’s (1936) and treat the model-theoretic definition as a theoretical definition of ‘logical consequence’. The questions raised are whether the Tarskian model-theoretic definition of logical consequence leads to a good theory and whether it improves our understanding of logical consequence. In order to sketch a framework for thinking about this question, we review the key moves in the Tarskian analysis. In what follows, K is an arbitrary set of sentences from a language L, and X is any sentence from L. First, Tarski observes what he takes to be the salient features of the common concept and makes the following claim.
(1) X is a logical consequence of K if and only if (1) it is not possible for all the K to be true and X false, (2) this is due to the forms of the sentences, and (3) this is known a priori.

Tarski’s deep insight was to see the criteria, listed in bold, in terms of the technical notion of truth in a structure. The key step in his analysis is to embody the above criteria (1)-(3) in terms of the notion of a possible interpretation of the non-logical terminology in sentences. Substituting for what is in bold in (1) we get

(2) X is a logical consequence of K if and only if there is no possible interpretation of the non-logical terminology of the language according to which all the sentences in K are true and X is false.

The third step of the Tarskian analysis of logical consequence is to use the technical notion of truth in a structure or model to capture the idea of a possible interpretation. That is, we understand there is no possible interpretation of the non-logical terminology of the language according to which all of the sentences in K are true and X is false in terms of: Every model of K is a model of X, i.e., K \models X.

To elaborate, as reflected in (2), the analysis turns on a selection of terms as logical constants. This is represented model-theoretically by allowing the interpretation of the non-logical terminology to change from one structure to another, and by making the interpretation of the logical constants invariant across the class of structures. Then, relative to a set of terms treated as logical, the Tarskian, model-theoretic analysis is committed to

(3) X is a logical consequence of K if and only if K \models X

and

(4) X is a logical truth, i.e., it is logically impossible for X to be false, if and only if \models X.

As a theoretical definition, we expect the \models -relation to reflect the essential features of the common concept of logical consequence. By Tarski’s lights, the \models -consequence relation should be necessary, formal, and a priori. Note that model theory by itself does not provide the means for drawing a boundary between the logical and the non-logical. Indeed, its use presupposes that a list of logical terms is in hand. For example, taking Sister and Female to be constants, the
consequence relation from (A) ‘Sister(kelly, paige)’ to (B) ‘Female(kelly)’ is necessary, formal and \textit{a priori}. So perhaps (B) should be a logical consequence of (A). The fact that (B) is not a model-theoretic consequence of (A) is due to the fact that the interpretation of the two predicates can vary from one structure to another. To remedy this we could make the interpretation of the two predicates invariant so that ‘∀x(∃ySister(x,y)→Female(x))’ is true in all structures, and, therefore if (A) is true in a structure, (B) is too. The point here is that the use of models to capture the logical consequence relation requires a prior choice of what terms to treat as logical. This is, in turn, reflected in the identification of the terms whose interpretation is constant from one structure to another.

So in assessing the success of the Tarskian model-theoretic definition of logical consequence for a language L, two issues arise. First, does the model-theoretic consequence relation reflect the salient features of the common concept of logical consequence? Second, is the boundary in L between logical and non-logical terms correctly drawn? In other words: what in L qualifies as a logical constant? These are central questions in the philosophy of logic and their significance is at least partly due to the prevalent use of model theory in logic to represent logical consequence in a variety of languages. In what follows, I sketch some responses to the two questions that draw on contemporary work in philosophy of logic. I begin with the first question.

\textit{4.5.1 Does the model-theoretic consequence relation reflect the salient features of the common concept of logical consequence?}

As indicated above, the $\models$-consequence relation is formal. Also, a brief inspection of the above justifications that $K \models X$ obtain for given $K$ and $X$ reveals that the $\models$-consequence relation is \textit{a priori}. Does the $\models$ consequence relation capture the modal element in the common concept of logical consequence? There are critics who argue that the model-theoretic account lacks the conceptual resources to rule out the possibility of there being logically possible situations in which sentences in $K$ are true and $X$ is false but no structure $U$ such that $U \models K$ and not $U \models X$. 
Kneale (1961) is an early critic, Etchemendy (1988, 1999a, 1999b) offers a sustained and multi-faceted attack. We follow Etchemendy. Consider the following three sentences.

(1) (Female(shannon) & ~Married(shannon, matt))
(2) (~Female(matt) & Married(beth, matt))
(3) ~Female(beth)

(3) is neither a logical nor a model-theoretic consequence of (1) and (2). However, in order for a structure to make (1) and (2) true but not (3) its domain must have at least three elements. If the world contained, say, just two things, then there would be no such structure and (3) would be a model-theoretic consequence of (1) and (2). But in this scenario, (3) would not be a logical consequence of (1) and (2) because it would still be logically possible for the world to be larger and in such a possible situation (1) and (2) can be interpreted true and (3) false. The problem raised for the model-theoretic account of logical consequence is that we do not think that the class of logically possible situations varies under different assumptions as to the cardinality of the world’s elements. But the class of structures surely does since they are composed of worldly elements. This is a tricky criticism. Let’s look at it from a slightly different vantagepoint.

We might think that the extension of the logical consequence relation for an interpreted language such as our language M about the McKeons is necessary. For example, it can’t be the case that for some K and X, even though X isn’t a logical consequence of a set K of sentences, X could be. So, on the supposition that the world contains less, the extension of the logical consequence relation should not expand. However, the extension of the model-theoretic consequence does expand. For example, (3) is not, in fact, a model-theoretic consequence of (1) and (2), but it would be if there were just two things. This is evidence that the model-theoretic characterization has failed to capture the modal notion inherent in the common concept of logical consequence.

In defense of Tarski (see Ray (1999) and Sher (1991) for defenses of the Tarskian analysis against Etchemendy), one might question the force of the criticism because it rests on the
supposition that it is possible for there to be just finitely many things. How could there be just two things? Indeed, if we countenance an infinite totality of necessary existents such as abstract objects (e.g., pure sets), then the class of structures will be fixed relative to an infinite collection of necessary existents, and the above criticism that turns on it being possible that there are just \( n \) things for finite \( n \) doesn’t go through (for discussion see McGee (1999)). One could reply that while it is metaphysically impossible for there to be merely finitely many things it is nevertheless logically possible and this is relevant to the modal notion in the concept of logical consequence. This reply requires the existence of primitive, basic intuitions regarding the logical possibility of there being just finitely many things. However, intuitions about possible cardinalities of worldly individuals—not informed by mathematics and science—tend to run stale. Consequently, it is hard to debate this reply: one either has the needed logical intuitions, or not.

What is clear is that our knowledge of what is a model-theoretic consequence of what in a given \( L \) depends on our knowledge of the class of \( L \)-structures. Since such structures are furniture of the world, our knowledge of the model-theoretic consequence relation is grounded on knowledge of substantive facts about the world. Even if such knowledge is \textit{a priori}, it is far from obvious that our \textit{a priori} knowledge of the logical consequence relation is so substantive. Recall the \textit{locked room} metaphor in section 2.2.1. One might argue that knowledge of what follows from what shouldn’t turn on worldly matters of fact, even if they are necessary and \textit{a priori}. If correct, this is a strike against the model-theoretic definition. However, this standard logical positivist line has been recently challenged by those who see logic penetrated and permeated by metaphysics (e.g., Putnam (1971), Almog (1989), Sher (1991), Williamson (1999)). We illustrate the insight behind the challenge with a simple example. Consider the following two sentences.

(1) \( \exists x \ ( \text{Female}(x) \ & \ \text{Sister}(x, \ evan)) \)
(2) \( \exists x \text{Female}(x) \)
(2) is a logical consequence of (1), i.e., there is no domain for the quantifiers and no interpretation of the predicates and the individual constant in that domain which makes (1) true and not (2). Why? Because on any interpretation of the non-logical terminology, (1) is true just in case the intersection of the set of objects that satisfy Female(x) and the set of objects that satisfy Sister(x, evan) is non-empty. If this obtains, then the set of objects that satisfy Female(x) is non-empty and this makes (2) true. The basic metaphysical truth underlying the reasoning here is that for any two sets, if their intersection is non-empty, then neither set is the empty set. This necessary and a priori truth about the world, in particular about its set-theoretic part, is an essential reason why (2) follows from (1). This approach, reflected in the model-theoretic consequence relation (see Sher (1996)), can lead to an intriguing view of the formality of logical consequence reminiscent of the pre-Wittgensteinian views of Russell and Frege. Following the above, the consequence relation from (1) to (2) is formal because the metaphysical truth on which it turns describes a formal (structural) feature of the world. In other words: it is not possible for (1) to be true and (2) false because

for any extensions of P, P', it is the case that if for some object x, it is true that (P(x) & P'(x, evan)), then for some object x, it is true that P(x).

According to this vision of the formality of logical consequence, the consequence relation between (1) and (2) is formal because what is in bold expresses a formal feature of reality. Russell writes that, “Logic, I should maintain, must no more admit a unicorn than zoology can; for logic is concerned with the real world just as truly as zoology, though with its more abstract and general features” (Russell, (1919) p. 169). If we take the abstract and general features of the world to be its formal features, then Russell’s remark captures the view of logic that emerges from anchoring the necessity, formality and a priority of logical consequence in the formal features of the world. The question arises as to what counts as a formal feature of the world. If
we say that all set-theoretic truths depict formal features of the world, including claims about how many sets there are, then this would seem to justify making

$$\exists x \forall y \neg(x = y)$$

(i.e., there are at least two individuals) a logical truth since it is necessary, a priori, and a formal truth. To reflect model-theoretically that such sentences, which consist just of logical terminology, are logical truths we would require that the domain of a structure simply be the collection of the world’s individuals. See Sher (1991) for an elaboration and defense of this view of the formality of logical truth and consequence. See Shapiro (1993) for further discussion and criticism of the project of grounding our logical knowledge on primitive intuitions of logical possibility instead of on our knowledge of metaphysical truths.

