Endogenous supply of fiat money

Luis Araujo\textsuperscript{a,b,*}, Braz Camargo\textsuperscript{c}

\textsuperscript{a}Department of Economics, Michigan State University, 110 Marshall Adams Hall, East Lansing, MI 48824-1038, USA
\textsuperscript{b}Fucape, Av. Fernando Ferrari 1358, Vitoria ES 29075-010, Brazil
\textsuperscript{c}Department of Economics, Social Science Centre, University of Western Ontario, London, Ont., Canada N6A 5C2

Received 17 February 2006; final version received 18 August 2006
Available online 11 December 2006

Abstract

We consider whether reputation concerns can discipline the behavior of a long-lived self-interested agent who has a monopoly over the provision of fiat money. We obtain that when this agent can commit to a choice of money supply, there is a monetary equilibrium where it never overissues. We show, however, that monetary equilibria with no overissue do not exist when there is no commitment. This happens because the incentives this agent has to maintain a reputation for providing valuable currency disappear once its reputation is high enough. More generally, we prove that in the absence of commitment overissue happens infinitely often in any monetary equilibrium. We conclude by showing that imperfect memory can restore the positive result obtained with commitment.

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\textit{JEL classification:} C73; D82; D83; E00

\textit{Keywords:} Endogenous money; Reputation

1. Introduction

Frictions in trade are necessary if money is to be valued as a medium of exchange. The standard approach to model these frictions is to assume that trade is anonymous and decentralized. Under these assumptions, a large body of work has shown that money is feasible when its supply is exogenous. However, if the amount of money in circulation is determined by self-interested agents, this can lead to the so-called dynamic inconsistency problem: if money has value, any agent with the ability to print money faces a temptation to overissue, as any deviation from a

\textsuperscript{*}Corresponding author. Fax: +1 517 432 1068.
E-mail addresses: araujolu@msu.edu (L. Araujo), bcamargo@uwo.ca (B. Camargo).
pre-specified plan of action is likely to go unnoticed in a decentralized economy. In other words, money may not be feasible if its supply is endogenous.

The existing literature on endogenous money deals with the dynamic inconsistency problem by assuming a form of record keeping: the behavior of note issuers can be publicly monitored. See Berentsen [5], Cavalcanti et al. [6], Cavalcanti and Wallace [7], Martin and Schreft [16], Ritter [18], and Williamson [19], for example.¹ In this article we depart from this approach and address the dynamic inconsistency problem in a decentralized economy where money is issued by a single self-interested agent and its choice of money supply is private. The absence of record keeping means that agents can only learn about the money supply from their own experience, i.e., information is decentralized. Hence, the monitoring of the money supplier’s behavior is private and imperfect. This assumption about information is a natural one in an economy where trade is decentralized.

The starting point of our analysis is a simple version of the model introduced in Kiyotaki and Wright [13], modified in a number of ways. First, as indicated above, the money supply is privately determined in each period. Moreover, the money supplier is either patient or impatient, and this is also its private information. Second, the other agents in the economy can now decide between staying in autarky or entering the market and transacting with the help of money. The money supplier’s revenue from money issue in a given period is proportional to how much new currency it prints and to the measure of agents who choose the market at this point in time. In particular, holding everything else constant, this revenue is higher if there is overissue. Finally, autarky is always better than the market if money is always overissued, which happens when the money supplier is impatient, but the opposite is true when overissue never takes place.

Since the market is always worse than autarky when the money supplier is impatient, the patient money supplier faces a trade-off between short-run gains from overissue and long-run losses due to a decrease in its reputation for providing valuable currency. Indeed, if it overissues, the agents in the economy become more convinced that the money supplier they face is impatient, leading to a smaller revenue from money issue in future periods. The idea that reputation concerns may help solve the dynamic inconsistency problem is not new. Klein [14] considers an environment where such trade-off is present. In his model, however, this trade-off is assumed rather than derived, and it turns out that this has important consequences.

Notice that in general the choice of money supply affects both the frequency of trade meetings (the extensive margin) and the terms of trade in such meetings (the intensive margin). However, since in our environment money and goods are indivisible and there is an unit upper bound on money holdings, the only margin that is affected is the extensive one. This simplifies the analysis considerably, but preserves the trade-off between reputation and short-run gains from overissue.

We first consider the case where the choice of money supply in the first period is binding, the so-called full-commitment case. We show that in this situation there is an equilibrium where the patient money supplier never overissues as long as its discount factor is high enough. The intuition is simple. If the patient money supplier behaves as expected and never overissues, its reputation increases over time, which leads to a steady stream of revenue from money issue. If, instead, the patient supplier deviates and always overissues, its revenue from money issue increases in the short-run. However, its reputation for being patient disappears over time, and so its revenue from money issue decreases to zero in the long-run. Hence, if the patient money supplier cares enough about the future, deviating is not profitable.

¹ An exception is Monnet [17], who establishes the feasibility of endogenous fiat money without monitoring. The dynamic inconsistency problem is not present in his framework, though.
We then consider the no-commitment case, where the money supplier can change its behavior at any point in time. In this case, a policy where the patient supplier never overissues is not time-consistent. Indeed, if it never overissues, its reputation for being patient, and thus providing valuable currency, increases over time. Eventually, a point is reached where all agents in the market are so convinced that the money supplier they face is patient that any negative experience is attributed to bad luck. At this stage, the patient supplier would rather overissue. The cost of doing so, a reduction in future revenue from money issue due to a decrease in reputation, is almost zero, while the immediate benefit is substantial. In other words, the “reputational” cost of overissue for the patient money supplier eventually becomes negligible, at which point it has a profitable deviation.

In light of this negative result, a natural question to ask is what type of equilibria are possible in the no-commitment case. For instance, is it possible to have a monetary equilibrium where the net gain of choosing the market is bounded away from zero when the money supplier is patient? We show that the same logic that rules out the no-overissue equilibrium also rules out these other equilibria. A consequence of this result is that the patient money supplier must overissue infinitely many times in any monetary equilibrium.

The discussion so far suggests that a monetary equilibrium with no overissue may become possible if the reputational cost of overissue is somehow bounded away from zero. Motivated by this, we modify the no-commitment case by assuming that in every period a fraction of the population becomes uninformed. We show that an equilibrium where the patient money supplier never overissues is possible with this form of imperfect memory. The reason is that now the patient supplier has always an incentive to look after its reputation: whenever it overissues, the negative impact on its reputation is non-negligible.

Besides the literature on endogenous money, this work also belongs to the literature on reputation. A related paper in this literature is Mailath and Samuelson [15], who consider an environment where monitoring is private and imperfect. A crucial difference is that in our environment the agents have an outside option, staying in autarky.

This article is structured as follows. We introduce the basic setup in the next section and define a monetary equilibrium in Section 3. Section 4 considers the full-commitment case, Section 5 considers the no-commitment case, and Section 6 analyzes the no-commitment case with imperfect memory. Section 7 provides a discussion of our modeling choices. Section 8 concludes and several appendices collect details and proofs that are omitted from the main text.

2. Basic setup

We first introduce the baseline model and then compute payoffs.

2.1. Baseline model

Time is discrete and indexed by \( t \). The economy has one large infinitely lived agent, the money supplier. Its discount factor \( \delta \) is either zero or \( \delta_p > 0 \). In the first case we say the money supplier is impatient, while in the second case we say it is patient. The value of \( \delta \), however, is known to the money supplier only. The economy is also populated by a large number of small infinitely lived agents, the agents, that we describe in the paragraphs that follow.

The economy starts in \( t = 1 \) with a mass one of agents, all with the same prior belief \( \theta_0 \in (0, 1) \) that \( \delta = \delta_p \). Moreover, in every \( t > 1 \) each agent born in the previous period gives birth to another agent, who inherits his parent’s private history. So, a mass one of agents enters the economy in
each period. Later on, we see that an agent’s private history determines his belief about the money supplier’s discount factor (type). Hence, any agent born after \( t = 1 \) starts with the same belief about \( \delta \) as his parent. An agent in his first period of life is said to be newly born. Finally, any agent who is not newly born has a probability \( \gamma > 0 \) of dying at the beginning of every period. An agent can only die after giving birth, and when this happens his money holdings leave the economy.

All agents have the same discount factor \( \beta \in (0, 1) \), so that \( \beta = \beta(1 - \gamma) \) is their effective discount factor. They also have a type that is determined when they first enter the economy. There are \( K > 2 \) of these types, one for each type of good that can be produced in the economy. The probability that a newly born agent is of the type \( k \in \{1, \ldots, K\} \) is the same in every period, \( 1/K \). Agents of type \( k \) can only consume a type \( k \) good, their so-called preferred good.

Production works as follows. Each newly born agent receives a non-perishable endowment and makes a once and for all decision between moving to autarky or entering the market. In autarky, an agent uses his endowment as an input to a production technology. In each period there are \( n \geq 1 \) production opportunities, and each good produced yields utility \( a \). In the market, an agent uses his endowment in the production of indivisible and perishable goods. An agent of type \( k \) can only produce, at a cost \( c \) per unit, a good of type \( k+1 \mod K \), his so-called endowment good. An agent in the market can hold at most one unit of either goods or money at any point in time.

The money supplier derives utility from the consumption of all \( K \) goods, but cannot produce any of them. It has, however, the technology to print indivisible units of fiat money. These units provide no direct benefit, but can be offered in exchange for goods. Each newly born agent who enters the market is approached by the money supplier with a certain probability \( m \), in which case he receives one unit of fiat money in exchange for one unit of his endowment good. The value of \( m \) is restricted to \( \{m_L, m_H\} \), with \( \frac{1}{2} \leq m_L < m_H < 1 \), and is determined by the money supplier in each period.\(^2\) No agent observes this choice. If \( \mu \) is the measure of newly born agents who enter the market in a given period, the money supplier’s flow payoff from choosing \( m \) in this period is \( \mu m \). From now on, we say that the money supplier overissues when it chooses \( m_H \).

The market is organized as follows. There are \( K \) sectors, each one specialized in the exchange of one of the \( K \) available goods. Agents can identify sectors, but inside each sector they are randomly and anonymously matched in pairs. Since \( K > 2 \), there are no double coincidence of wants meetings. An agent, however, can trade his endowment good for money and use money to buy his preferred good. If an agent wants money, he goes to the sector that trades his endowment good and searches for an agent with money. If he has money, he goes to the sector that trades his preferred good and searches for an agent who can produce it. When a single coincidence of wants meeting takes place, the buyer transfers his money to the seller and the latter produces one unit of his endowment good for the buyer, who consumes it to obtain utility \( u > c \). In any period, an agent in the market faces a number of meetings that is equal to the number \( n \) of per-period production opportunities that are available in autarky. From now on we refer to \( n \) as the number of market meetings in a period.

An implicit assumption in the above description of the market is that there is a positive measure of agents in it at any point in time. Since once in the market an agent does not leave it, a necessary and sufficient condition for this is that a positive measure of agents enters the market in period

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\(^2\) In what follows we show that the assumption that \( \frac{1}{2} \leq m_L < m_H < 1 \), together with the indivisibility of money and the unit upper bound on money holdings, implies that an increase in the amount of money in circulation reduces the agents’ expected payoff in the market. Thus, the interests of the money supplier and the agents are not aligned.
one. When the measure of agents in the market is zero, i.e., when the market is “empty”, money does not circulate and the market flow payoff is zero.

Notice that we take the behavior of the agents and the money supplier in the market as given. It is possible, in a natural way, to model the market environment as a game involving the agents and the money supplier. This game has an equilibrium where the agents always exchange their endowment for one unit of money if approached by the money supplier and, as long as their effective discount factor $\beta$ is high enough, their behavior in the market is as described.