Part of the difficulty in reaching a consensus with respect to whether or not the model-theoretic consequence relation reflects the salient features of the common concept of logical consequence is that philosophers and logicians differ over what the features of the common concept are. Some offer accounts of the logical consequence relation according to which it is not a priori (e.g., see Koslow (1999), Sher (1991) and see Hanson (1997) for criticism of Sher) or deny that it even need be strongly necessary (Smiley 1995, 2000, section 6). Here we illustrate with a quick example.

Given that we know that a McKeon only admires those who are older (i.e., we know that (a)

$$\forall x \forall y (\text{Admires}(x,y) \rightarrow \text{OlderThan}(y,x)))$$

wouldn’t we take (2) to be a logical consequence of (1)?

1. Admires (paige, kelly)
2. OlderThan (kelly, paige)

A Tarskian response is that (2) is not a consequence of (1) alone, but of (1) plus (a). So in thinking that (2) follows from (1), one assumes (a). A counter suggestion is to say that (2) is a logical consequence of (1) for if (1) is true, then necessarily-relative-to-the-truth-of-(a) (2) is
true. The modal notion here is a weakened sense of necessity: \textit{necessity relative to the truth of a collection of sentences}, which in this case is composed of (a). Since (a) is not \textit{a priori}, neither is the consequence relation between (1) and (2). The motive here seems to be that this conception of modality is inherent in the notion of logical consequence that drives deductive inference in science, law, and other fields outside of the logic classroom. This supposes that a theory of logical consequence must not only account for the features of the intuitive concept of logical consequence but also reflect the intuitively correct deductive inferences. After all, the logical consequence relation is the foundation of deductive inference: it is not correct to deductively infer B from A unless B is a logical consequence of A. Referring to our example, in a conversation where (a) is a truth that is understood and accepted by the conversants, the inference from (1) to (2) seems legit. Hence, this should be supported by an accompanying concept of logical consequence. This idea of construing the common concept of logical consequence in part by the lights of basic intuitions about correct inferences is reflected in the Relevance logician’s objection to the Tarskian account. The Relevance logician claims that X is not a logical consequence of K unless K is relevant to X. For example, consider the following pairs of sentences.

(1) (Female(evan) & ~Female(evan))  
(2) Admires(kelly, shannon)  
(1) Admires(kelly, paige)  
(2) (Female(evan) v ~Female(evan))

In the first pair, (1) is logically false, and in the second (2) is a logical truth. Hence it isn’t possible for (1) to be true and (2) false. Since this seems to be formally determined and a priori, for each pair (2) is a logical consequence of (1) according to Tarski. Against this Anderson and Belnap write, “the fancy that relevance is irrelevant to validity [i.e. logical consequence] strikes us as ludicrous, and we therefore make an attempt to explicate the notion of relevance of A to B” (Anderson and Belnap (1975) pp. 17-18). The typical support for the relevance conception of logical consequence draws on intuitions regarding correct inference, e.g. it is counterintuitive to think that it is correct to infer (2) from (1) in either pair for what does being a female have to do with who one admires? Would you think it correct to infer, say, that \textit{Admires(kelly, shannon)} on
the basis of \( (\text{Female(evan)} \& \sim \text{Female(evan)}) \)? For further discussion of the different types of relevance logic and more on the relevant philosophical issues see Haack (1978) pp. 198-203 and Read (1995) pp. 54-63. The bibliography in Haack (1996) pp. 264-265 is helpful. We say more about relevance logic below on pp. 65-68.

Our question is, does the model-theoretic consequence relation reflect the essential features of the common concept of logical consequence? Our discussion illustrates at least two things. First, it isn’t obvious that the model-theoretic definition of logical consequence reflects the Tarskian portrayal of the common concept. One option, not discussed above, is to deny that the model-theoretic definition is a theoretical definition and argue for its utility simply on the basis that it is extensionally equivalent with the common concept (see Shapiro (1998)). Our discussion also illustrates that Tarski’s identification of the essential features of logical consequence is disputed. One reaction, not discussed above, is to question the presupposition of the debate and take a more pluralist approach to the common concept of logical consequence. On this line, it is not so much that the common concept of logical consequence is vague as it is ambiguous. At minimum, to say that a sentence X is a logical consequence of a set K of sentences is to say that X is true in every circumstance [i.e. logically possible situation] in which the sentences in K are true. “Different disambiguations of this notion arise from taking different extensions of the term ‘circumstance’ ” (Restall (2002 p. 427). If we disambiguate the relevant notion of ‘circumstance’ by the lights of Tarski, ‘Admires(kelly, paige)’ is a logical consequence of ‘(Female(evan) \& \sim \text{Female(evan)})’. If we follow the Relevance logician, then not. There is no fact of the matter about whether or not the first sentence is a logical consequence of the second.

2.5.2 What is a logical constant?

We turn to the second, related issue of what qualifies as a logical constant. Tarski writes, “No objective grounds are known to me which permit us to draw a sharp boundary between [logical and non-logical terms]. It seems possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stand in sharp contrast to ordinary usage (1936) p. 418-419.
And at the end of his (1936), he tells us that the fluctuation in the common usage of the concept of consequence would be accurately reflected in a relative concept of logical consequence, i.e. a relative concept “which must, on each occasion, be related to a definite, although in greater or less degree arbitrary, division of terms into logical and extra logical” (p.420). Unlike the relativity described in the previous paragraph, which speaks to the features of the concept of logical consequence, the relativity contemplated by Tarski concerns the selection of logical constants. Tarski’s observations of the common concept do not yield a sharp boundary between logical and non-logical terms. It seems that the sentential connectives and the quantifiers of our language M about the McKeons qualify as logical if any terms of M do. We’ve also followed many logicians and included the identity predicate as logical (See Quine (1986) for considerations against treating ‘=’ as a logical constant). But why not include other predicates such as ‘OlderThan’?

(1) OlderThan(kelly,paige)  
(2) ~OlderThan(paige,kelly)  
(3) ~OlderThan(kelly, kelly)

Then the consequence relation from (1) to (2) is necessary, formal, and a priori and the truth of (3) is necessary, formal and also a priori. If treating ‘OlderThan’ as a logical constant does not do violence to our intuitions about the features of the common concept of logical consequence and truth, then it is hard to see why we should forbid such a treatment. By the lights of the relative concept of logical consequence, there is no fact of the matter about whether (2) is a logical consequence of (1) since it is relative to the selection of ‘OlderThan’ as a logical constant.

On the other hand, Tarski hints that even by the lights of the relative concept there is something wrong in thinking that B follows from A and B only relative to taking ‘and’ as a logical constant. Rather, B follows from A and B we might say absolutely since ‘and’ should be on everybody’s list of logical constants. But why do ‘and’ and the other sentential connectives, along with the identity predicate and the quantifiers have more of a claim to logical constancy than, say, ‘OlderThan’? Tarski (1936) offers no criteria of logical constancy that help answer this question.
On the contemporary scene, there are three general approaches to the issue of what qualifies as a logical constant. One approach is to argue for an inherent property (or properties) of logical constancy that some expressions have and others lack. For example, topic neutrality is one feature traditionally thought to essentially characterize logical constants. The sentential connectives, the identity predicate, and the quantifiers seem topic neutral; they seem applicable to discourse on any topic. The predicates other than identity such as ‘OlderThan’ do not appear to be topic neutral, at least as standardly interpreted, e.g., ‘OlderThan’ has no application in the domain of natural numbers. One way of making the concept of topic neutrality precise is to follow Tarski’s suggestion in his (1986) that the logical notions expressed in a language L are those notions that are invariant under all one-one transformations of the domain of discourse onto itself. A one-one transformation of the domain of discourse onto itself is a one-one function whose domain and range coincide with the domain of discourse. And a one-one function is a function that always assigns different values to different objects in its domain (i.e., for all \( x \) and \( y \) in the domain of \( f \), if \( f(x)=f(y) \), then \( x=y \)).

Consider ‘OlderThan’. By Tarski’s lights, the notion expressed by the predicate is its extension, i.e. the set of ordered pairs \(<d, d'>\) such that \( d \) is older than \( d' \). Recall that the extension is:

\[
{<\text{Beth, Matt}>, <\text{Beth, Shannon}>, <\text{Beth, Kelly}>, <\text{Beth, Paige}>, <\text{Beth, Evan}>, <\text{Matt, Shannon}>, <\text{Matt, Kelly}>, <\text{Matt, Paige}>, <\text{Matt, Evan}>, <\text{Shannon, Kelly}>, <\text{Shannon, Paige}>, <\text{Shannon, Evan}>, <\text{Kelly, Paige}>, <\text{Kelly, Evan}>, <\text{Paige, Evan}>}. 
\]

If ‘OlderThan’ is a logical constant its extension (the notion it expresses) should be invariant under every one-one transformation of the domain of discourse (i.e. the set of McKeons) onto itself. A set is invariant under a one-one transformation \( f \) when the set is carried onto itself by the transformation. For example, the extension of ‘Female’ is invariant under \( f \) when for every \( d, d' \) is a female if and only if \( f(d) \) is. ‘OlderThan’ is invariant under \( f \) when \( <d, d'> \) is in the extension of ‘OlderThan’ if and only if \( <f(d), f(d')> \) is. Clearly, the extensions of Female and the
Older than relation are not invariant under every one-one transformation. For example, Beth is older than Matt, but \( f(Beth) \) is not older than \( f(Matt) \) when \( f(Beth) = Evan \) and \( f(Matt) = Paige \). Compare the identity relation: it is invariant under every one-one transformation of the domain of McKeons because it holds for each and every McKeon. The invariance condition makes precise the concept of topic neutrality. Any expression whose extension is altered by a one-one transformation must discriminate among elements of the domain, making the relevant notions topic-specific. The invariance condition can be extended in a straightforward way to the quantifiers and sentential connectives (see McCarthy (1981) and McGee (1997)). Here I illustrate with the existential quantifier. Let \( \Psi \) be a well-formed formula with ‘\( x \)’ as its free variable. \( \exists x \Psi \) has a truth-value in the intended structure \( U^M \) for our language \( M \) about the McKeons. Let \( f \) be an arbitrary one-one transformation of the domain \( D \) of McKeons onto itself. The function \( f \) determines an interpretation \( I' \) for \( \Psi \) in the range \( D' \) of \( f \). The existential quantifier satisfies the invariance requirement for \( U^M \Vdash \exists x \Psi \) if and only if \( U \Vdash \exists x \Psi \) for every \( U \) derived by a one-one transformation \( f \) of the domain \( D \) of \( U^M \) (we say that the \( U \)’s are isomorphic with \( U^M \)).