To summarize, the timing in each period is as follows. Agents born in the previous period give birth. Then, each agent who is not newly born leaves the economy with probability $\gamma$, taking his money holdings away with him. After that, the newly born agents make their market–autarky decisions. The ones who decide to enter the market receive one unit of money with probability $m$. Only then the market meetings take place and payoffs are collected.

2.2. Payoffs

Suppose that in period $t$ the choice of $m$ by the money supplier is $m_t$ and the fraction of agents in the market with (one unit of) money is $\eta_t$. Notice that if $\eta_t > 0$, then $\eta_t \in [m_L, m_H]$, since the money supplier’s choice of $m$ is constrained to $[m_L, m_H]$. Moreover, $\eta_t > 0$ implies that $\eta_k > 0$ for all $k > t$; i.e., if the market is not empty in $t$, then it is not empty in all subsequent periods.

Let $w^j_{q,t}$ be the expected lifetime payoff for an agent in the market in period $t$ with $q \in \{0, 1\}$ units of money right before this period’s $j$th market meeting, and let $w_{q,t}^{n+1} = \hat{\beta}w_{q,t+1}^1$. Assume, without loss, that agents do not discount the future within a period. Then, if $\eta_t > 0$,

\[
\begin{align*}
    w^j_{1,t} &= \eta_tw^j_{1,t+1} + (1 - \eta_t)[u + w^j_{0,t+1}], \\
    w^j_{0,t} &= \eta_t[w^j_{1,t} - c] + (1 - \eta_t)w^j_{0,t} + 1.
\end{align*}
\]

(1)

An agent with money before his $j$th market meeting in $t$ has probability $\eta_t$ of meeting another agent with money, in which case trade occurs and he obtains utility $u$. A similar interpretation holds for the second equation.

For each $t \in \mathbb{N}$, we can rewrite (1) as $w^j_t = B(\eta_t)w^{j+1}_{t+1} + b(\eta_t)$, with $j \in \{1, \ldots, n\}$, where

\[
B(\eta) = \left( \begin{array}{c} \eta & 1 - \eta \\ \eta & 1 - \eta \end{array} \right), \quad w^j_t = \left( \begin{array}{c} w^j_{1,t} \\ w^j_{0,t} \end{array} \right), \quad \text{and} \quad b(\eta) = \left( \begin{array}{c} (1 - \eta)u \\ -\eta c \end{array} \right).
\]

(2)

Solving this last system of equations recursively, we obtain that if $\eta_t > 0$, then

\[
w^1_t = b(\eta_t) + (n - 1)B(\eta_t)b(\eta_t) + B(\eta_t)\hat{\beta}w^1_{t+1} = [I + (n - 1)B(\eta_t)]b(\eta_t) + B(\eta_t)\hat{\beta}w^1_{t+1}.
\]

(3)

Notice that we made use of the fact that $B^2(\eta) = B(\eta)$ for all $\eta \in (0, 1)$. In what follows we omit the superscript from $w^1_t, w^1_{1,t}$, and $w^1_{0,t}$. In particular, from now on $w_{q,t}$ denotes the lifetime expected payoff from entering the market in $t$ with $q$ units of money. Now solving (3) recursively, we obtain that

\[
w_t = C(\eta_t)b(\eta_t) + \sum_{k=1}^{\infty} \hat{\beta}^k B(\eta_t) \cdots B(\eta_{t+k-1})C(\eta_{t+k})b(\eta_{t+k})
\]

(4)
if the market is not empty in \( t \).\(^3\) If the market is empty in \( t \), \( w_t = \beta^t w_{t_0} \), where \( t_0 \) is the first period when the market is not empty and \( t_0 = +\infty \) if the market is always empty.

Let \( v_t \) be the lifetime expected reward from entering the market in \( t \) and \( a(m) \) be the \( 2 \times 1 \) row vector \((m, 1 - m)\). Assume, without loss, that a newly born agent who decides to enter the market does not discount the time between this decision and his first meeting in the market. Then,

\[
v_t = m_t(w_{1,t} - c) + (1 - m_t)w_{0,t} = a(m_t)w_t - m_t c.
\]

When necessary, we indicate the dependence of \( w_{q,t}, w_t, \) and \( v_t \) on the sequence \( \{\eta_{t+k-1}\}_{k=1}^\infty \) by writing \( w_{q,t} = w_{q,t}(\{\eta_{t+k-1}\}) \), \( w_t = w_t(\{\eta_{t+k-1}\}) \), and \( v_t = v_t(\{\eta_{t+k-1}\}) \). Notice that we omit the dependence of these quantities on \( n \), the number market meetings in one period. In Appendix A we show that \( v_t \) is a strictly decreasing function of \( \eta_k \) for \( k \geq t \) when \( \eta_t > 0 \).

Consider now the case where \( m_t = \eta_t = m \in \{m_L, m_H\} \) for all \( t \in \mathbb{N} \), so that the market environment is stationary, and denote the expected lifetime utility from entering the market with \( q \) units of money by \( w_q(m) \). Since \( B(\eta)C(\eta) = nB(\eta) \) for all \( \eta \in (0, 1) \), Eq. (4) implies that

\[
\begin{align*}
    w_1(m) &= (1 - m)u + (n - 1)m(1 - m)(u - c) + \frac{1}{n} m (1 - m)(u - c),
    \\
    w_0(m) &= -mc + (n - 1)m(1 - m)(u - c) + \beta(1 - \beta)^{-1}nm(1 - m)(u - c).
\end{align*}
\]

Hence, the expected lifetime payoff from choosing the market in this particular case is

\[
v(m) = (1 - \beta)^{-1} nm(1 - m)(u - c) - mc.
\]

Let \( v_A = (1 - \beta)^{-1}na \) denote the lifetime expected payoff from choosing autarky. Notice that both \( v(m) \) and \( v_A \) are proportional to \( n \).

**Assumption 1.** \( \frac{1}{n} v(m_H) < \frac{1}{n} v_A < \frac{1}{n} v(m_L) \) for all \( n \in \mathbb{N} \).

Observe that the impatient money supplier, being myopic, always chooses \( m_H \), whether it can commit to its period one choice of \( m \) or not. Also observe that if a positive measure of agents enters the market in period 1, and the money supplier’s choice of \( m \) is the same in every period, then \( \eta_t = m \) for all \( t \). Hence, Assumption 1 implies that no matter the number of market meetings in a period, if a positive measure of agents enters the market in the first period, then: (i) the market is always worse than autarky when the money supplier is impatient; (ii) the market is always better than autarky when the money supplier chooses \( m_L \) in every period.

Assumption 1 also implies that there is a unique \( \theta \in (0, 1) \) such that \( \theta v(m_L) + (1 - \theta)v(m_H) = v_A \). Denote this value of \( \theta \) by \( \theta_M \). Notice that \( \theta_M \) depends on \( n \), but is bounded away from one since \( mc/n \) converges to zero as \( n \) increases to infinity. Therefore, we need that \( \theta_0 \geq \theta_M \), otherwise no agent would ever enter the market. Indeed, \( v_t(\{\eta_{t+k-1}\}) \) is a strictly decreasing function of \( \eta_k \) for all \( k \geq t \) when \( \eta_t > 0 \). Moreover, the market flow payoff is zero when the market is empty. So, the highest payoff a newly born agent can obtain if he chooses the market is \( v(m_L) \).

**Assumption 2.** \( \theta_0 \geq \theta_M \) for all \( n \in \mathbb{N} \).

\(^3\) In particular, \( w_{1,t} - c - w_{0,t} = (1 - \eta_t)(u - c) > 0 \) if \( \eta_t \in (0, 1) \). Hence, a newly born agent who enters the market is always willing to accept money in exchange for his endowment good if he knows that the market is not empty. Moreover, irrespective of the amount of money in circulation, an agent is always willing to produce in exchange for a note as long as \( \beta > c/[1(1 - m_H)u + m_H c] \).
3. Equilibrium

We first describe the histories and strategies for the money supplier and the agents. We then discuss how a strategy profile is mapped into aggregate behavior. Next, we show how agents use their experience to update their beliefs about the money supplier’s type. Finally, we define what a monetary equilibrium is and establish a preliminary result.

3.1. Strategies

Let \( H_t \) denote the set of possible period \( t \) histories for the money supplier. Then, \( H_1 \) is the singleton set containing the empty history and \( H_t = ([0, 1] \times \{m_L, m_H\})^{t−1} \) if \( t > 1 \). The money supplier’s history in \( t > 1 \) is the sequence of its previous choices of \( m \) together with the list of measures of agents who entered the market in the periods preceding \( t \). A strategy for the money supplier is a sequence \( \tau^t = \{\tau_t^s\} \) of contingent plans, where \( \tau_t^s : [0, \delta_p) \times H_t \rightarrow [0, 1] \) is the Borel measurable function mapping the money supplier’s type and period \( t \) history into the probability that it chooses \( m_L \) in \( t \). Since it is always optimal for the impatient money supplier to overissue, we restrict attention to strategies \( \tau^t = \{\tau_t^s\} \) such that \( \tau_t^s(0, \cdot) \equiv 0 \) for all \( t \in \mathbb{N} \).

When making his market-autarky decision, the only piece of information a newly born agent has is the private history he inherits from his parent. Let \( H_t \), with typical element \( h^t \), denote the set of all possible histories for an agent born in \( t \). By assumption, \( H_1 \) is the singleton set containing the empty history. We describe \( H_t \) for \( t > 1 \) in the next two paragraphs. A strategy for an agent born in \( t \) is a Borel measurable function \( s_t : H_t \rightarrow [0, 1] \), where \( s_t(h^t) \) is the probability that he chooses the market given a private history \( h^t \).

An agent’s history in his first period of life is his decision together with his subsequent experience in this period. If he chooses autarky, he observes nothing. If he goes to the market, his experience consists of how many units of money he receives from the money supplier and the money holdings of his partners, if the market is not empty. Define a family to be the collection of all agents whose genealogy can be traced back to a given agent born in \( t = 1 \) of the empty history. We describe \( H_t \) for \( t > 1 \) in the next two paragraphs. A strategy for an agent born in \( t \) is a Borel measurable function \( s_t : H_t \rightarrow [0, 1] \), where \( s_t(h^t) \) is the probability that he chooses the market given a private history \( h^t \).

Let \( \Pi = \{A, \emptyset\} \cup \{M, [0, 1] \times \{e, 0, \ldots, n\}\} \), where \( A \) denotes the event that autarky is chosen, \( \emptyset \) denotes the event that nothing is observed, \( M \) denotes the event that the market is chosen, \( q \in [0, 1] \) is the number of notes received from the money supplier, \( e \) denotes the event that the market is empty, and \( r \in [0, \ldots, n] \) is the number of market meetings an agent faces in his first period of life where a partner has one unit of money. Then, \( H_t = \Pi^{t−1} \) for \( t > 1 \). In what follows, we denote an arbitrary element of \( \Pi \) by \( \pi = (d, \omega) \), where \( d \in \{A, M\} \) and \( \omega \in [0, 1] \times \{e, 0, \ldots, n\} \).