For example, consider the following existential quantification.

\[
\exists x \text{Female}(x)
\]

This is true in the intended structure for our language \( M \) about the McKeons (i.e., \( U^M \Vdash \exists x \text{Female}(x)[g_0] \)) ultimately because the set of elements that satisfy ‘Female(x)” on some variable assignment that extends \( g_0 \) is non-empty (recall that Beth, Shannon, Kelly, and Paige are females). The cardinality of the set of McKeons that satisfy an \( M \)-formula is invariant under every one-one transformation of the domain of McKeons onto itself. Hence, for every \( U \) isomorphic with \( U^M \), the set of elements from \( D^U \) that satisfy ‘Female(x)” on some variable assignment that extends \( g_0 \) is non-empty and so

\[
U \Vdash \exists x \text{Female}(x) [g_0].
\]
Speaking to the other part of the invariance requirement given at the end of the previous paragraph, clearly for every \( U \) isomorphic with \( U^M \), if \( U \models \exists x \text{Female}(x) [g_0] \), then \( U^M \models \exists x \text{Female}(x) [g_0] \) (since \( U \) is isomorphic with itself). Crudely, the topic neutrality of the existential quantifier is confirmed by the fact that it is invariant under all one-one transformations of the domain of discourse onto itself.

Key here is that the cardinality of the subset of the domain \( D \) that satisfies an \( L \)-formula under an interpretation is invariant under every one-one transformation of \( D \) onto itself. For example, if at least two elements from \( D \) satisfy a formula on an interpretation of it, then at least two elements from \( D' \) satisfy the formula under the \( I' \) induced by \( f \). This makes not only ‘All’ and ‘Some’ topic neutral, but also any cardinality quantifier such as ‘Most’, ‘Finitely many’, ‘Few’, ‘At least two’ etc… The view suggested in Tarski (1986, p. 149) is that the logic of a language \( L \) is the science of all notions expressible in \( L \) which are invariant under one-one transformations of \( L \)’s domain of discourse. For further discussion, defense of, and extensions of the Tarskian invariance requirement on logical constancy see in addition to McCarthy (1981) and McGee (1997) see (Sher (1989), (1991)).

A second approach to what qualifies as a logical constant is to not make topic neutrality a necessary condition for logical constancy. This undercuts at least some of the significance of the invariance requirement. Instead of thinking that there is an inherent property of logical constancy, we allow the choice of logical constants to depend, at least in part, on the needs at hand, as long as the resulting consequence relation reflects the essential features of the intuitive, pre-theoretic concept of logical consequence. I take this view to be very close to the one that we are left with by default in Tarski (1936). The approach, suggested in Prior (1976) and developed in related but different ways in Hanson (1996) and Warmbrod (1999), amounts to regarding logic in a strict sense and loose sense. Logic in the strict sense is the science of what follows from what relative to topic neutral expressions, and logic in the loose sense is the study of what follows
from what relative to both topic neutral expressions and those topic centered expressions of interest that yield a consequence relation possessing the salient features of the common concept.

Finally, a third approach the issue of what makes an expression a logical constant is to simply reject the view of logical consequence as a formal consequence relation, thereby nullifying the need to distinguish logical terminology in the first place (see Etchemendy (1983) and the conclusion in (1999b), and see Bencivenga (1999)). We just say, for example, that X is a logical consequence of a set K of sentences if the supposition that all of the K are true and X false violates the meaning of component terminology. Hence, ‘Female(kelly)’ is a logical consequence of ‘Sister(kelly, paige)’ simply because the supposition otherwise violates the meaning of the predicates. Whether or not ‘Female’ and ‘Sister’ are logical terms doesn’t come into play.

This concludes our discussion of the status of the model-theoretic characterization of logical consequence. We now complete our presentation of a full logic for language M by offering an account of the deductive or proof-theoretic consequence relation in M.

5. **Deductive Consequence**

In defining logical consequence for M in terms of the deductive consequence relation, we appeal to a natural deductive system that originates in the work of the mathematician Gerhard Gentzen (1934) and the logician Fredrick Fitch (1952). We will refer to the deductive system as N (for ‘natural deduction’). For an in-depth introductory presentation of a natural deductive system very similar to N see Barwise and Etchemendy J (2001). N is a collection of inference rules. A proof of X from K that appeals exclusively to the inference rules of N is a formal deduction or formal proof. A formal proof is associated with a pair <K, X> where K is a set of sentences from L and X is an L-sentence. The set K of sentences is the basis of the deduction, and X is the conclusion. We shall spend some time in what follows later explaining the nature of a formal deduction. But for now, we say that a formal deduction from K to X is a finite sequence S of sentences ending with X such that each sentence in S is either an assumption, deduced from a sentence (or more) in K, or deduced from previous sentences in S in accordance with one of N’s
inference rules. These inference rules are introduction and elimination rules, defined for each logical constant of our language M. An introduction rule introduces a logical constant into a proof by allowing us to derive a sentence in which the logical constant appears. From a sentence in which a logical constant occurs, an elimination rule for the constant allows us to derive a sentence that has at least one less occurrence of the logical constant. An introduction rule for a logical constant is useful for deriving a sentence that contains the constant, and the elimination rule is useful for deriving a sentence from another in which the constant appears.

Formal proofs are not only epistemologically significant for securing knowledge, but also the derivations making up formal proofs may serve as models of the informal deductive reasoning that we perform. Indeed, a primary value of a formal proof is that it can serve as a model of ordinary deductive reasoning that explains the force of such reasoning by representing the principles of inference required to get to X from K. This is nice for, after all, like Molière’s M. Jourdain, who spoke prose all his life without knowing it, we frequently reason all the time without being aware of the principles underlying what we are doing.

Gentzen, one of the first logicians to present a natural deductive system, makes clear that a primary motive for the construction of his system is to reflect as accurately as possible the actual logical reasoning involved in mathematical proofs. He writes, My starting point was this: The formalization of logical deduction especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs...In contrast, I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a ‘calculus of natural deduction’ (1934), p. 68.

Natural deductive systems are distinguished from other deductive systems by their usefulness in modeling ordinary, informal deductive inferential practices. Paraphrasing Gentzen, we may say that if one is interested in seeing logical connections between sentences in the most natural way possible, then a natural deductive system is a good choice for defining the deductive consequence relation.
5.1 Deductive System N

In stating N’s rules, we begin with the simpler inference rules and give a sample formal deduction of them in action. Then we turn to the inference rules that employ what we shall call sub-proofs. In the statement of the rules, we let ‘P’ and ‘Q’ be any sentences from our language M. We shall number each line of a formal deduction with a positive integer. We let ‘k’, ‘l’, ‘m’, ‘n’, ‘o’, and ‘p’ be any positive integers such that k < m, and l < m, and m < n < o < p < q.

&-Intro

\[
\begin{array}{l}
  k. P \\
  l. Q \\
  m. (P \& Q) \quad &-\text{Intro: } k, l.
  \end{array}
\]

&-Elim

\[
\begin{array}{l}
  k. (P \& Q) \\
  m. P \quad &-\text{Elim: } k \\
  m. Q \quad &-\text{Elim: } k
  \end{array}
\]

&-Intro allows us to derive a conjunction from both of its two parts (called conjuncts). According to the &-Elim rule we may derive a conjunct from a conjunction. To the right of the sentence derived using an inference rule is the justification. Steps in a proof are justified by identifying both the lines in the proof used and by citing the appropriate rule. The vertical lines serve as proof margins, which, as you will shortly see, help in portraying the structure of a proof when it contains embedded sub-proofs.

~ -Elim

\[
\begin{array}{l}
  k. \sim P \\
  m. P \quad \sim\text{-Elim: } k
  \end{array}
\]

The ~ -Elim rule allows us to drop double negations and infer what was subject to the two negations.
v-Intro

<table>
<thead>
<tr>
<th>k. P</th>
<th>m. (P v Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>v-Intro: k</td>
<td>v-Intro: k</td>
</tr>
</tbody>
</table>

By v-Intro we may derive a disjunction from one of its parts (called disjuncts).

$\Rightarrow$ -Elim

| k. (P$\Rightarrow$Q) |
| l. P                |
| m. Q                |

$\Rightarrow$-Elim: k, l

The $\Rightarrow$-Elim rule corresponds to the principle of inference called modus ponens: from a conditional and its antecedent one may infer the consequent.

Here’s a sample deduction using the above inference rules. The formal deduction—the sequence of sentences 4-11—is associated with the pair

$<\{(\text{Female(paige) }\&\text{ Female (kelly)}), (\text{Female(paige) }\Rightarrow\text{~Sister(paige, kelly)}), (\text{Female(kelly) }\Rightarrow\text{~Sister(paige, shannon)})\}, ((\text{Sister(paige, kelly) }\&\text{ Sister(paige, shannon)}) \lor \text{Male(evan)})>.$

The first element is the set of basis sentences and the second element is the conclusion. We number the basis sentences and list them (beginning with 1) ahead of the deduction. The deduction ends with the conclusion.

1. (Female(paige) & Female (kelly))  Basis
2. (Female(paige)$\Rightarrow$~Sister(paige, kelly))  Basis
3. (Female(kelly)$\Rightarrow$~Sister(paige, shannon))  Basis
4. Female(paige)  &-Elim: 1
5. Female(kelly)  &-Elim: 1
6. ~Sister(paige,kelly)  $\Rightarrow$-Elim: 2,4
7. Sister(paige,kelly)  ~-Elim: 6
8. ~Sister(paige, shannon)  $\Rightarrow$-Elim: 3,5
9. Sister(paige, shannon)  ~-Elim: 8
10. (Sister(paige, kelly) & Sister(paige, shannon))  &-Intro: 7, 9
11. ((Sister(paige, kelly) & Sister(paige, shannon)) v Male(evan))  v-Intro: 10
Again, the column all the way to the right gives the explanations for each line of the proof.

Assuming the adequacy of N, the formal deduction establishes that the following inference is correct.

\[
\begin{align*}
&\text{(Female(paige) & Female (kelly))} \\
&\text{(Female(paige)\rightarrow\sim\text{Sister(paige, kelly))} \\
&\text{(Female(kelly)\rightarrow\sim\text{Sister(paige, shannon))} \\
\text{(therefore) ((Sister(paige, kelly) & Sister(paige, shannon)) v Male(evan))}
\end{align*}
\]

For convenience in building proofs, we expand M to include ‘\(\bot\)’, which we use as a symbol for a contradiction (e.g., ‘(Female(beth) & \sim\text{Female(beth)})’).

\[
\begin{align*}
\text{\(\bot\)-Intro} & & \text{\(\bot\)-Elim} \\
\text{k. P} & & \text{k. \(\bot\)} \\
\text{l. \sim P} & & \text{m. P \(\bot\)-Elim: k} \\
\text{m. \(\bot\) \(\bot\)-Intro: k, l} & & \\
\text{\(\bot\)-Intro: 2,3} & & \text{\(\bot\)-Elim: 4}
\end{align*}
\]

If we have derived a sentence and its negation we may derive \(\bot\) using \(\bot\)-Intro. The \(\bot\)-Elim rule represents the idea that any sentence P is deducible from a contradiction. So, from \(\bot\) we may derive any sentence P using \(\bot\)-Elim.

Here’s a deduction using the two rules.

\[
\begin{align*}
1. \text{Parent(beth, evan) & \sim Parent(beth, evan)} & \text{Basis} \\
2. \text{Parent(beth, evan)} & \text{&-Elim: 1} \\
3. \text{\sim Parent(beth, evan)} & \text{&-Elim: 1} \\
4. \text{\(\bot\)} & \text{\(\bot\)-Intro: 2,3} \\
5. \text{Parent(beth, shannon)} & \text{\(\bot\)-Elim: 4}
\end{align*}
\]

For convenience, we introduce a reiteration rule that allows us to repeat steps in a proof as needed.
We now turn to the rules for the sentential connectives that employ what we shall call sub-proofs. Consider the following inference.

1. \(\neg(Married(\text{shannon}, \text{kelly}) \& \text{OlderThan}(\text{shannon}, \text{kelly}))\)
2. \(\text{Married}(\text{shannon}, \text{kelly})\)

(therefore) \(\neg\text{Olderthan}(\text{shannon}, \text{kelly})\)

Here is an informal deduction of the conclusion from the basis sentences.

Proof: Suppose that ‘Olderthan(\text{shannon}, \text{kelly})’ is true. Then, from this assumption and basis sentence 2 it follows that ‘((\text{Shannon is married to Kelly}) \& (\text{Shannon is taller than Kelly}))’ is true. But this contradicts the first basis sentence ‘\(\neg((\text{Shannon is married to Kelly}) \& (\text{Shannon is taller than Kelly}))\)’, which is true by hypothesis. Hence our initial supposition is false. We have derived that ‘\(\neg(\text{Shannon is married to Kelly})\)’ is true.

Such a proof is called a reductio ad absurdum proof (or reductio for short). Reductio ad absurdum is Latin for something like ‘reduction to the absurd’. In order to model this proof in N we introduce the \(\neg\)-Intro rule.

\[
\begin{array}{c}
\begin{array}{c}
\text{k. } P \\
\text{.} \\
\text{.} \\
\text{m. } P \text{ Reit: k}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{k. } P \text{ Assumption} \\
\text{.} \\
\text{.} \\
\text{m. } \bot
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{n. } \neg P \text{ \(\neg\)-Intro: k-m}
\end{array}
\end{array}
\]

The \(\neg\)-Intro rule allows us to infer the negation of an assumption if we have derived a contradiction, symbolized by ‘\(\bot\)’, from the assumption. The indented proof margin (k-l) signifies a sub-proof. In a sub-proof the first line is always an assumption (and so requires no justification), which is cancelled when the sub-proof is ended and we are back out on a line that sits on a wider proof margin. The effect of this is that we can no longer appeal to any of the lines in the sub-proof to generate later lines on wider proof margins. No deduction ends in the middle of a sub-proof.

Here is a formal analogue of the above informal reductio.
1. ~\(\text{Married(shannon, kelly)} \land \text{OlderThan(shannon, kelly)}\) \hspace{1cm} \text{Basis}
2. \(\text{Married(shannon, kelly)}\) \hspace{1cm} \text{Basis}

\[
\begin{align*}
3. & \text{OlderThan(shannon, kelly)} \hspace{1cm} \text{Assumption} \\
4. & \text{(Married(shannon, kelly)} \land \text{OlderThan(shannon, kelly)}) \hspace{1cm} \& \text{Intro: 2, 3} \\
5. & \bot \hspace{1cm} \bot \text{Intro: 1, 4} \\
6. & \sim \text{Olderthan(shannon, kelly)} \hspace{1cm} \sim \text{Intro: 3-5}
\end{align*}
\]

We signify a sub-proof with the indented proof margin line; the start and finish of a sub-proof is indicated by the start and break of the indented proof margin. An assumption, like a basis sentence, is a supposition we suppose true for the purposes of the deduction. The difference is that whereas a basis sentence may be used at any step in a proof, an assumption may only be used to make a step within the sub-proof it heads. At the end of the sub-proof, the assumption is discharged. We now look at more sub-proofs in action and introduce another of N’s inference rules. Consider the following inference.

\[
\begin{align*}
1. & \text{(Male(kelly)} \lor \text{Female(kelly))} \\
2. & \text{(Male(kelly)} \rightarrow -\text{Sister(kelly, paige))} \\
3. & \text{(Female(kelly)} \rightarrow -\text{Brother(kelly, evan))} \\
\text{(therefore)} & (-\text{Sister(kelly,paige) v -Brother(kelly, evan))}
\end{align*}
\]

\textbf{Informal Proof}

By assumption ‘(Male(kelly) v Female(kelly))’ is true, i.e. by assumption at least one of the disjuncts is true.

Suppose that ‘Male(kelly)’ is true. Then by modus ponens we may derive that ‘~Sister(kelly, paige)’ is true from this assumption and the basis sentence 2. Then ‘(~Sister(kelly, paige) v ~Brother(kelly, evan))’ is true.

Suppose that ‘Female(kelly)’ is true. Then by modus ponens we may derive that ‘~Brother(kelly, evan)’ is true from this assumption and the basis sentence 3. Then ‘(~Sister(kelly, paige) v ~Brother(kelly, evan))’ is true.

So in either case we have derived that ‘(~Sister(kelly, paige) v ~Brother(kelly, evan))’ is true. Thus we have shown that this sentence is a deductive consequence of the basis sentences.

We model this proof in N using the v-Elim rule.
v-Elim

k. (P v Q)
   m. P   Assumption
       .
       .
   n. R
   o. Q   Assumption
       .
       .
p. R
q. R   v-Elim: k, m-n, o-p

The v-Elim rule allows us to derive a sentence from a disjunction by deriving it from each disjunct, possibly using sentences on earlier lines that sit on wider proof margins.

The following formal proof models the above informal one.

1. (Male(kelly) v Female(kelly))   Basis
2. (Male(kelly)→/~Sister(kelly, paige))   Basis
3. (Female(kelly)→/~Brother(kelly, evan))   Basis
4. Male(kelly)   Assumption
5. ~Sister(kelly, paige)   →-Elim: 2, 4
6. (~Sister(kelly, paige) v ~Brother(kelly, evan))   v-Intro: 5
7. Female(kelly)   Assumption
8. ~Brother(kelly, evan)   →-Elim: 3, 7
9. (~Sister(kelly, paige) v ~Brother(kelly, evan))   v-Intro: 8
10. (~Sister(kelly, paige) v ~Brother(kelly, evan))   v-Elim: 1, 4-6,7-9

Here is N’s representation of the principle of the disjunctive syllogism: for any sentences P and Q, from ¬(P v Q) and ¬P ⊨ to infer Q.
Now we introduce the \( \rightarrow \)-Intro rule by considering the following inference.

1. \((\text{Olderthan}(\text{shannon, kelly}) \rightarrow \text{OlderThan}(\text{shannon, paige}))\)
2. \((\text{OlderThan}(\text{shannon, paige}) \rightarrow \text{OlderThan}(\text{shannon, evan}))\)
(therefore) \((\text{Olderthan}(\text{shannon, kelly}) \rightarrow \text{OlderThan}(\text{shannon, evan}))\)

Informal proof

Suppose that \( \text{OlderThan}(\text{shannon, kelly}) \). From this assumption and basis sentence 1 we may derive, by modus ponens, that \( \text{OlderThan}(\text{shannon, paige}) \). From this and basis sentence 2 we get, again by modus ponens, that \( \text{OlderThan}(\text{shannon, evan}) \). Hence, if \( \text{OlderThan}(\text{shannon, kelly}) \), then \( \text{OlderThan}(\text{shannon, evan}) \).