Identify the set of families with the unit interval and let \( \Delta_t \) denote the set of Borel measurable functions from \( H_t \) into \([0, 1] \). Loosely speaking, a strategy profile for the agents is an equivalence class of sequences \( \tau^a = \{\tau_t^a\} \), where \( \tau_t^a \) maps \([0, 1] \) into \( \Delta_t \) and two sequences \( \tau_1^a = \{\tau_1^a, t\} \) and \( \tau_2^a = \{\tau_2^a, t\} \) are considered to be the same if for all \( t \in \mathbb{N} \) the functions \( \tau_t^{a, t} \) and \( \tau_t^{b, t} \) differ on a set of Lebesgue measure zero in \([0, 1] \). We interpret \( \tau_t^a(i) \in \Delta_t \) as the strategy of the generation \( t \) member of the family labeled by \( i \in [0, 1] \). The details can be found in Appendix B.

3.2. Aggregate behavior

Suppose \( \tau^q \) is such that the patient money supplier follows a pure strategy and let \( \tau^q = \{\tau_t^q\} \) be a strategy profile for the agents. Since there is no aggregate uncertainty, the measure \( \mu_1 \) of
agents who enter the market in period 1 is deterministic. It is also independent of the money supplier’s type. Denote the money supplier’s choice of m in period 1 by \(m_1\). It is deterministic by assumption. Together with \(\tau^t\), it induces a probability measure \(\lambda_2\) over the Borel sets of \(H_2\) such that \(\lambda_2(D)\) is the fraction of agents born in \(t = 2\) with private histories in \(D \subseteq H_2\). Note that \(\lambda_2\) is a function of the money supplier’s type.

Let \(m_2\) denote the money supplier’s choice of \(m\) in period 2. If the supplier is impatient, \(m_2 = m_H\). If the supplier is patient, \(m_2\) is a deterministic function of \((\mu_1, m_1)\). Once more because there is no aggregate uncertainty, the pair \((\lambda_2, \tau^{t}_2)\) completely determines the measure \(\mu_2\) of agents who enter the market in \(t = 2\). Unlike \(\mu_1, \mu_2\) is a function of the money supplier’s type. To finish, observe that the list \((m_1, m_2, \mu_1, \mu_2)\) together with \(\lambda_2\) determine a Borel probability measure \(\lambda_3\) over \(H_3\) such that \(\lambda_3(D)\) is the fraction of agents born in \(t = 3\) that have private histories in \(D \subseteq H_3\). Like \(\lambda_2, \lambda_3\) depends on the money supplier’s type.

Continuing with this process, we obtain sequences \(\{m_t(\tau^t, \tau^a, \delta)\}\) and \(\{\mu_t(\tau^t, \tau^a, \delta)\}\) such that if \(\delta\) is the money supplier’s discount factor, then \(m_t(\tau^t, \tau^a, \delta)\) is the money supplier’s choice of \(m\) in \(t\) and \(\mu_t(\tau^t, \tau^a, \delta)\) is the fraction of agents born in \(t\) who enter the market. Notice that \(m_t(\tau^t, \tau^a, 0) = m_H\) for all \(t\). We also obtain a sequence \(\{\lambda_t(\tau^t, \tau^a, \delta)\}\) such that \(\lambda_t(\tau^t, \tau^a, \delta)\) is the Borel probability measure over \(H_t\) with the property that \(\lambda_t(\tau^t, \tau^a, \delta)(D) = \lambda_t(D|\tau^t, \tau^a, \delta)\) is the fraction of agents born in \(t > 1\) with private histories in \(D \subseteq H_t\) when the money supplier’s discount factor is \(\delta\). The important point is that if the patient money supplier uses a pure strategy, then both its behavior over time and the aggregate behavior of the agents over time are deterministic.

### 3.3. Belief updating

Let \(m_t(\delta)\) be the choice of \(m\) in \(t\), \(\mu_t(\delta)\) be the measure of newly born agents who enter the market in \(t\), and \(\eta_t(\delta)\) be the fraction of agents in the market in \(t\) that have money, all as a function of the money supplier’s type. Since \(m_t(0) = m_H\) for all \(t\), either \(\eta_t(0) = 0\), which happens when the market is empty in \(t\), or \(\eta_t(0) = m_H\). On the other hand, \(\eta_t(\delta_p)\) can change over time even when the market is not empty. In fact, \(\eta_t(\delta_p) = 0\) if \(\sum_{k=1}^{t}(1 - \gamma)^{t-k}\mu_k(\delta_p)\), the mass of agents in the market in period \(t\) when the money supplier is patient, is equal to zero, while

\[
\eta_t(\delta_p) = \frac{\sum_{k=1}^{t}(1 - \gamma)^{t-k}\mu_k(\delta_p)m_k(\delta_p)}{\sum_{k=1}^{t}(1 - \gamma)^{t-k}\mu_k(\delta_p)}
\]

(8)

if \(\sum_{k=1}^{t}(1 - \gamma)^{t-k}\mu_k(\delta_p) > 0\). Moreover, there may be periods when the market is empty if the money supplier is of one of type, but not of the other. As a consequence, the belief an agent born in \(t\) has that the money supplier is patient depends not only on his private history \(h^t \in H_t\), but also on the sequences \(\{\mu_t(0)\}, \{\mu_t(\delta_p)\}, \{m_t(0)\}\) and \(\{m_t(\delta_p)\}\). Denote this belief by \(\theta(h^t; \{\mu_t\}, \{m_t\})\). When there is no risk of confusion, we omit its dependence on \(\{\mu_t(\delta)\}\) and \(\{m_t(\delta)\}\).

Let \(\Omega = \{0, 1\} \times \{e, 0, \ldots, n\}\) be the set of events an agent can experience in his first period of life if he chooses the market and define \(X_t(\delta; \{\mu_t\}, \{m_t\})\) to be the random variable on \(\Omega\) such that if \(q \in \{0, 1\}\) and \(r \in \{0, \ldots, n\}\), then

\[
\Pr\{X_t(\delta; \{\mu_t\}, \{m_t\}) = (q, e)\} = \begin{cases} m_t(\delta)^q(1 - m_t(\delta))^{1-q} & \text{if } \sum_{k=1}^{t}(1 - \gamma)^{t-k}\mu_k(\delta) = 0, \\ 0 & \text{otherwise} \end{cases}
\]

(9)
and
\[
\Pr\{X_t(\delta; \mu_t, \{m_t\}) = (q, r)\} = \begin{cases} 
\left(\frac{n}{r}\right) m_t(\delta)q(1 - m_t(\delta))(1 - q)^{n-1} \eta_t(\delta)^r & \text{if } \sum_{k=1}^t (1 - \gamma)^{t-k} \mu_k(\delta) > 0, \\
(1 - \eta_t(\delta))^{n-r} & \text{otherwise.}
\end{cases}
\]

Observe that if the money supplier’s discount factor is \(\delta\), then: (i) \(m_t(\delta)q(1 - m_t(\delta))(1 - q)^{n-1}\eta_t(\delta)^r\) is the probability an agent born in \(t\) has of receiving \(q \in \{0, 1\}\) units of money upon entering the market; and (ii) \(\left(\frac{n}{r}\right) \eta_t(\delta)^r(1 - \eta_t(\delta))^{n-r}\) is the probability he has of meeting \(r \in \{0, \ldots, n\}\) agents with one unit of money in the market in \(t\) if \(\eta_t(\delta) > 0\). We also omit the dependence of \(X_t\) on \(\mu_t(\delta)\) and \(\{m_t(\delta)\}\) when there is no chance of confusion.

The belief \(\theta(h')\) can then be computed as follows. In the first period, \(\theta(h^1) = \theta_0\), the prior belief that the money supplier is patient. Now fix \(t \geq 1\) and assume that \(\theta(h')\) is defined for all \(h' \in H_t\). Moreover, let \(h_{t+1} = (h^t, \pi)\), with \(\pi = (d, \omega) \in \Pi\), be an element of \(H_{t+1}\). If \(d = A\), then \(\theta(h^t, \pi) = \theta(h^t)\). If \(d = M\), so that \(\omega \in \Omega\), then
\[
\theta(h^t, \pi) = \frac{\theta(h^t) \Pr\{X_t(\delta_p) = \omega\}}{\theta(h^t) \Pr\{X_t(\delta_p) = \omega\} + (1 - \theta(h^t)) \Pr\{X_t(0) = \omega\}}
\]
in case the denominator is positive. In case the denominator is zero, set \(\theta(h^t, \pi)\) equal to \(\theta(h^t)\).

### 3.4. Equilibria

It is clear that in order to define an equilibrium where the patient money supplier uses a pure strategy, we have to take into account that: (i) the agents need \(\{\mu_t(\delta)\}\) and \(\{m_t(\delta)\}\) to compute the expected payoff from choosing the market, since this payoff depends on the belief that the money supplier is patient; (ii) the sequences \(\mu_t(0)\), \(\mu_t(\delta_p)\), and \(m_t(\delta_p)\) depend on the aggregate behavior of the agents. Hence, the requirement that correct expectations about \(\mu_t(0)\), \(\mu_t(\delta_p)\), and \(m_t(\delta_p)\) are held needs to be included in such a definition.

**Definition 1.** Let \(\tau^s\) be a strategy for the money supplier and \(\tau^a\) be a strategy profile for the agents. Moreover, let \(\Theta : \bigcup_{t=1}^{\infty} H_t \to [0, 1]\) be a belief updating rule for the agents, i.e., \(\Theta(h')\) is the belief an agent born in \(t\) with history \(h'\) has that the money supplier is patient. The list \(\sigma = (\tau^s, \tau^a, \Theta, \{\mu_t(0)\}, \{\mu_t(\delta_p)\}, \{m_t(\delta_p)\})\) is a (deterministic) equilibrium if:

(a) \(\tau^s = \{\tau^s_t\}\) is such that \(\tau^s_t(\delta_p, \cdot) \in \{0, 1\}\) and \(\tau^s_t(0, \cdot) \equiv 0\) for all \(t\);
(b) \(\Theta^s(h) = \theta(h; \{\mu_t\}, \{m_t\})\) for all \(h \in \bigcup_{t=1}^{\infty} H_t\);
(c) the agents hold correct expectations about the sequences \(\{\mu_t(0)\}, \{\mu_t(\delta_p)\}, \{m_t(\delta_p)\}\). In other words, \(\mu_t(\delta) = \mu_t(\tau^s, \tau^a, \delta)\) and \(m_t(\delta_p) = m_t(\tau^s, \tau^a, \delta_p)\) for all \(\delta \in \{0, \delta_p\}\) and \(t \in \mathbb{N}\);
(d) the patient money supplier’s behavior is sequentially rational. In particular,
\[
\{m_t(\tau^s, \tau^a, \delta_p)\} \in \arg \max \left\{ (1 - \delta_p) \sum_{t=1}^{\infty} \delta_p^{t-1} \mu_t(\tau^s, \tau^a, \delta_p) m_t; \{m_t\} \in \{m_L, m_H\}^\infty \right\};
\]
(e) the decision rules of almost all agents are optimal given the belief updating rule \(\Theta\).

---

4 But not the sequence \(\{m_t(0)\}\), which is constant and equal to \(m_H\) regardless of how the agents behave.
When the patient money supplier uses a mixed strategy, the evolutions of both \( m_t(\delta_p) \) and \( \mu_t(\delta_p) \) are no longer necessarily deterministic. If this is the case, the belief a newly born agent has about the money supplier’s type is not enough to determine his lifetime expected payoff from choosing the market. He also needs a (history dependent) conjecture about how \( m_t(\delta_p) \) and \( \mu_t(\delta_p) \) are going to evolve starting with his period of birth. This means that the equilibrium concept introduced above needs to be modified if one wants to consider this more general case. One exception is when the patient money supplier’s behavior fails to be deterministic only off the equilibrium path, in which case the definition given above is appropriate. We restrict attention to deterministic equilibria in this article.