The structure of this proof is that of a conditional proof: a deduction of a conditional from a set of basis sentence which starts with the assumption of the antecedent, then a derivation of the consequent, and concludes with the conditional. To build conditional proofs in \( N \), we rely on the \( \rightarrow \)-Intro rule.

\[ \rightarrow \text{-Intro} \]

\[
\begin{align*}
\text{k. } P & \quad \text{Assumption} \\
\text{m. } Q \\
\text{n. } (P \rightarrow Q) & \quad \text{-Intro: k-m}
\end{align*}
\]

According to the \( \rightarrow \)-Intro rule we may derive a conditional if we derive the consequent \( Q \) from the assumption of the antecedent \( P \), and, perhaps, other sentences occurring earlier in the proof on wider proof margins. Again, such a proof is called a conditional proof.

We model the above informal conditional proof in \( N \) as follows.
1. (Olderthan(shannon, kelly) \(\rightarrow\) OlderThan(shannon, paige)) Basis
2. (OlderThan(shannon, paige) \(\rightarrow\) OlderThan(shannon, evan)) Basis
   3. OlderThan(shannon, kelly) Assumption
   4. OlderThan(shannon, paige) \(\rightarrow\)-Elim: 1,3
   5. OlderThan(shannon, evan) \(\rightarrow\)-Elim: 2,4
   6. (Olderthan(shannon, kelly) \(\rightarrow\) Olderthan(shannon, evan)) \(\rightarrow\)-Intro: 3-5

Mastery of a deductive system facilitates the discovery of proof pathways in hard cases and increases one’s efficiency in communicating proofs to others and explaining why a sentence is a logical consequence of others. For example, (1) if Beth is not Paige’s parent, then it is false that if Beth is a parent of Shannon, Shannon and Paige are sisters. Suppose (2) that Beth is not Shannon’s parent. Then we may conclude that Beth is Paige’s parent. Of course, knowing the type of sentences involved is helpful for then we have a clearer idea of the inference principles that may be involved in deducing that Beth is a parent of Paige. Accordingly, we represent the two basis sentences and the conclusion in M, and then give a formal proof of the latter from the former.

1. (~Parent(beth, paige) \(\rightarrow\) ~(Parent(beth, shannon) \(\rightarrow\) Sister(shannon, paige))) Basis
2. ~Parent(beth, shannon) Basis
   3. ~ Parent(beth, paige) Assumption
   4. ~(Parent(beth, shannon) \(\rightarrow\) Sister(shannon, paige)) \(\rightarrow\)-Elim: 1, 3
   5. Parent(beth, shannon) Assumption
   6. \(\bot\) \(\bot\)-Intro: 2, 5
   7. Sister(shannon, paige) \(\bot\)-Elim: 6
   8. (Parent(beth, shannon) \(\rightarrow\) Sister(shannon, paige)) \(\rightarrow\)-Intro: 5-7
   9. \(\bot\) \(\bot\)-Intro: 4, 8
10. ~ ~ Parent(beth, paige) ~ -Intro: 3-9
11. Parent(beth, paige) ~-Elim: 10
Because we derived a contradiction at line 9, we got ‘~ ~ Parent(beth, paige)’ at line 10, using ~-Intro, and then we derived ‘Parent(beth, paige)’ by ~-Elim. Look at the conditional proof (lines 5-7) from which we derived line 8. Pretty neat, huh? Lines 2 and 5 generated the contradiction from which we derived ‘Sister(shannon, paige)’ at line 7 in order to get the conditional at line 8. This is our first example of a sub-proof (5-7) embedded in another sub-proof (3-9). It is unlikely that independent of the resources of a deductive system, a reasoner would be able to readily build the informal analogue of this pathway from the basis sentences to the sentence at line 11. Again, mastery of a deductive system such as N can increase the efficiency of our performances of rigorous reasoning and cultivate skill at producing elegant proofs (proofs that take the least number of steps to get from the basis to the conclusion).

We now introduce the intro and elim rules for the identity symbol and the quantifiers. Let \( n \) and \( n' \) be any names, and \( \Omega n \) and \( \Omega n' \) be any well-formed formula in which \( n \) and \( n' \) appear and that have no free variables.

\[
\begin{array}{c}
\text{= -Intro} & \text{= -Elim} \\
\hline
\text{k. } n=n & \text{k. } \Omega n \\
\text{=Intro} & \text{l. } n=n' \\
\text{m. } \Omega n' & \text{= -Elim: k. 1}
\end{array}
\]

The =-Intro rule allows us to introduce \( \Omega n=n \) at any step in a proof. Since \( \Omega n=n \) is deducible from any sentence, there is no need to identify the lines from which line k is derived. In effect, the = -Intro rule confirms that ‘(paige=paige)’, ‘(shannon=shannon)’, ‘(kelly=kelly)’, etc... may be inferred from any sentence(s). The =-Elim rule tells us that if we have proven \( \Omega n \) and \( \Omega n' \), then we may derive \( \Omega n' \) which is gotten from \( \Omega n \) by replacing \( n \) with \( n' \) in some but possibly not all occurrences. The =-Elim rule represents the principle known as the indiscernibility of identicals, which says that if \( \Omega n=n \), then whatever is true of the referent of \( n \) is true of the referent of \( n' \). This principle grounds the following inference

\[
\begin{align*}
1 & \sim\text{Sister(beth, kelly)} \\
2. & \text{(beth=shannon)} \\
(\text{therefore}) & \sim\text{Sister(shannon, kelly)}
\end{align*}
\]
The indiscernibility of identicals is fairly obvious. If I know that Beth isn’t Kelly’s sister and that Beth is Shannon (perhaps ‘Shannon’ is an alias) then this establishes, with the help of the indiscernibility of identicals, that Shannon isn’t Kelly’s sister. Now we turn to the quantifier rules.

Let \( \forall \Omega \neg \) be a formula in which \( v \) is the only free variable, and let \( n \) be any name.

The \( \exists \)-Intro rule, which represents the principle of inference known as existential generalization, tells us that if we have proven \( \neg \Omega n \neg \), then we may derive \( \exists \forall \Omega v \neg \) which results from \( \neg \Omega n \neg \) by replacing \( n \) with a variable \( v \) in some but possibly not all of its occurrences and prefixing the existential quantifier. According to this rule, we may infer, say, ‘\( \exists x \) Married(\( x \), matt)’ from the sentence ‘Married(beth, matt)’. By the \( \exists \)-Elim rule, we may reason from a sentence that is produced from an existential quantification by stripping the quantifier and replacing the resulting free variable in all of its occurrences by a name which is new to the proof. Recall that the language M has an infinite number of constants, and the name introduced by the \( \exists \)-Elim rule may be one of the \( w_i \). We regard the assumption at line 1, which starts the embedded sub-proof, as
saying “Suppose $n$ is an arbitrary individual from the domain of discourse such that $\neg \Omega n \wedge \neg.$” To illustrate the basic idea behind the $\exists$-Elim rule, if I tell you that Shannon admires some McKeon, you can’t infer that Shannon admires any particular McKeon such as Matt, Beth, Shannon, Kelly, Paige, or Evan. Nevertheless we have it that she admires somebody. The principle of inference corresponding to the $\exists$-Elim rule, called existential instantiation, allows us to assign this ‘somebody’ an arbitrary name new to the proof, say, ‘$w_1$’ and reason within the relevant sub-proof from ‘Shannon admires $w_1$’. Then we cancel the assumption and infer a sentence that doesn’t make any claims about $w_1$. For example, suppose that (1) Shannon admires some McKeon. Let’s call this McKeon ‘$w_1$’, i.e., assume (2) that Shannon admires a McKeon named ‘$w_1$’. By the principle of inference corresponding to $\lor$-intro we may derive (3) that Shannon admires $w_1$ or $w_1$ admires Kelly. From (3), we may infer by existential generalization (4) that for some McKeon $x$, Shannon admires $x$ or $x$ admires Kelly. We now cancel the assumption (i.e., cancel (2)) by concluding (5) that some McKeon $x$, Shannon admires $x$ or $x$ admires Kelly from (1) and the subproof (2)-(4), by existential instantiation. Here is the above reasoning set out formally.

1. $\exists x \text{Admires(shannon, x)}$ Basis  

$[w_1]$  2. $\text{Admires(shannon, } w_1)$ Assumption  

3. $\text{Admires(shannon, } w_1) \lor \text{Admires}(w_1, \text{ kelly})$ $\lor$-Intro: 2  

4. $\exists x (\text{Admires(shannon, x)} \lor \text{Admires}(x, \text{ kelly}))$ $\exists$-Intro: 3  

5. $\exists x (\text{Admires(shannon, x)} \lor \text{Admires}(x, \text{ kelly}))$ $\exists$-Elim: 1, 2-4  

The string at the assumption of the sub-proof (line 2) says “Suppose that ‘$w_1$’ is an arbitrary McKeon such that ‘$\text{Admires(shannon, } w_1)$’ is true.” This is not a sentence of $M$, but of the meta-language for $M$, i.e., the language used to talk about $M$. Hence, the $\exists$-Elim rule (as well as the $\forall$-Intro rule introduced below) has a meta-linguistic character.
∀-Intro

| [n] | k. Assumption
| .  |
| .  |
| m. \(\Omega n\) |

∀-Elim

| k. \(\forall \Omega v\) |
| m. \(\Omega n\) \(\forall\) Elim: k |

\(\forall\) Intro: k-l

\(n\) must be unique to the subproof

The \(\forall\)-Elim rule corresponds to the principle of inference known as universal instantiation: to infer that something holds for an individual of the domain if it holds for the entire domain. The \(\forall\)-Intro rule allows us to derive a claim that holds for the entire domain of discourse from a proof that the claim holds for an arbitrary selected individual from the domain. The assumption at line k reads in English “Suppose \(n\) names an arbitrarily selected individual from the domain of discourse.”

As with the \(\exists\)-Elim rule, the name introduced by the \(\forall\)-Intro rule may be one of the \(w_i\). The \(\forall\)-Intro rule corresponds to the principle of inference often called universal generalization.

For example, suppose that we are told that (1) if a McKeon admires Paige, then that McKeon admires himself/herself, and that (2) every McKeon admires Paige. To show that we may correctly infer that every McKeon admires himself/herself we appeal to the principle of universal generalization, which (again) is represented in N by the \(\forall\)-Intro rule. We begin by assuming that (3) a McKeon is named ‘\(w_1\)’. All we assume about \(w_1\) is that \(w_1\) is one of the McKeons. From (2), we infer that (4) \(w_1\) admires Paige. We know from (1), using the principle of universal instantiation (the \(\forall\)-Elim rule in N), that (5) if \(w_1\) loves Paige then \(w_1\) loves \(w_1\). From (4) and (5) we may infer that (6) \(w_1\) loves \(w_1\) by modus ponens. Since \(w_1\) is an arbitrarily selected individual (and so what holds for \(w_1\) holds for all McKeons) we may conclude from (3)-(6) that (7) every
McKeon loves himself/herself follows from (1) and (2) by universal generalization. This reasoning is represented by the following formal proof.

1. \( \forall x(\text{Admires}(x, \text{paige}) \to \text{Admires}(x, x)) \)  
   \( \text{Basis} \)
2. \( \forall x \text{Admires}(x, \text{paige}) \)  
   \( \text{Basis} \)

\[ w_1 \]
3. \[ \text{Assumption} \]
4. \( \text{Admires}(w_1, \text{paige}) \)  
   \( \forall \text{-Elim: 2} \)
5. \( (\text{Admires}(w_1, \text{paige}) \to \text{Admires}(w_1, w_1)) \)  
   \( \forall \text{-Elim: 1} \)
6. \( \text{Admires}(w_1, w_1) \)  
   \( \to \text{-Elim: 4,5} \)
7. \( \forall x \text{Admires}(x, x) \)  
   \( \forall \text{-Intro: 3-6} \)

Line 3, the assumption of the sub-proof, corresponds to the English sentence “Let \( w_1 \) be an arbitrary McKeon.” The notion of a name referring to an arbitrary individual from the domain of discourse, utilized by both the \( \forall \text{-Intro} \) and \( \exists \text{-Elim} \) rules in the assumptions that start the respective sub-proofs, incorporates two distinct ideas. One, relevant to the \( \exists \text{-Elim} \) rule, means “some specific object, but I don’t know which”, while the other, relevant to the \( \forall \text{-Intro} \) rule means “any object, it doesn’t matter which” (See Pelletier (1999), p. 118-120 for discussion).

\( K = \{ \text{All McKeons admire those who admire somebody, Some McKeon admires a McKeon}\} \)
\( X = \text{Paige admires Paige} \)

Here’s a proof that \( K \) produces \( X \).

1. \( \forall x(\exists y \text{Admires}(x, y) \to \forall z \text{Admires}(z, x)) \)  
   \( \text{Basis} \)
2. \( \exists x \exists y \text{Admires}(x, y) \)  
   \( \text{Basis} \)

\[ w_1 \]
3. \[ \text{Assumption} \]
4. \( (\exists y \text{Admires}(w_1, y) \to \forall z \text{Admires}(z, w_1)) \)  
   \( \forall \text{-Elim: 1} \)
5. \( \forall z \text{Admires}(z, w_1) \)  
   \( \to \text{-Elim: 3,4} \)
6. \( \text{Admires}(\text{paige}, w_1) \)  
   \( \forall \text{-Elim: 5} \)
7. \( \exists y \text{Admires}(\text{paige}, y) \)  
   \( \exists \text{-Intro: 6} \)
8. \( (\exists y \text{Admires}(\text{paige}, y) \to \forall z \text{Admires}(z, \text{paige})) \)  
   \( \forall \text{-Elim: 1} \)
9. \( \forall z \text{Admires}(z, \text{paige}) \)  
   \( \to \text{-Elim: 7,8} \)
10. \( \text{Admires}(\text{paige}, \text{paige}) \)  
    \( \forall \text{-Elim: 9} \)
11. Admires(paige, paige) ∃-Elim: 2, 3-10

An informal correlate put somewhat succinctly, runs as follows.

Let’s call the unnamed admirer, mentioned in (2), \( w_1 \). From this and (1), every McKeon admires \( w_1 \) and so Paige admires \( w_1 \). Hence, Paige admires somebody. From this and (1) it follows that everybody admires Paige. So, Paige admires Paige. This is our desired conclusion.

Even though the informal proof skips steps and doesn’t mention by name the principles of inference used, the formal proof guides its construction.

5.2 The deductive-theoretic definition and the common concept of logical consequence

To make it official, we now characterize logical consequence in terms of deducibility in \( N \).

A sentence \( X \) of \( M \) is a logical consequence of a set \( K \) of sentences from \( M \) iff \( X \) is deducible in \( N \) from \( K \).

We now inquire into the status of the above characterization of the logical consequence relation in terms of deductive consequence, as fixed by \( N \). It is the case that the \( \models \) and \( \vdash_N \) relations are extensionally equivalent. That is, for any set \( K \) of \( M \)-sentences and \( M \)-sentence \( X \), \( K \vdash_N X \) if and only if \( K \models X \). A soundness proof establishes \( K \vdash_N X \) only if \( K \models X \), and a completeness proof establishes \( K \vdash_N X \) if \( K \models X \). With these two proofs in hand, we say that our deductive system \( N \) is complete and sound with respect to the model-theoretic consequence relation. The question arises: which characterization of the logical consequence relation is more basic?

5.2.1 Tarski’s criticism of the deductive-theoretic definition

Tarski (1936) claims that the deductive-theoretic characterization of the common concept of logical consequence is inadequate and so, by implication, we shouldn’t regard it as a theoretical definition. Here’s a rendition of his reasoning, focusing on the \( \vdash_N \)-consequence relation defined on a language for arithmetic, which allows us to talk about the natural numbers 0, 1, 2, 3, and so on. Let ‘\( P \)’ be a predicate defined over the domain of natural numbers and let ‘\( \text{NatNum} \ (x) \)’ abbreviate ‘\( x \) is a natural number’. According to Tarski, intuitively,

\[
\forall x (\text{NatNum}(x) \rightarrow P(x))
\]

is a logical consequence of the infinite set \( S \) of sentences.
However, the universal quantification is not a $\vdash_N$-consequence of the set $S$. The reason why is that the $\vdash_N$-consequence relation is compact: for any sentence $X$ and set $K$ of sentences, $X$ is a $\vdash_N$-consequence of $K$, if and only if $X$ is a $\vdash_N$-consequence of some finite subset of $K$. Proofs in $N$ are objects of finite length; recall that a deduction is a finite sequence of sentences. Since the universal quantification is not a $\vdash_N$-consequence of any finite subset of $S$, it is not a $\vdash_N$-consequence of $S$. By the completeness of system $N$, it follows that

$\forall x (\text{NatNum}(x) \to P(x))$

is not a $\vdash$-consequence of $S$ either. Consider the structure $U$ whose domain is the set of McKeons. Let all numerals name Beth. Let the extension of ‘NatNum’ be the entire domain, and the extension of ‘$P$’ be just Beth. Then each element of $S$ is true in $U$, but ‘$\forall x (\text{NatNum}(x) \to P(x))$’ is not true in $U$. Note that the sentences in $S$ only say that $P$ holds for 0, 1, 2, and so on, and not also that 0, 1, 2, etc… are all the elements of the domain of discourse. The above interpretation takes advantage of this fact by reinterpreting all numerals as names for Beth.

What Tarski’s illustration shows is that what is called the $\omega$-rule is a correct inference rule.

The $\omega$-rule:

$\{ P(0), P(1), P(2), \ldots \}$

to infer

$\forall x (\text{NatNum}(x) \to P(x))$

with respect to any predicate $P$. Any inference guided by this rule is correct even though it can’t be represented in a deductive system as this notion has been construed here.
However, we can reflect model-theoretically the intuition that \( \forall x (\text{NatNum}(x) \rightarrow P(x)) \) is a logical consequence of set \( S \) by doing one of two things. We can add to \( S \) the functional equivalent of the claim that 1, 2, 3, etc., are all the natural numbers there are on the basis that this is an implicit assumption of the view that the universal quantification follows from \( S \). Or we could add ‘NatNum’ and all numerals to our list of logical terms. On either option it still won’t be the case that \( \forall x (\text{NatNum}(x) \rightarrow P(x)) \) is a \( \models_N \) –consequence of the set \( S \). There is no way to accommodate the intuition that \( \forall x (\text{NatNum}(x) \rightarrow P(x)) \) is a logical consequence of \( S \) in terms of a compact consequence relation. Tarski takes this to be a reason to favor his model-theoretic account of logical consequence, which, as indicated above, can reflect the above intuition over any compact consequence relation such as \( \models_N \).

Compactness is not a salient feature of logical consequence conceived deductively. This suggests that no compact consequence relation is definitive of the intuitive notion of deducibility. So, assuming that deductive system \( N \) is correct (i.e., deducibility is co-extensive in \( M \) with the \( \models_N \) relation), we can’t treat

\[
X \text{ is intuitively deducible from } K \text{ if and only if } K \models_N X
\]

as a definition of deducibility in \( M \) since

\[
X \text{ is a deductive consequence of } K \text{ if and only if } X \text{ is deducible in a correct deductive system from } K
\]

is not true with respect to languages for which deducibility is not captured by any compact consequence relation (i.e., not captured by any deduction-system account of it). Some (e.g., Quine) demur using a language for purposes of science in which deducibility is not completely represented by a deduction-system account of it because of epistemological considerations. Nevertheless, as Tarski (1936) argues, the fact that there cannot be deduction-system account of some intuitively correct principles of inference is reason for taking a model-theoretic characterization of logical consequence to be more fundamental than any characterization of it in
terms of a deductive system sound and complete with respect to the model-theoretic characterization.