Suppose \( \sigma \) is an equilibrium and let \( N_1(\sigma) = \{ t \in \mathbb{N} : \mu_t(0) > 0, \mu_t(\delta_p) > 0 \} \) be the set of periods where, regardless of the money supplier’s type, a positive measure of agents enters the market. Lemma 1 implies that \( \mathbb{N} \setminus N_1(\sigma) = N_0(\sigma) = \{ t \in \mathbb{N} : \mu_t(0) = \mu_t(\delta_p) = 0 \} \). One consequence of this result is that the market is never completely informative about the money supplier’s type in any equilibrium. Its proof is in Appendix C.

**Lemma 1.** Suppose \( \sigma \) is an equilibrium. Then, for all \( t \in \mathbb{N} \), \( \mu_t(0) = 0 \) if, and only if, \( \mu_t(\delta_p) = 0 \).

For any equilibrium \( \sigma \), the fraction of agents who are in the market in period \( t \) as a function of the money supplier’s type is

\[
\frac{\gamma}{1 - (1 - \gamma)^t} \sum_{k=1}^{t} (1 - \gamma)^{t-k} \mu_k(\delta)
\]

(12)

Hence, if \( N_1(\sigma) \) is finite, this fraction converges to zero regardless of the money supplier’s type. In other words, any equilibrium where a positive measure of agents enters the market only in a finite number of periods has monetary trade collapsing in the long-run. For this reason, we say an equilibrium \( \sigma \) is monetary only when \( N_1(\sigma) \) is infinite.

**Definition 2.** An equilibrium \( \sigma \) is monetary if \( N_1(\sigma) \) is infinite.

### 4. The full-commitment case

In this section we assume that the money supplier can commit to its period 1 choice of \( m \), i.e., once it chooses the value of \( m \) in the first period, it cannot change it afterwards. This is equivalent to using the equilibrium notion introduced in the previous section, but reducing the set of strategies of the money supplier to \( \{ \tau^s_{LL}, \tau^s_{LH}, \tau^s_{HL}, \tau^s_{HH} \} \), where \( \tau^s_{kl} \) is the strategy for the money supplier where it chooses \( m_k \) after every history if it is impatient and it chooses \( m_l \) after every history if it is patient.

We show that when \( \delta_p \) is close enough to 1, a monetary equilibrium where the money supplier chooses \( \tau^{s,c} = \tau^s_{HL} \) exists. Moreover, the sequence \( \{ \mu_t(\delta_p) \} \) is bounded away from zero in this equilibrium, which implies that the subset of the population that transacts with money when the money supplier is patient does not vanish over time.

For this, let \( \Theta^c \) be such that: (i) \( \Theta^c(h^1) = \theta_0 \); (ii) if \( h^{t+1} = (h^t, d, \omega) \), then \( \Theta^c(h^{t+1}) = \Theta^c(h^t) \) when \( d = A \) or \( d = M \) and \( \omega \in \{0, 1\} \times \{e\} \), and

\[
\Theta^c(h^{t+1}) = \frac{\Theta^c(h^t) m^{q+r}_L (1 - m_L)^{n+1-q-r}}{\Theta^c(h^t) m^{q+r}_L (1 - m_L)^{n+1-q-r} + (1 - \Theta^c(h^t)) m^{q+r}_H (1 - m_H)^{n+1-q-r}}
\]

(13)
when \( d = M \) and \( \omega = (q, r) = \{0, 1\} \times \{0, \ldots, n\} \). Recall that \( e \) denotes the event that the market is empty. Now define \( \tau^{a,c} = \{\tau^{a,c}_t\} \) to be such that \( \tau^{a,c}_t(\cdot) \equiv s^{a,c}_t \), where

\[
s^{a,c}_t(h) = \begin{cases} 
1 & \text{if } \Theta^c(h) \geq \theta_M , \\
0 & \text{if } \Theta^c(h) < \theta_M .
\end{cases}
\] (14)

Recall that \( \theta_M \) is the value of \( \theta \) for which \( \theta v(M_L) + (1 - \theta) v(M_H) \) is equal to \( v_A \). To finish, let \( \mu^c_t(\delta_p) = \mu_t(\tau^{c,e}, \tau^{a,c}, \delta_p) \), \( \mu^c_t(0) = \mu_t(\tau^{c}, \tau^{a,c}, 0) \), \( m^{c}_t(\delta_p) \equiv m_L \) and \( m^{c}_t(0) \equiv m_H \). Notice that \( \mu^c_t(\delta_p) = \mu^c_t(0) = 1 \).

**Theorem 1.** For each number \( n \) of per-period market meetings there exists \( \delta \in (0, 1) \) such that \( \sigma^c = (\tau^{c,e}, \tau^{a,c}, \Theta^c, \{\mu^c_t(\delta_p)\}, \{\mu^c_t(0)\}, \{m^c_t(\delta_p)\}) \) is a monetary equilibrium if \( \delta_p > \delta \). Moreover, the sequence \( \{\mu^c_t(\delta_p)\} \) is always bounded away from zero.

**Proof.** Fix \( n \). Notice first that expectations are trivially satisfied. Under \( \tau^{c,e} \), a positive measure of agents enters the market in period 1. Hence, if the money supplier follows \( \tau^{c,e} \), the fraction of agents in the market that have money is constant over time: \( m_L \) when the money supplier is patient and \( m_H \) when the money supplier is impatient. Therefore, \( \Theta^c(h^t) = \Theta(h^t; \{\mu^c_t\}, \{m^c_t\}) \) for all \( h \in \bigcup_{t=1}^{\infty} H_t \). Moreover, \( v(m_L) \) is the lifetime expected payoff from entering the market when the money supplier is patient and \( v(m_H) \) is the same payoff when the money supplier is impatient, and so \( s^{c}_t \) is an optimal strategy for an agent born in \( t \). Now observe, by [3, Proposition 2], that \( \mu^c_t(0) \rightarrow 0 \) and there exists \( \underline{\mu} = \underline{\mu}(n) > 0 \) such that \( \mu^c_t(\delta_p) \rightarrow \underline{\mu} \). In particular, the sequence \( \{\mu^c_t(\delta_p)\} \) is bounded away from zero regardless of \( n \). Since \( (1 - \delta_p) \sum_{t=1}^{\infty} \delta_p^{-1} \mu^c_t(\delta_p) m_L \) is the patient money supplier’s payoff when it chooses \( m_L \) and \( (1 - \delta_p) \sum_{t=1}^{\infty} \delta_p^{-1} \mu^c_t(0) m_H \) is its payoff when it chooses \( m_H \), we can then conclude that there is \( \delta \in (0, 1) \) such that it is optimal for the patient money supplier to choose \( m_L \) when \( \delta_p > \delta \). This follows from the fact that if \( \{xt\} \) is a numerical sequence with limit \( x_{\infty} \), then \( \lim_{\delta \rightarrow 1^-} (1 - \delta) \sum_{t=1}^{\infty} \delta_p^{-1} x_t = x_{\infty} \). Consequently, \( \sigma^c \) is an equilibrium when \( \delta_p > \delta \). Notice that \( \delta \) depends on \( n \), since the sequences \( \{\mu^c_t(\delta_p)\} \) do. To finish, observe that \( \mathbb{N}_1(\sigma^c) = \mathbb{N}_1 \), and so \( \sigma^c \) is monetary. \( \square \)

5 The setting in [3] is slightly different from the setting considered in this article. It is straightforward to adapt the proof of their Proposition 2 to our environment.

5. The no-commitment case

By restricting the money supplier to make an once and for all decision on the value of \( m \) in period 1, we rule out any considerations about the time-consistency of its behavior. In this section we investigate what happens when the money supplier can change its decision of \( m \) at the beginning of every period. In what follows, we refer to the belief that the money supplier is patient as the belief only. The distribution of these beliefs among the newly born agents is the money supplier’s reputation.

It turns out that for some of the arguments that we make below it is convenient to reinterpret the decision problems of the successive generations of agents in this economy as follows. Associate to each family a myopic decision maker, the family lawyer, who is now responsible for the decisions of the members of this family; i.e., in each \( t \) he decides whether the generation \( t \) member of the family he represents enters the market or not. Assume that the period \( t \) flow payoff and private history of a family lawyer are, respectively, the expected lifetime payoff and private history of the

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5 The setting in [3] is slightly different from the setting considered in this article. It is straightforward to adapt the proof of their Proposition 2 to our environment.
generation \( t \) member of his family. In particular, a strategy profile \( \tau^a = \{ \tau^a_i \} \) for the agents is also a strategy profile for the lawyers (and vice versa): \( \tau^a(i) = \{ \tau^a_i(i) \} \) is the strategy for the lawyer representing the family indexed by \( i \). In this way, a family lawyer behaves optimally if, and only if, the decisions he makes over time coincide with the decisions the members of his family would make if they were to optimally choose between the market and autarky. Notice that the problem faced by the family lawyers is a two-armed bandit where one arm, the autarky, has known payoffs, and the other arm, the market, has (in principle) non-stationary payoffs.

We start by arguing, somewhat informally, that \( \sigma^c \) cannot be an equilibrium when there is no commitment. To see why, notice first that autarky is absorbing in \( \sigma^c \). Hence, for the generation \( t \) member of a family to enter the market, it must be that all previous generations of his family did the same. Moreover, the market is informative about the money supplier’s type in this equilibrium. Therefore, if the money supplier is patient, for any \( \theta \in [0, 1) \) the fraction of agents who enter the market in \( t \) with beliefs greater than \( \theta \) can be made as close to one as needed, as long as \( t \) is taken to be sufficiently large.\(^6\) In particular, for each \( k \in \mathbb{N} \) there exists \( \hat{t} \in \mathbb{N} \) with the property that a fraction close to one of the families with generation \( \hat{t} \) members entering the market has their next \( k \) generations making the same decision whether the patient money supplier chooses \( m_L \) in \( \hat{t} \) or not. In other words, the patient money supplier’s reputation is so high in \( \hat{t} \) that its choice of money supply in this period has a negligible impact on its revenue from money issue in the next \( k \) periods. Hence, by choosing \( k \) large enough (how large \( k \) needs to be depends on \( \delta_p \)), the patient money supplier has a profitable deviation in \( \hat{t} \).

Notice that the above argument relies on the fact that autarky is absorbing in \( \sigma^c \). It is easy to show that if we modify \( \tau^{a,c} \) to allow the agents to randomize when indifferent between the market and autarky, then \( \sigma^c \) remains an equilibrium in the full-commitment case, but autarky is no longer absorbing.\(^7\) The market is still informative about the money supplier’s type, though, and so the measure of newly born agents who are indifferent between the market and autarky converges to zero over time. Otherwise, there is a positive measure of family lawyers who send an infinite number of members of their families to the market, but who remain indifferent between the market and autarky, a contradiction. Hence, the reasoning of the previous paragraph is also valid in this case.

Now let \( \sigma = (\tau^e, \tau^a, \Theta, \{\mu_t(0)\}, \{\mu_t(\delta_p)\}, \{m_t(\delta_p)\}) \) be such that \( m_t(\delta_p) = m_L \). By Lemma 1, \( \sigma \) is an equilibrium only if there is \( t \geq 1 \) such that \( \mu_t(\delta_p) > 0, \mu_t(0) > 0, \) and \( \mu_t(\delta_p) = \mu_t(0) = 0 \) for all \( t < t \). Hence, from period \( t \) on, the fraction of agents in the market with money is \( m_L \) when the money supplier is patient and \( m_H \) when the money supplier is impatient. Therefore, it is optimal for a newly born agent to enter the market in \( t \geq \hat{t} \) if his belief is greater than \( \theta_M \) and to stay in autarky if his belief is smaller than \( \theta_M \). Consequently, if \( \sigma \) is to be an equilibrium, then the only instances when the decisions of the agents can differ from the ones of \( \sigma^c \) are when \( t < t \) or when their beliefs are equal to \( \theta_M \). Because what happens before \( t \) only affects the beliefs of a measure zero of agents, we can set \( t = 1 \) without loss of generality. We are then in the case of the last paragraph, and so \( \sigma \) cannot be an equilibrium.

To summarize, regardless of the number \( n \) of per-period market meetings, there is no equilibrium in the no-commitment case where the patient money supplier never overissues. In face of this negative result, it is natural to ask what type of monetary equilibria are possible in this case. The following theorem is a partial answer to this question. Recall that an equilibrium \( \sigma \) is monetary

\(^6\) This follows from the consistency of Bayes estimates in the discrete case.

\(^7\) More generally, we could let the agents follow asymmetric strategies where their market–autarky decisions differ from the ones of \( \tau^{a,c} \) when they are indifferent between the market and autarky.
only when \( \mathbb{N}_1(\sigma) \), the set of periods when a positive measure of agents enters the market, is infinite. A consequence of Theorem 2, and the main result of this section, is that in any monetary equilibrium the patient money supplier overissues infinitely many times in \( \mathbb{N}_1(\sigma) \).

**Theorem 2.** Suppose \( \sigma \) is a monetary equilibrium and let \( v_t \) be expected lifetime payoff from entering the market in \( t \) when the money supplier is patient. Then, regardless of the number of market meetings in one period, \( \liminf_{t \in \mathbb{N}_1(\sigma)} v_t = v_A \).

The proof of this result is in Appendix D. A rough sketch of it is as follows. Suppose \( \sigma \) is monetary. Then, \( v_t > v_A \) for all \( t \in \mathbb{N}_1(\sigma) \), so that \( \liminf_{t \in \mathbb{N}_1(\sigma)} v_t \geq v_A \). Hence, \( \{\eta_t(\delta_p)\} \) cannot converge to \( m_H \), otherwise \( v_t \) eventually becomes smaller than \( v_A \). Recall that \( \eta_t(\delta_p) \) is the fraction of agents in the market in \( t \) that have money when the supplier is patient. In particular, the market is informative about the money supplier's type in any monetary equilibrium. Suppose then, by contradiction, that \( \liminf_{t \in \mathbb{N}_1(\sigma)} v_t > v_A \). In this case, the sequence \( \{\mu_t(\delta_p)\}_{t \in \mathbb{N}_1(\sigma)} \) is bounded away from zero. A consequence of this fact is that \( m_t(\delta_p) = m_L \) infinitely many times in \( \mathbb{N}_1(\sigma) \), otherwise \( \{\eta_t(\delta_p)\} \) converges to \( m_H \). This, in turn, implies that there is a profitable deviation for the patient money supplier. The existence of such a deviation follows from a reasoning similar to the one used to argue that the patient money supplier has a profitable deviation in \( \sigma' \).

The lack of commitment implies that in any monetary equilibrium \( \sigma \), the gain \( v_t - v_A \) from choosing the market cannot be bounded away from zero in \( \mathbb{N}_1(\sigma) \). This does not mean that this gain necessarily converges to zero in this set, which would be the case when \( \limsup_{t \in \mathbb{N}_1(\sigma)} v_t = v_A \). What prevents this more negative result is the existence of an outside option for the agents. In periods where the gain from entering the market is small, even small changes in the patient money supplier’s reputation can have relatively large effects on its revenue from money issue, as the market–autarky decision is very knife-edge. This has the potential of disciplining the patient money supplier’s behavior to a certain extent. Not enough, however, to prevent the fraction of agents with money in the market from eventually becoming greater than \( m_L \) in any monetary equilibrium, which is a consequence of the result that follows.

**Corollary 1.** Let \( \mathbb{N}_1(\sigma, m_k) = \{ t \in \mathbb{N}_1(\sigma) : m_t(\delta_p) = m_k \} \). Then, \( \mathbb{N}_1(\sigma, m_H) \) is an infinite set if \( \sigma \) is a monetary equilibrium.

**Proof.** Fix the number \( n \) of per-period market meetings. Since \( a(m)B(\eta) = a(\eta) \) and \( a(m)C(\eta) = a(m) + (n - 1)a(\eta) \) for all \( m, \eta \in (0, 1) \), Eqs. (4) and (5) in Section 2 imply that

\[
v_t = a(m_t(\delta_p))b(\eta_t(\delta_p)) + (n - 1)a(\eta_t(\delta_p))b(\eta_t(\delta_p)) - m_t(\delta_p)c
\]

\[+n \sum_{k=1}^{\infty} \hat{\beta}^k a(\eta_{t+k}(\delta_p))b(\eta_{t+k}(\delta_p)). \tag{15}\]

Suppose then, by contradiction, that \( \mathbb{N}_1(\sigma, m_H) \) is finite. Because \( \eta_t(\delta_p) \) only changes value when \( t \in \mathbb{N}_1(\sigma) \) and in any \( t > 1 \) a positive mass of agents leaves the economy, taking their money holdings away with them, it must be that \( \eta_t(\delta_p) \to m_L \). Consequently, since \( a(m)b(m) = m(1 - m)(u - c) \), \( \{v_t\}_{t \in \mathbb{N}_1(\sigma)} \) converges to

\[
n \left\{ a(m_L)b(m_L) + \sum_{k=1}^{\infty} \hat{\beta}^k a(m_L)b(m_L) \right\} - m_t c = v(m_L) > v_A, \tag{16}\]

a contradiction. \( \square \)
6. Imperfect memory

The analysis so far suggests that in order to have an equilibrium where the patient money supplier never overissues there must be something that prevents its reputation among the newly born agents from increasing too much when it always chooses $m_L$. Put differently, we need a mechanism that provides the patient money supplier with the incentive to always invest in its reputation by never choosing $m_H$.

With this in mind, we modify our environment by assuming that in any period $t$ there is a probability $\rho > 0$ that a newly born agent does not inherit his parent’s private history. Instead, this agent’s market–autarky decision is based on his prior belief that the money supplier is patient, that we assume to be $\theta_0$. In this way, regardless of what the patient money supplier does, there is always a positive measure of newly born agents for whom its reputation is not high. The equilibrium notion we use is still the one introduced in Section 3. We only have to change the definition of $H_t$ for $t > 1$ to incorporate the fact that memory is imperfect. Now $H_t = \bigcup_{k=1}^{t-1} \Pi^k \cup \{N\}$ for $t > 1$.

The way we interpret an element $h^t$ of this set is the following. If an agent born in $t$ inherits no history, $h^t = N$. If the last ancestor of the agent born in $t$ who fails to inherit a history is born $k$ periods before, $h^t \in \Pi^k$.

Let $\Theta^*: \bigcup_{t=1}^{\infty} H_t \rightarrow [0, 1]$ be the belief updating rule given by

$$
\Theta^*(h) = \begin{cases} 
\theta_0 & \text{if } h = N, \\
\Theta^c(h) & \text{otherwise,}
\end{cases}
$$

(17)

where $\Theta^c$ is the belief updating rule of Section 4. Define $\tau^{a,*} = \{\tau^a_t\}$ to be such that $\tau^a_t(\cdot) \equiv s^c_t$, where $s^c_t$ is given by (14) with $\Theta^*$ in the place of $\Theta^c$. Now let $\mu^*_t(\delta) = \mu_t(\tau^{*,*}, \tau^{a,*}, \delta)$, $m^*_t(0) \equiv m_H$, and $m^*_t(\delta_p) \equiv m_L$, where $\tau^{*,*} = \tau^*_{HL}$ is the strategy profile for the money supplier introduced in Section 4. Notice that $\Theta^*$ and $\{\mu^*_t(\delta)\}$ depend on the number $n$ of per-period market meetings. We make the dependence of the latter on $n$ explicit by writing $\mu^*_t(\delta) = \mu^*_t(\delta, n)$ in the remainder of this section. The next theorem shows that imperfect memory works as desired when $n$ is large enough; i.e., there is a monetary equilibrium where the patient money supplier never overissues as long as its discount factor is not too small.\footnote{We would like to thank one referee for drawing our attention to a mistake in our original proof.}

Theorem 3. Let $\underline{\delta} = (m_H - m_L)(1 - \rho)[m_H - (1 - \rho)m_L]^{-1}$ and suppose

$$(1 - \gamma)m_L + \gamma m_H > \ln \left(\frac{1 - m_L}{1 - m_H}\right) \left[\ln \left(\frac{m_H}{m_L} \cdot \frac{1 - m_L}{1 - m_H}\right)\right]^{-1}.
$$

(18)

There is $n_1 \in \mathbb{N}$ such that if $n \geq n_1$, then $\sigma^* = (\tau^{*,*}, \tau^{a,*}, \Theta^*, \{\mu^*_t(0)\}, \{\mu^*_t(\delta_p)\})$ is a monetary equilibrium in the no-commitment case with imperfect memory when $\delta_p > \underline{\delta}$.

Let $g : [0, 1) \rightarrow \mathbb{R}$ be given by

$$
g(\rho) = \frac{m_H - m_L}{(1 - \rho)[m_H - (1 - \rho)m_L]}.
$$

(19)

Notice that $g(0) = 1$, $g$ has a global minimum at $\rho = \frac{1}{4} = m_H/4m_L$, and $\lim_{\rho \rightarrow 1} g(\rho) = +\infty$. Hence, there is $\overline{\rho} < 1$, that depends on $m_H - m_L$, such that $\delta < 1$ if, and only if $\rho \in (0, \overline{\rho})$. The restriction that $\rho$ cannot be too large is intuitive. If the probability that private histories are
not passed from one generation to the next is large, the patient money supplier derives no benefit from choosing \( m_L \), as any good reputation it builds is quickly eroded.

Now let \( h : [0, 1 - m_L) \rightarrow \mathbb{R} \) be such that

\[
h(x) = \ln \left( \frac{1 - m_L}{1 - m_L - x} \right) \left[ \ln \left( \frac{m_L + x}{m_L} \cdot \frac{1 - m_L}{1 - m_L - x} \right) \right]^{-1},
\]

(20)

where \( x = m_H - m_L \). It is possible, but tedious, to show that \( h \) is convex and

\[
h(x) = m_L + \frac{[m_L(1 + m_L) - (1 - m_L)^2]}{2(1 - m_L)} x + o(x^2).^9
\]

(21)

Therefore, (18) can only be satisfied if \( m_H \) is sufficiently close to \( m_L \), and it is necessary that

\[
\gamma > \frac{[m_L(1 + m_L) - (1 - m_L)^2]}{2(1 - m_L)}.
\]

(22)

The right-hand side of (22) is increasing in \( m_L \), its lowest value is \( \frac{1}{2} \) (when \( m_L = \frac{1}{2} \)), and it equals one when \( m_L = \frac{3}{5} \). Hence, we need \( \gamma > \frac{1}{2} \) and \( m_L < \frac{3}{5} \) if (18) is to be satisfied. This happens, for example, when \( m_L = \frac{1}{2}, m_H = \frac{3}{5} \), and \( \gamma = .505 \). With these values of \( m_L \) and \( m_H \), \( \delta \) is smaller than 1 if, and only if, \( \rho \in (0, \frac{4}{5}) \).