5.2.2 **Is N a correct deductive system?**

In the above, we assumed that deductive system N is correct and discussed the status of the characterization of logical consequence, conceived deductive-theoretically, in terms of N. The question arises whether N is correct. That is, is the case that X is intuitively deducible from K if and only if $K \vdash_N X$? The biconditional holds only if both (1) and (2) are true.

1. If sentence X is intuitively deducible from set K of sentences, then $K \vdash_N X$.
2. If $K \vdash_N X$, then sentence X is intuitively deducible from set K of sentences.

N is not correct if either (1) or (2) is false. The truth of (1) and (2) is relevant to the correctness of the characterization of logical consequence in terms of system N, because any adequate deductive-theoretic characterization of logical consequence must identify the logical terms of the relevant language and account for their inferential properties. (1) is false if the list of logical terms in M is incomplete. In such a case, there will be a sentence X and set K of sentences such that X is intuitively deducible from set K because of at least one inferential property of logical terminology unaccounted for by N and so false that $K \vdash_N X$. In this case, N would be incorrect because it wouldn’t completely account for the inferential machinery of Language M. (2) is false if there are deductions in N that are intuitively incorrect. Is this the case? In order to fine-tune the question note that the sentential connectives, the identity symbol, and the quantifiers of M are intended to correspond to or, and, not, if…then (the indicative conditional), is identical with, some, and all. Hence, N is a correct deductive system only if the intro and elim rules of N reflect the inferential properties of the ordinary language expressions. In what follows, we sketch three views that are critical of the correctness of system N because they reject (2).

5.2.2.1 **Relevance logic**
Not everybody accepts it as a fact that any sentence is deducible from a contradiction, and so some question the correctness of the $\bot$-Elim rule. Consider the following informal proof of $Q$ from $\neg P \& \neg P \bot$, for sentences $P$ and $Q$, as a rationale for the $\bot$-Elim rule.

From (1) $P$ and $\neg P$ we may correctly infer (2) $P$, from which it is correct to infer (3) $P$ or $Q$. We derive (4) $\neg P$ from (1). (5) $P$ follows from (3) and (4).

The proof seems to be composed of valid modes of inference. Critics of the $\bot$-Elim rule are obliged to tell us where it goes wrong. Here we follow the relevance logicians Anderson and Belnap (1962) pp.105-108 (for discussion, see Read (1995) pp. 54-60). In a nutshell, Anderson and Belnap claim that the proof is defective because it commits a fallacy of equivocation. The move from (2) to (3) is correct only if $or$ has the sense of at least one. For example, from Kelly is female it is legit to infer that at least one of the two sentences Kelly is female and Kelly is older than Paige is true. On this sense of $or$ given that Kelly is female, one may infer that Kelly is female or whatever you like. However, in order for the passage from (3) and (4) to (5) to be legitimate the sense of $or$ in (3) is if not-…then. For example from if Kelly is not female, then Kelly is not Paige’s sister and Kelly is not female it is correct to infer Kelly is not Paige’s sister.

Hence, the above “support” for the $\bot$-Elim rule is defective for it equivocates on the meaning of $or$.

Two things to highlight. First, Anderson and Belnap think that the inference from (2) to (3) on the if not-…then reading of $or$ is incorrect. Given that Kelly is female it is problematic to deduce that if she is not then Kelly is older than Paige—or whatever you like. Such an inference commits a fallacy of relevance for Kelly not being female is not relevant to her being older than Paige. The representation of this inference in system N appeals to the $\bot$-Elim rule, which is rejected by Anderson and Belnap. Second, the principle of inference underlying the move from (3) and (4) to (5)—from $P$ or $Q$ and $\neg P$ to infer $Q$—is called the principle of the disjunctive syllogism. Anderson and Belnap claim that this principle is not generally valid when $or$ has the
sense of *at least one*, which it has when it is rendered by ‘v’ (e.g., see above p. 18). If Q is relevant to P, then the principle holds on this reading of *or*.

It is worthwhile to note the essentially informal nature of the debate. It calls upon our pre-theoretic intuitions about correct inference. It would be quite useless to cite the proof in N of the validity of disjunctive syllogism (p. 51) against Anderson and Belnap for it relies on the $\bot$-Elim rule whose legitimacy is in question. No doubt, pre-theoretical notions and original intuitions must be refined and shaped somewhat by theory. Our pre-theoretic notion of correct deductive reasoning in ordinary language is not completely determinatant and precise independently of the resources of a full or partial logic (see Shapiro (1991) Chapters 1 and 2 for discussion of the interplay between theory and pre-theoretic notions and intuitions). Nevertheless, hardcore intuitions regarding correct deductive reasoning do seem to drive the debate over the legitimacy of deductive systems such as N and over the legitimacy of the $\bot$-Elim rule in particular. Anderson and Belnap write that denying the principle of the disjunctive syllogism, regarded as a valid mode of inference since Aristotle, “…. will seem hopelessly naïve to those logicians whose logical intuitions have been numbed through hearing and repeating the logicians fairy tales of the past half century, and hence stand in need of further support” (p. 108). The possibility that intuitions in support of the general validity of the principle of the disjunctive syllogism have been shaped by a bad theory of inference is motive enough to consider argumentative support for the principle and to investigate deductive systems for relevance logic.

A natural deductive system for relevance logic has the means for tracking the relevance quotient of the steps used in a proof and allows the application of an introduction rule in the step from A to B “only when A is relevant to B in the sense that A is *used* in arriving at B” (Anderson and Belnap 1962, p. 90). Consider the following proof in system N.
Recall that the rationale behind the $\rightarrow$-Intro rule is that we may derive a conditional if we derive the consequent $Q$ from the assumption of the antecedent $P$, and, perhaps, other sentences occurring earlier in the proof on wider proof margins. The defect of this rule, according to Anderson and Belnap is that “from” in “from the assumption of the antecedent $P$” is not taken seriously. They seem to have a point. By the lights of the $\rightarrow$-Intro rule, we have derived line 4 but it is hard to see how we have derived the sentence at line 3 from the assumption at step 2 when we have simply reiterated the basis at line 3. Clearly, $\neg\text{Married(beth,matt)}$ was not used in inferring $\text{Admires(evan,beth)}$ at line 3. The relevance logician claims that the $\rightarrow$-Intro rule in a correct natural deductive system should not make it possible to prove a conditional when the consequent was arrived at independently of the antecedent. A typical strategy is to use classes of numerals to mark the relevance conditions of basis sentences and assumptions and formulate the intro and elim rules to tell us how an application of the rule transfers the numerical subscript(s) from the sentences used to the sentence derived with the help of the rule. Label the basis sentences, if any, with distinct numerical subscripts. Let $a, b, c, etc...$ range over classes of numerals. The $\rightarrow$ rules for a relevance natural deductive system may be represented as follows.

$\rightarrow$-Elim

| k. $(P \rightarrow Q)_a$ |
| l. $P_b$ |
| m. $Q_{a,b}$ $\rightarrow$-Elim: k, l |

$\rightarrow$-Intro

| k. $P_{\{k\}}$ Assumption |
| . |
| . |
| m. $Q_b$ |
| n. $(P \rightarrow Q)_{b-{k}}$ $\rightarrow$-Intro: k-l, provided $k \in b$ |
The numerical subscript of the assumption at line k must be new to the proof. This is insured by using the line number for the subscript.

In the directions for the $\rightarrow$-Intro rule, the proviso that $k \in b$ insures that the antecedent P is used in deriving the consequent Q. Anderson and Belnap require that if the line $m$ that results from the application of either rule is the conclusion of the proof the relevance markers be discharged.

Here is a sample proof of the above two rules in action.

1. Admires(evan, paige)$_1$ Assumption
   2. (Admires(evan, paige)$\rightarrow$Married(beth,matt))$_2$ Assumption
   3. ~Married(beth,matt)$_{1,2}$ $\rightarrow$-Elim: 1,2
   4. ((Admires(evan, paige)$\rightarrow$Married(beth,matt))$\rightarrow$Married(beth,matt))$_1$ $\rightarrow$-Intro: 2-3
   5. (Admires(evan,paige)$\rightarrow$
      ((Admires(evan, paige)$\rightarrow$Married(beth,matt))$\rightarrow$Married(beth,matt))) $\rightarrow$-Intro: 1-4

For further discussion see Anderson and Belnap (1962). For a comprehensive discussion of relevance deductive systems see their (1975). For a more up-to-date review of the relevance logic literature see Dunn (1986).

5.2.2.2 Intuitionist Logic

We now consider the correctness of the $\sim$-Elim rule and consider the rule in the context of using it along with the $\sim$-Intro rule.

~ -Intro $\sim$ -Elim

k. P Assumption k. $\sim$P
  m. P $\sim$-Elim: k
n. $\sim$P $\sim$-Intro: k-l
Here is a typical use in classical logic of the ~ Intro and ~-Elim rules. Suppose that we derive a contradiction from the assumption that a sentence P is true. So, if P were true, then a contradiction would be true which is impossible. So P cannot be true and we may infer that not-P. Similarly, suppose that we derive a contradiction from the assumption that not-P. Since a contradiction cannot be true, not-P is not true. **Then we may infer that P is true by ~-Elim.**

The intuitionist logician rejects the reasoning given in bold. If a contradiction is derived from not-P we may infer that not-P is not true, i.e. that not-not-P is true, but it is incorrect to infer that P is true. Why? Because the intuitionist rejects the presupposition behind the ~-Elim rule, which is that for any proposition P there are two alternatives: P and not-P. The grounds for this are the intuitionistic conceptions of truth and meaning.

According to intuitionistic logic, truth is an epistemic notion: the truth of a sentence P consists of our ability to verify it. To assert P is to have a proof of P, and to assert not-P is to have a refutation of P. This leads to an epistemic conception of the meaning of logical constants. The meaning of a logical constant is characterized in terms of its contribution to the criteria of proof for the sentences in which it occurs. Compare with classical logic: the meaning of a logical constant is semantically characterized in terms of its contribution to the determination of the truth conditions of the sentences in which it occurs. For example, the classical logician accepts a sentence of the form \( \neg P \lor Q \) only when she accepts that at least one of the disjuncts is true. On the other hand, the intuitionistic logician accepts \( \neg P \lor Q \) only when she has a method for proving P or a method for proving Q. But then the Law of Excluded Middle no longer holds, because a sentence of the form *P or not-*P is true, i.e. assertible, only when we are in a position to prove or refute P, and we lack the means for verifying or refuting all sentences. The alleged problem with the ~-Elim rule is that it illegitimately extends the grounds for asserting P on the basis of not-not-P since a refutation of not-P is not *ipso facto* a proof of P.
Since there are finitely many McKeons and the predicates of M seem well defined, we can work through the domain of the McKeons to verify or refute any M-sentence and so there doesn’t seem to be an M-sentence that is neither verifiable not refutable. However, consider a language about the natural numbers. Any sentence that results by substituting numerals for the variables in ‘x = y + z’ is decidable. This is to say that for any natural numbers x, y, and z, we have an effective procedure for determining whether or not x is the sum of y and z. Hence, for all x, y, and z, we may assert that x = y + z or the contrary. Let ‘A(x)’ abbreviate ‘if x is even and greater than 2 then there exists primes y and z such that x = y + z’. Since there are algorithms for determining of any number whether or not it is even, greater than 2, or prime, the hypothesis that the open formula ‘A(x)’ is satisfied by a given natural number is decidable for we can effectively determine for all pairs of numbers less than x whether they are prime. However, there is no known method for verifying or refuting Goldbach’s conjecture, for all x, A(x). Even though, for each numeral n standing for a natural number, the sentence [A(n)] is decidable (i.e., we can determine which of [A(n)] or [not-A(n)] is true, the sentence ‘for all x, A(x)’ is not. That is, we are not in a position to hold that either Goldbach’s conjecture is true or it is not. Clearly, verification of the conjecture via an exhaustive search of the domain of natural numbers is not possible since the domain is non-finite. Minus a counterexample or proof of Goldbach’s conjecture, the intuitionist demurs from asserting that either Goldbach’s conjecture is true or it is not. This is just one of many examples where the intuitionist thinks that the law of excluded middle fails.

In sum, the legitimacy of the ~-Elim rule requires a realist conception of truth as verification transcendent. On this conception, sentences have truth-values independently of the possibility of a method for verifying them. Intuitionistic logic abandons this conception of truth in favor of an epistemic conception according to which the truth of a sentence turns on our ability to verify it. Hence, the inference rules of an intuitionistic natural deductive system must be coded in such a way to reflect this notion of truth. For example, consider an intuitionistic language in which a,
range over proofs, ‘a: P’ stand for ‘a is a proof of P’, and ‘(a, b)’ stand for some suitable pairing of the proofs a and b. The & rules of an intuitionistic natural deductive system may look like the following.

\[\text{&-Intro} \quad \text{& -Elim}\]

\begin{align*}
\text{k. } & a: P \\
\text{l. } & b: Q \\
\text{m. } & (a, b): (P \& Q) \quad \&\text{-Intro: k, l.}
\end{align*}

\begin{align*}
\text{k. } & (a, b): (P \& Q) \\
\text{m. } & a: P \quad & \&\text{Elim: k} \\
\text{m. } & b: Q \quad & \&\text{Elim: k}
\end{align*}

Apart from the negation rules, it is fairly straightforward to dress the intro and elim rules of N with a proof interpretation as is illustrated above with the & rules. For the details see Van Dalen (1999). For further introductory discussion of the philosophical theses underlying intuitionistic logic see Read (1995) and Shapiro (2000). Tennant (1997) offers a more comprehensive discussion and defense of the philosophy of language underlying intuitionistic logic.

5.2.2.3 Free Logic

We now turn to the \(\exists\)-Intro and \(\forall\)-elim rules. Consider the following two inferences.

\begin{align*}
\text{(1) } & \text{Male(evan)} \\
\text{(therefore) } & \exists x \text{Male(x)} \\
\text{(2) } & \exists x \text{Male(x)} \\
\text{(therefore) } & \forall x \text{Male(x)} \\
\text{(3) } & \forall x \text{Male(x)} \\
\text{(4) } & \text{Male(evan)}
\end{align*}

Both are correct by the lights of our system N. Specifically, (2) is derivable from (1) by the \(\exists\)-Intro rule and we get (4) from (3) by the \(\forall\)-elim rule. Note an implicit assumption required for the legitimacy of these inferences: every individual constant refers to an element of the quantifier domain. If this existence assumption, which is built into the semantics for M and reflected in the two quantifier rules, is rejected, then the inferences are unacceptable. What motivates rejecting the existence assumption and denying the correctness of the above inferences?

Well, there are contexts in which singular terms are used without assuming that they refer to existing objects. For example, it is perfectly reasonable to regard the individual constants of a
language used to talk about myths and fairy tales as not denoting existing objects. For example, it seems inappropriate to infer that some actually existing individual is jolly on the basis that the sentence *Santa Claus is jolly* is true. Also, the logic of a language used to debate the existence of God should not presuppose that *God* refers to something in the world. The atheist doesn’t seem to be contradicting herself in asserting that God does not exist. Furthermore, there are contexts in science where introducing an individual constant for an allegedly existing object such as a planet or particle should not require the scientist to know that the purported object to which the term allegedly refers actually exists. A logic that allows non-denoting individual constants (terms that do not refer to existing things) while maintaining the existential import of the quantifiers (‘∀x’ and ‘∃x’ mean something like ‘for all existing individuals x’ and ‘for some existing individuals x’, respectively) is called a free logic. In order for the above two inferences to be correct by the lights of free logic, the sentence *Evan exists* must be added to the basis. Correspondingly, the ∃-Intro and ∀-elim rules in a natural deductive system for free logic may be portrayed as follows. Again, let ⊢ \( \Omega v \) be a formula in which \( v \) is the only free variable, and let \( n \) be any name.

\[
\begin{align*}
\text{∀-Elim} & \\
k. & \forall v \Omega v \\
l. & E!n \\
m. & \Omega n & \forall \text{Elim: } k, l
\end{align*}
\]

\[
\begin{align*}
\text{∃-Intro} & \\
k. & \Omega n \\
l. & E!n \\
m. & \exists v \Omega v & \exists \text{-Intro: } k, l
\end{align*}
\]

\( E!n \) abbreviates *n exists* and so we suppose that ‘E!’ is an item of the relevant language. The ∀-Intro and ∃-Elim rules in a free logic deductive system also make explicit the required existential presuppositions with respect to individual constants (for details see Bencivenga (1986), p. 387). Free logic seems to be a useful tool for representing and evaluating reasoning in contexts such as the above. Different types of free logic arise depending on whether we treat
terms that do not denote existing individuals as denoting objects that do not actually exists or as simply not denoting at all.

In sum, there are contexts in which it is appropriate to use languages whose vocabulary and syntactic, formation rules are independent of our knowledge of the actual existence of the entities the language is about. In such languages, the quantifier rules of deductive system N sanction incorrect inferences, and so at best N represents correct deductive reasoning in languages for which the existential presupposition with respect to singular terms makes sense. The proponent of system N may argue that only those expressions guaranteed a referent (e.g., demonstratives) are truly singular terms. On this view, advocated by Bertrand Russell at one time, expressions that may not have a referent such as Santa Claus, God, Evan, Bill Clinton, the child abused by Michael Jackson, are not genuinely singular expressions. For example, in the sentence Evan is male, Evan abbreviates a unique description such as the son of Matt and Beth. Then Evan is male comes to

There exists a unique x such that x is a son of Matt and Beth and x is male.

From this we may correctly infer that some are male. The representation of this inference in N appeals to both the \( \exists \)-Intro and \( \exists \)-Elim rules, as well as the \&-Elim rule. However, treating most singular expressions as disguised definite descriptions at worst generates counter-intuitive truth-value assignments (Santa Claus is jolly turns out false since there is no Santa Claus) and seems at best an unnatural response to the criticism posed from the vantagepoint of free logic.

For a short discussion of the motives behind free logic and a review of the family of free logics see Read (1995), Chapter 5. For a more comprehensive discussion and a survey of the relevant literature see Bencivenga (1986). Morscher and Hieke (2001) is a collection of recent essays devoted to taking stock of the past fifty years of research in free logic and outlining new directions.

This completes our discussion of the status of the characterization of the logical consequence relation in terms of deducibility in system N. In sum, Tarski argues that since logical
consequence is not compact according to the common concept, it cannot be defined in terms of
deducibility in a deductive system. So, there are intuitively correct principles of inference that
cannot be represented in deductive systems such as N (e.g., the \( \omega \)-rule on p.63). Is N correct as
far as it goes? In other words: Do the intro and elim rules of N represent correct principles of
inference? We sketched three motives for answering in the negative, each leading to a logic that
differs from the classical one developed here and which requires altering intro and elim rules of
N. For a comprehensive and very readable survey of proposed revisions to classical logic (those
discussed here and others) see Haack (1996). We now conclude our discussion of the concept of
logical consequence.

6. Conclusion

The central concern of this article has been the conditions that must be met in order for a
sentence to be a logical consequence of others. Using the first-order language M as the context
for our inquiry, we have discussed two approaches to answering this question: the model-
theoretic and deductive approaches. Each is motivated by a distinct development of the common
concept of logical consequence. The issue of the nature of logical consequence, which intersects
with other areas of philosophy, is still a matter of debate. Any full coverage of the topic would
involve study of the logical consequence relation between sentence from other types of languages
such as modal languages (containing necessity and possibility operators) (see Hughes and
Cresswell (1996)) and second-order languages (containing variables that range over properties)
(see Shapiro (1991)).

7. Bibliography

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