The restriction that \( \gamma > \frac{1}{2} \) is somewhat severe, as it implies that \( \hat{\beta} < \frac{1}{2} \), but we only need it to be satisfied when \( n \) is large. Moreover, by setting the cost \( c \) of production low enough, we can make sure that it is always optimal for an agent in the market to produce in exchange for money.

**Proof of Theorem 3.** We know, from the proof of Theorem 1, that expectations are trivially satisfied, \( \Theta^*(\cdot) \equiv \theta(\cdot; \{\mu_t^x\}, \{m_t^x\}) \), and agents behave optimally given the money supplier’s behavior and the behavior of the other agents. Hence, we just need to check that there is \( n_1 \in \mathbb{N} \) such that the patient money supplier has no profitable one-shot deviation if \( \delta_p > \hat{\delta} \) when \( n \geq n_1 \).

Fix a period \( k \) and suppose the money supplier is patient. Given any history \( h \in \mathcal{H}_k \) for the money supplier, we can compute the distribution of beliefs across the agents born in \( k \) induced by \( h \). This distribution depends on \( n \). Denote it by \( \pi_k(h, n) \) and let \( D_k(\theta, n) \) be the money supplier’s gain from a one-shot deviation in \( k \) if all agents born in this period have the same belief \( \theta \). The money supplier’s gain from a one-shot deviation in \( k \), as a function of \( h \) and \( n \), is then given by

\[
D_k(h, n) = \int D_k(\theta, n) \pi_k(d\theta|h, n).
\]

(23)

Because memory is imperfect, for each \( h \in \mathcal{H}_k \) there exists a distribution \( \tilde{\pi}_k(h, n) \) such that

\[
D_k(h, n) = (1 - \rho) \int D_k(\theta) \tilde{\pi}_k(d\theta|h, n) + \rho D_k(\theta_0, n).
\]

(24)

Consider an agent born in \( k \) with belief \( \theta \). For each \( t \in \mathbb{N} \), denote by \( \xi_{t,k}(m, \theta, n) \) the probability, when the number of per-period market meetings is \( n \), that the generation \( k + t \) member of this agent’s family enters the market in \( k + t \) if: (i) the money supplier chooses \( m_L \) in all periods but \( k \) and chooses \( m \in \{m_L, m_H\} \) in \( k \); (ii) private histories are always transmitted from one generation

\[^9\text{Recall that } f(x) = o(x^2) \text{ if } \lim_{x \to 0} x^{-2} f(x) = 0.\]
to the next. Notice that \( \tilde{\xi}_{t,k}(m_L, \theta, n) = \tilde{\xi}_{t,1}(m_L, \theta, n) \) for all \( k \geq 2, \theta \in [0, 1], \) and \( n \in \mathbb{N}, \) as the fraction of agents in the market that have money is always \( m_L, \) when the money supplier chooses \( m_L \) in every period. Now let \( \phi_{t,k}(m, \theta, n) \) be the same probability when private histories can fail to be transmitted. Then,

\[
\phi_{t,k}(m, \theta, n) = (1 - \rho)^t \tilde{\xi}_{t,k}(m, \theta, n) + \sum_{j=1}^{t-1} \rho (1 - \rho)^j \tilde{\xi}_{j,k+(t-1)-j}(m_L, \theta_0, n) + \rho. \tag{25}
\]

Indeed, with probability \((1 - \rho)^t\) private histories are always passed from one generation to the next until period \( k + t, \) while with probability \( \rho (1 - \rho)^j\) the last period before \( k + t \) when private histories fail to be transmitted is \( k + (t - 1) - j, \) with \( j \in \{0, \ldots, t - 1\}. \) Moreover, any agent who does not inherit a private history from his father enters the market.

Let \( \zeta \) be the indicator function of \([\theta_M, 1].\) Then, for all \( k \in \mathbb{N}, \)

\[
D_k(\theta, n) = (m_H - m_L)\zeta(\theta) + m_L \sum_{t=1}^{\infty} \delta_p[\phi_{t,k}(m_H, \theta, n) - \phi_{t,k}(m_L, \theta, n)]
\]

\[= (m_H - m_L)\zeta(\theta) + m_L \sum_{t=1}^{\infty} \delta_p(1 - \rho)^t[\tilde{\xi}_{t,k}(m_H, \theta, n) - \tilde{\xi}_{t,k}(m_L, \theta, n)], \tag{26}\]

where the second equality follows from (25). Now observe that for each \( k, t, n \in \mathbb{N}, \) \( \tilde{\xi}_{t,k}(m_H, \theta, n) \geq \tilde{\xi}_{t,k}(m_H, \theta, n) \) for all \( \theta \in [0, 1] \) and \( \tilde{\xi}_{t,k}(m_L, 1, n) = \tilde{\xi}_{t,k}(m_H, 1, n) = 1. \) Hence, \( D_k(\theta, n) \leq D_k(1, n) \) for all \( k, n \in \mathbb{N} \) and \( \theta \in [0, 1], \) which implies that

\[
D_k(h, n) \leq (1 - \rho)D_k(1, n) + \rho D_k(\theta_0, n) = D_k(n) \tag{27}
\]

for all \( h \in \mathcal{H}_k \) regardless of \( k \) and \( n. \) Therefore, it is enough to show that there is \( n_1 \in \mathbb{N} \) such that \( D_k(n) \) is negative for all \( k \in \mathbb{N} \) if \( n \geq n_1 \) and \( \delta_p > \delta. \)

For this observe that when the money supplier is patient and the number of market meetings in one period is \( n, \) the on-the-equilibrium path amount of money in the market in period \( k - 1 \) is \( \sum_{t=1}^{k-1} (1 - \gamma)^{k-1-t} \mu^*_t(\delta_p, n)m_L. \) Consequently, if the patient money supplier does a one-shot deviation in period \( k, \) the fraction of agents in the market that have money increases from \( m_L \) to

\[
\frac{(1 - \gamma) \sum_{t=1}^{k-1} (1 - \gamma)^{k-1-t} \mu^*_t(\delta_p, n)m_L + \mu^*_k(\delta_p, n)m_H}{(1 - \gamma) \sum_{t=1}^{k-1} (1 - \gamma)^{k-1-t} \mu^*_t(\delta_p, n) + \mu^*_k(\delta_p, n)} \eta(k, n) \tag{28}
\]

in this period.

Since all agents in period 1 have the same belief \( \theta_0 \) about the money supplier’s type and there is no aggregate uncertainty, \( \mu^*_t(\delta_p, n) = 1 \) for all \( n \in \mathbb{N} \) and \( \mu^*_t(\delta_p, n) = \phi_{t-1,1}(m_L, \theta_0, n) \) for all \( t \geq 2 \) and \( n \in \mathbb{N}. \) In particular,

\[
\eta(k, n) \geq m_L + \frac{\mu^*_k(\delta_p, n)}{1 - (1 - \gamma)^{k+1}} \gamma(m_H - m_L) > m_L + \mu^*_k(\delta_p, n)\gamma(m_H - m_L). \tag{29}
\]

Now let \( \varepsilon > 0 \) and suppose \( \bar{t} \geq 2 \) is such that \( \sum_{j=1}^{\bar{t}-1} \rho (1 - \rho)^j > 1 - \varepsilon/2. \) Notice that in the limiting case when \( n = \infty, \) the market meetings in any one period became completely informative about the money supplier’s choice of \( m \) in this period. Hence,

\[
\lim_{n \to \infty} \tilde{\xi}_{t,1}(m_L, \theta_0, n) = 1 \quad \forall t \in \mathbb{N}, \tag{30}
\]
and so there is \( n_0 \in \mathbb{N} \) such that \( \xi_{t,k}(m_L, \theta_0, n) = \xi_{t,1}(m_L, \theta, n) > 1 - \varepsilon/2\kappa \) for all \( k \in \mathbb{N} \) and \( j \in \{1, \ldots, \tilde{t} - 1\} \) if \( n \geq n_0 \), where \( \kappa = (1 - \rho) \). Because

\[
\phi_{k-1,1}(m_L, \theta_0, n) = (1 - \rho)^{k-1} \xi_{k-1,1}(m_L, \theta_0, n) \\
+ \sum_{j=1}^{k-2} \rho(1 - \rho)^j \xi_{j,k-1-j}(m_L, \theta_0, n) + \rho, \tag{31}
\]

we can then conclude that \( n \geq n_0 \) implies that: (i) if \( k \in \{1, \ldots, \tilde{t}\} \), then

\[
\phi_{k-1,1}(m_L, \theta_0, n) > [ (1 - \rho)^{k-1} + \sum_{j=1}^{k-2} \rho(1 - \rho)^j ] \left( 1 - \frac{\varepsilon}{2\kappa} \right) + \rho \\
= 1 - \frac{(1 - \rho)\varepsilon}{2\kappa} = 1 - \frac{\varepsilon}{2^j}, \tag{32}
\]

(ii) if \( k > \tilde{t} \), then

\[
\phi_{k-1,1}(m_L, \theta_0, n) > \sum_{j=1}^{\tilde{t}-1} \rho(1 - \rho)^j \left( 1 - \frac{\varepsilon}{2\kappa} \right) + \rho > 1 - \varepsilon. \tag{33}
\]

Consequently, (18) and (29) jointly imply that there is \( n_0 \in \mathbb{N} \) such that

\[
\eta(k, n) > \ln \left( \frac{1 - m_L}{1 - m_H} \right) \left[ 1 + \ln \left( \frac{m_H}{m_L} \cdot \frac{1 - m_L}{1 - m_H} \right) \right]^{-1} \tag{34}
\]

for all \( k \in \mathbb{N} \) when \( n \geq n_0 \).

To finish, note that

\[
D_k(n) = (m_H - m_L) - \rho m_L \sum_{t=1}^{\infty} \delta_p(1 - \rho)^t [\xi_{t,1}(m_L, \theta_0, n) - \xi_{t,k}(m_H, \theta_0, n)]. \tag{35}
\]

In Appendix E we show that the fact that there is \( n_0 \in \mathbb{N} \) such that (34) is satisfied for all \( k \in \mathbb{N} \) when \( n \geq n_0 \) implies that \( \xi_{t,k}(m_H, \theta_0, n) \) converges to zero as \( n \to \infty \) uniformly in \( t \) and \( k \). By condition (30), we can then conclude that

\[
\lim_{n \to \infty} \sup_{k \in \mathbb{N}} | D_k(n) - (m_H - m_L) + m_L \delta_p(1 - \rho) \frac{1 - \delta_p(1 - \rho)}{1 - \delta_p(1 - \rho)} | = 0. \tag{36}
\]

Because \( \delta_p > \frac{1}{2} \) implies that \( m_L \delta_p(1 - \rho)[1 - \delta_p(1 - \rho)]^{-1} > m_H - m_L \), the desired result holds. \( \square \)

7. Modeling options

We assume that money and goods are indivisible and there is an unit upper bound on money holdings. This assumption greatly simplifies the analysis while still capturing the idea that in a fully decentralized economy an agent learns from his market experience. If goods and/or money are divisible, the interaction between the agents in the market becomes complex. In order to determine the terms of trade in a single coincidence of wants meeting, we have to solve a bargaining problem.
with two-sided incomplete information. The interaction between the agents and the money supplier is also more complex. Now agents want to bargain with the money supplier over how much they should produce in exchange for money. Nevertheless, the trade-off between reputation and short-run gains from overissue is still present in this more general setting, and similar conclusions should hold.

Our simplifying assumptions come at a cost, though. If money is indivisible and there is an upper bound on its holdings, a constant injection of money in the economy is not feasible unless there is population growth. Otherwise, the effect of continuous money creation on the extensive margin destroys the value of money in the long-run even when the bank is patient. At the same time, we need to ensure that the money supplier can be punished for misbehavior, which in our setting is possible only if future generations are less willing to enter the market. Hence, the assumption that the market experience of one generation gets passed to the succeeding generations.

Another simplifying assumption is that the money supplier’s choice of \( m \) is restricted to two values only. Our model can be extended to the case where the choice set of the money supplier is finite, with the same conclusions. There is, however, little gain from doing so. What matters for the agents in this setting is not the exact amount of money in circulation, but whether the market is better or worse than autarky. Hence, a model where the money supplier is constrained to two choices, one that makes the market better than autarky and one that makes it worse, is enough for our purposes.

8. Conclusion

This work contributes to the literature on endogenous money. It addresses the feasibility of fiat money when its supply is determined by a single self-interested agent. This is done in an environment where trade is decentralized and agents are anonymous and have heterogenous preferences, so that money is essential. We depart from previous work by assuming that: (i) there is uncertainty about the money supplier’s preferences, so that there is a role for reputation; (ii) there is no technology that allows the money supplier to be publicly monitored, so that information is decentralized and its flow is constrained by the same technology that hinders trade. The main feature of our model is that the money supplier faces a trade-off between short-run gains from overissue and long-run losses due to a decrease in its reputation for providing valuable currency.

We show that if the patient money supplier can commit to a choice of money supply, then a monetary equilibrium where it does not overissue exists as long as it is sufficiently patient. This equilibrium, however, is not time-consistent when the patient money supplier cannot commit to a plan of action. The reason for this is that the patient money supplier’s incentives to maintain a good reputation by not overissuing disappear once this reputation becomes too high. In other words, the trade-off between gains from overissue and reputation is only significant for moderate reputations. Following this insight, we show that if memory is imperfect the patient money supplier’s incentive to maintain a good reputation never disappears, and so a monetary equilibrium with no overissue is possible. The shortcoming of this approach is that even though reasonable, it is somewhat ad hoc.

An interesting question is whether there are other mechanisms that can discipline the money supplier’s behavior. In this regard, we believe that there are two possible directions for study. First, it may be that inconvertibility is at the root of the overissue problem. As Friedman and Schwartz point out, “historically, producers of money have established confidence by promising convertibility into some dominant money, generally, specie. Many examples can be cited of fairly
long-continued and successful producers of private moneys convertible into specie” [10, p. 45].

The second alternative, which we are currently investigating, is to introduce competition among money issuers. According to von Hayek, one of its main advocates, “convertibility is a safeguard necessary to impose upon a monopolist, but unnecessary with competing suppliers who cannot maintain themselves in the business unless they provide money at least as advantageous to the user as anybody else” [12, p. 111].

Acknowledgments

We would like to thank two anonymous referees for their comments and suggestions. We would also like to thank participants of the conference in honor of Neil Wallace, as well as seminar participants at Chicago, FGV-RJ, FGV-SP, IBMEC-RJ, and Rochester for their comments. Financial support from the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged.

Appendix A

Notice that $B(\eta_1)B(\eta_2) = B(\eta_2)$ for all $\eta_1, \eta_2 \in (0, 1)$. Hence,

$$w_t(\eta_{t+k+1}) = C(\eta_t)b(\eta_t) + \sum_{k=1}^{\infty} \hat{p}^k B(\eta_{t+k-1}) C(\eta_{t+k}) b(\eta_{t+k})$$

$$= C(\eta_t)b(\eta_t) + \sum_{k=1}^{\infty} \hat{p}^k [B(\eta_{t+k-1}) + (n-1)B(\eta_{t+k})]b(\eta_{t+k}).$$  \hspace{1cm} (A.1)

Now observe that

$$B(\eta_1)b(\eta_2) = \left( \begin{array}{c} \eta_1(1-\eta_2)u - (1-\eta_1)\eta_2c \\ \eta_1(1-\eta_2)u - (1-\eta_1)\eta_2c \end{array} \right)$$  \hspace{1cm} (A.2)

for all $\eta_1, \eta_2 \in (0, 1)$. Therefore, $\eta_t > 0$ implies that

$$\frac{\partial w_t}{\partial \eta_t} = \frac{\partial}{\partial \eta_t} [C(\eta_t)b(\eta_t)] = \frac{\partial b(\eta_t)}{\partial \eta_t} + (n-1)\frac{\partial}{\partial \eta_t} [B(\eta_t)b(\eta_t)]$$

$$= \left( \begin{array}{c} -u + (n-1)(1-2\eta_t)(u-c) \\ -c + (n-1)(1-2\eta_t)(u-c) \end{array} \right)$$  \hspace{1cm} (A.3)

and, for $k \geq 1$,

$$\frac{\partial w_t}{\partial \eta_{t+k}} = \hat{p}^k \left\{ B(\eta_{t+k-1}) \frac{\partial b(\eta_{t+k})}{\partial \eta_{t+k}} + \hat{p} \frac{\partial B(\eta_{t+k})}{\partial \eta_{t+k}} b(\eta_{t+k+1}) 

+ (n-1) \frac{\partial}{\partial \eta_{t+k}} [B(\eta_{t+k})b(\eta_{t+k})] \right\}$$

$$< \hat{p}^k \left[ \frac{(n-1)(1-2\eta_{t+k}) + (1-\eta_{t+k+1}-\eta_{t+k-1})(u-c)}{(n-1)(1-2\eta_{t+k}) + (1-\eta_{t+k+1}-\eta_{t+k-1})(u-c)} \right],$$  \hspace{1cm} (A.4)

10The reason why convertibility may work together with reputation is that its failure can act as a signal that the money supplier is impatient, thus disciplining the behavior of the patient money supplier.
Since
\[
\frac{\partial B(\eta_{t+k})}{\partial \eta_{t+k}} b(\eta_{t+k+1}) = \left( (1 - \eta_{t+k+1})u + \eta_{t+k+1}c \right).
\]
(A.5)

Because \( \eta_k \geq \frac{1}{2} \) for all \( k \geq t \) when \( \eta_t > 0 \), it is easy to see that \( \partial v_t / \partial \eta_{t+k} \leq 0 \) for all \( t \in \mathbb{N} \) and \( k \geq 0 \) when \( \eta_t > 0 \), the desired result. Recall that \( v_t = a(m_t)w_t - m_t c \), where \( a(m) = (m + 1 - m) \).

Appendix B

Let \( \Omega_t \) be the vector space of bounded Borel measurable functions from \( H_t \) into \( \mathbb{R} \) endowed with the supremum norm and denote its dual by \( \Omega^*_t \). Now let \( ba(H_t) \) be the vector space of all signed charges of bounded variation defined over the Borel sets of \( H_t \) and endow this set with the total variation norm. A well-known result, see [1, Theorem 13.4], is that \( \Omega^*_t = ba(H_t) \).

A function \( \tau_t : [0, 1] \rightarrow \Omega_t \) is strongly Lebesgue measurable if it is Lebesgue measurable and there exist \( E \subset [0, 1] \) and \( X \subset \Omega_t \), where \( E \) is Lebesgue measurable with measure one and \( X \) is norm separable, such that \( \tau_t(i) \in X \) for all \( i \in E \). When \( \tau_t(i) \in \Delta_t \) for almost all \( i \in [0, 1] \), we write \( \tau_t : [0, 1] \rightarrow \Delta_t \). If \( \tau^1_t, \tau^2_t : [0, 1] \rightarrow \Omega_t \) are two strongly Lebesgue measurable functions such that \( \{ i : \tau^1_t(i) \neq \tau^2_t(i) \} \) has Lebesgue measure zero, we say that \( \tau^1_t \) and \( \tau^2_t \) are equivalent.

Definition. A strategy profile \( \tau^a \) for the agents is an equivalence class of infinite sequences \( \{ \tau^a_t \} \), where \( \tau^a_t : [0, 1] \rightarrow \Delta_t \) is strongly Lebesgue measurable for all \( t \in \mathbb{N} \) and two sequences \( \tau \) and \( \tau' \) are equivalent if their corresponding elements are equivalent.

Notice that if \( \{ s_t \} \) is such that \( s_t \in \Delta_t \) for all \( t \in \mathbb{N} \), then \( \tau^a = \{ \tau^a_t \} \) with \( \tau_t \equiv s_t \) is strongly Lebesgue measurable, and hence a possible strategy profile.

Appendix C

Lemma 1. Suppose \( \sigma \) is an equilibrium. Then, for all \( t \in \mathbb{N} \), \( \mu_t(0) = 0 \) if, and only if, \( v_t(\delta_p) = 0 \).

Proof. Let \( \underline{t} = \inf\{ t \in \mathbb{N} : \mu_t(0) > 0 \text{ or } \mu_t(\delta_p) > 0 \} \), where the infimum of an empty set is taken to be plus infinity. Assume, without loss, that \( \underline{t} < +\infty \). First observe that both \( \mu_t(0) \) and \( \mu_t(\delta_p) \) are positive. Indeed, since \( \mu_t(0) = \mu_t(\delta_p) = 0 \) for \( t < \underline{t} \), \( \lambda_t(D(\tau^\delta, \tau^a, \delta)) > 0 \) if, and only if, \( D = \{ A, \emptyset, A, \emptyset, \ldots, A, \emptyset \} \). Hence, \( \mu_{\underline{t}}(0) = \mu_{\underline{t}}(\delta_p) \), and so both must be positive by the definition of \( \underline{t} \). Consequently, the measure of agents in the market is greater than zero regardless of the money supplier’s type if \( t > \underline{t} \). Because \( m_L, m_H \in (0, 1) \), any event in \( H_t \) with \( t > \underline{t} \) happens with positive probability when the money supplier is patient if, and only if, it happens with positive probability when the money supplier is impatient. In particular, this is true for the set of all private histories that lead an agent born in \( t > \underline{t} \) to enter the market.

Appendix D

Theorem 2. Suppose \( \sigma \) is a monetary equilibrium and let \( v_t \) be expected lifetime payoff from entering the market in \( t \) when the money supplier is patient. Then, regardless of the number of market meetings in one period, \( \liminf_{t \in \mathbb{N}_1(\sigma)} v_t = v_A \).
Proof. Since $\sigma$ is monetary, $\mathbb{N}_1(\sigma)$ is infinite. Notice that $t \in \mathbb{N}_1(\sigma)$ only if $v_t \geq v_A$, and so $\liminf_{t \in \mathbb{N}_1(\sigma)} v_t \geq v_A$. Suppose then, by contradiction, that $\liminf_{t \in \mathbb{N}_1(\sigma)} v_t > v_A$. We show that the patient money supplier has a profitable deviation if this is the case. We divide the argument in three parts.

Step 1: We first show that $\{\eta_t(\delta_p)\}$ cannot converge to $m_H$. Recall that

$$v_t = a(m_t(\delta_p))b(\eta_t(\delta_p)) + (n - 1)a(\eta_t(\delta_p))b(\eta_t(\delta_p)) - m_t(\delta_p)c + \sum_{k=1}^{\infty} \tilde{h}^k a(\eta_{t+k}(\delta_p))b(\eta_{t+k}(\delta_p)).$$

Suppose, by contradiction, that $\{\eta_t(\delta_p)\}$ converges to $m_H$. Since $a(m_H)b(\eta) > a(m_L)b(\eta)$ for all $\eta \in (0, 1)$, Eq. (D.1) implies that $\limsup_t v_t \leq v(m_H)$, and so there is $t_1 \in \mathbb{N}$ such that $v_t < v_A$ if $t \geq t_1$. This, however, implies that $\mathbb{N}_1(\sigma)$ is finite, a contradiction. Hence, there is $m \in [m_L, m_H)$ and a subsequence $\{\eta_{t_j}(\delta_p)\}$ of $\{\eta_t(\delta_p)\}$ that converges to $m$. Let $k(t) = \max\{t' \leq t : t' \in \mathbb{N}_1(\sigma)\}$. Since $\eta_t(\delta_p) = \eta_{t(t)}(\delta_p)$ for all $t \in \mathbb{N}$, $\eta_{t(t)}(\delta_p) \to m$ as well. Assume, without loss, that

$$\{\eta_t(\delta_p)\}_{t \in \mathbb{N}_1(\sigma)} \to m.$$  

(D.2)

Step 2: Observe that $\liminf_{t \in \mathbb{N}_1(\sigma)} v_t > v_A$ implies that there are $v > v_A$ and $t_1 \in \mathbb{N}$ such that $v_t \geq v$ if $t \in \mathbb{N}_1(\sigma)$ and $t \geq t_1$. Assume, without loss, that $v_t \geq v$ for all $t \in \mathbb{N}_1(\sigma)$ and let $\bar{\theta} \in (0, 1)$ be such that $\bar{\theta}v/2 + (1 - \bar{\theta})v(m_H) = v_A$. Then, any agent born in $t \in \mathbb{N}_1(\sigma)$ with belief in $[\bar{\theta}, 1]$ enters the market. Alternatively, a family lawyer with belief $\theta \in [\bar{\theta}, 1]$ in $\mathbb{N}_1(\sigma)$ sends the generation $t$ member of his family to the market. A straightforward modification of Theorem 5.1 in [4] shows that at least one of the two subsequences $\{\mu_t(\delta_p)\}_{t \in \mathbb{N}_1(\sigma)}$ and $\{\mu_0(0)\}_{t \in \mathbb{N}_1(\sigma)}$ is bounded away from zero. In what follows we prove that $\{\mu_t(\delta_p)\}_{t \in \mathbb{N}_1(\sigma)}$ is bounded away from zero.

For this, let $H_\infty = \Pi^\infty$ and denote a typical element of this set by $h_\infty$. A standard argument shows that for each type $\delta$, the strategy $\pi^*(i)$ of family $i$’s lawyer induces a Borel probability measure $\lambda_\infty(i, \delta)$ on $H_\infty$. Since $\pi^*_t$ is strongly Lebesgue measurable for all $t \in \mathbb{N}$, $\lambda_\infty(i, \delta)(D) = \lambda_\infty(D|i, \delta)$ is a Lebesgue measurable function of $i$ for each Borel subset $D$ of $H_\infty$.

11 Now let $d_t(h_\infty)$ be the list of market–autarky decisions in $h_\infty$ up to period $t$ and $d_\infty(h_\infty)$ be the list of all market–autarky decisions in $h_\infty$. Moreover, let $\#M d_t(h_\infty)$ be the number of elements of $d_t(h_\infty)$ that are equal to $M$ and let $E_{kt} = \{h_\infty \in H_\infty : \#M d_t(h_\infty) \geq k - 1\}$, where $t \in \bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}$. Recall that $M$ denotes the event that the market is chosen. Notice that $E_{k\infty} = \bigcup_{k=1}^{\infty} E_{kt}$. To finish, let $p_t(k|i, \delta) = \lambda_\infty(E_{kt}|i, \delta)$ and $n_t(k|i, \delta) = p_t(k + 1|i, \delta) - p_t(k + 2|i, \delta)$, where once again $t \in \bar{\mathbb{N}}$ and $k \in \mathbb{N}$.

Suppose then, by contradiction, that $\{\mu_t(\delta_p)\}_{t \in \mathbb{N}_1(\sigma)}$ is not bounded away from zero, so that $\{\mu_t(0)\}_{t \in \mathbb{N}_1(\sigma)}$ is. By definition, if the money supplier’s discount factor is $\delta$, $n_t(k|i, \delta) = \int n_t(k|i, \delta) d\delta$, with $t \in \mathbb{N}_{1}$, is the measure of families for which $k$ of its members choose the market up to period $t$ and $n_\infty(k|i, \delta) = \int n_\infty(k|i, \delta) d\delta$ is the measure of families for which $k$ of its members enter the market. Notice that for each $k$, $\delta$, and $i$, $p_t(k|i, \delta) \to p_\infty(k|i, \delta)$. Hence, $n_t(k|\delta) \to n_\infty(k|\delta)$ by the dominated convergence theorem. Since $\mu_t(0) = \sum_{k=1}^{\infty} n_t(k|0) - n_{t-1}(k - 1|0)$, where $n_0(k|\delta) = 0$ by definition, it must be that $\sum_{k=1}^{\infty} n_\infty(k)$, the measure of families for which only a finite number of their members enter the market, is less than one. Otherwise, $\{\mu_t(0)\}$ converges to zero. Denote this set of families by $\mathcal{F}$.
Now let \( \theta_t(i) \) be the belief of the member of family \( i \) that is born in \( t \) and suppose that all members of \( i \) who enter the market do not use their initial money holdings—whether they receive money from the money supplier upon entering the market or not—to update beliefs. If \( i \in \mathcal{F} \), then (D.2) together with an application of Kolmogorov’s Strong Law of Large Numbers implies that \( \{\theta_t(i)\} \) converges to zero almost surely.\(^{12}\) Therefore, by [2, Lemma 1], \( \{\theta_t(i)\} \) converges to zero almost surely even when the members of \( i \) who enter the market use their initial money holdings to update beliefs. We know, however, that it is optimal to choose autarky if \( \theta_t(i) < \theta_M \), since \( v(m_L) \) is the highest payoff possible from choosing the market when the money supplier is patient. Hence, \( \mathcal{F} \) cannot have a positive measure. However, the argument from the previous paragraph shows that \( \mathcal{F} \) must have measure less than one, a contradiction. We can then conclude that \( \{\mu_t(\delta_p)\}_{t \in \mathbb{N}_1(\sigma)} \) is bounded away from zero.

To finish this step, notice that \( \mathbb{N}_1(\sigma, m_L) \) must be infinite. Suppose not. Because \( \{\mu_t(\delta_p)\}_{t \in \mathbb{N}_1(\sigma)} \) is bounded away from zero, \( \lim_{t \to \infty} \sum_{k \in \mathbb{N}_1(\sigma) \cap [1, \ldots, t]} \mu_k(\delta_p) = +\infty \). Hence, \( \eta_t(\delta_p) = \sum_{k \in \mathbb{N}_1(\sigma, m_L) \cap [1, \ldots, t]} \mu_k(\delta_p) m_L + \sum_{k \in \mathbb{N}_1(\sigma, m_H) \cap [1, \ldots, t]} \mu_k(\delta_p) m_H \to m_H \),

\[ (D.3) \]

since \( A_t \) converges to zero. This, however, contradicts (D.2).

**Step 3:** Let \( ca(\mathbb{N}) \) be the Banach space of all signed (countably additive) measures defined over the subsets of \( \mathbb{N} \) endowed with the total variation norm \( \| \| \). Define \( \chi_t(i, \delta) \in ca(\mathbb{N}) \), with \( t \in \mathbb{N} \), to be such that if \( B \subseteq \mathbb{N} \), then

\[ \chi_t(i, \delta)(B) = \chi_t(B|i, \delta) = \begin{cases} 0 & \text{if } B = \emptyset, \\ \sum_{k \in B} n_t(k|i, \delta) & \text{otherwise}. \end{cases} \]

Now let \( B \in 2^\mathbb{N} \) be such that if \( B \in \mathcal{B} \), then either \( B \) is finite or \( B = \{k, k+1, \ldots\} \) for some \( k \in \mathbb{N} \). It is obvious that \( \{\chi_t(B|i, \delta)\} \) converges to \( \chi_\infty(B|i, \delta) \) if \( B \) is finite. If \( B = \{k, k+1, \ldots\} \) for some \( k \in \mathbb{N} \), then \( \chi_t(B|i, \delta) = p_t(k+1|i, \delta) \) for all \( t \in \mathbb{N} \), and so \( \{\chi_t(B|i, \delta)\} \) converges to \( \chi_\infty(B|i, \delta) \) as well. Hence, \( \{\chi_t(B|i, \delta)\} \) converges to \( \chi_\infty(B|i, \delta) \) for all \( B \in C \), the algebra generated by \( \mathcal{B} \). By [11, Theorem 3.1], if \( \Lambda_t(B|i, \delta) = \chi_t(B|i, \delta) - \chi_\infty(B|i, \delta) \), with \( t \in \mathbb{N} \), then \( \Lambda_t(B|i, \delta) \to 0 \) for all \( B \subseteq \mathbb{N} \).\(^{13}\) Therefore, \( \|\Lambda_t(i, \delta)\| \to 0 \) by Phillips’ lemma, see [8, p. 83]. Egoroff’s Theorem, see [9], now tells us that for all \( \epsilon > 0 \) there exists a set \( \mathcal{E}_\epsilon \) of families of measure at least \( 1 - \epsilon \) such that \( \{\chi_t(i, \delta)\} \) converges uniformly (in norm) to \( \chi_\infty(i, \delta) \) on \( \mathcal{E}_\epsilon \).

Assume, without any loss, that \( \{\chi_t(i, \delta)\} \) converges uniformly to \( \chi_\infty(i, \delta) \) on \( [0, 1] \).

Let \( K \) be such that \( \sum_{t=K+1}^{\infty} \delta^t_p < \mu(m_H - m_L)/2 \), where \( \mu \) is a lower bound for \( \{\mu_t(\delta_p)\}_{t \in \mathbb{N}_1(\sigma)} \).

Moreover, let \( \epsilon \in (0, 1 - \delta) \) be such that if an agent is born in \( t \in \mathbb{N}_1(\sigma) \) with belief in \( [1-\epsilon, 1] \), then the next \( K \) generations of his family always enter the market as long as their date of birth also lies in \( \mathbb{N}_1(\sigma) \). Eq. (D.2) together with [2, Lemma 1] imply that \( \{\theta_t(i)\} \) converges to one almost surely if an infinite number of members of family \( i \) enter the market. Consequently, there exists \( \mathcal{N} \) such

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12 An intuition for this result is the following. Consider an individual who tosses a coin infinitely many times, and this coin is either fair or biased towards heads. Even if this bias changes over time, as long as the limiting frequency of heads in the biased coin is greater than \( \frac{1}{2} \), this individual still learns the coins’ true type with probability one.

13 For the sake of completeness, we state Theorem 3.1 from [11] below. Notice that \( \mathbb{N} \) is metric space, hence a regular Hausdorff space, when endowed with the discrete topology, in which case its Borel \( \sigma \)-algebra is the power set \( 2^\mathbb{N} \). It is straightforward to see that any measure \( \mu \) on \( (\mathbb{N}, 2^\mathbb{N}) \) is such that \( \mu(U) = \sup\{\mu(K) : K \subseteq U \text{ is compact} \} \).
that if \( \#_M d_t(h^\infty) \geq \overline{N} \), then the probability that \( \theta_t(i) \geq 1 - \varepsilon \) is at least \( 1 - \mu(1 - \delta_p)(m_H - m_L)/2 \), as almost sure convergence implies convergence in measure.

Now let \( t_1 \in \mathbb{N} \) be such that \( |\mathbb{N}_1(\sigma) \cap \{1, \ldots, t_1\}| = \overline{N} \), where \( |B| \) denotes the cardinality of \( B \). Then, for all \( \varepsilon > 0 \) there exists \( t_2 \in \mathbb{N} \) such that if \( t \geq t_2 \), then \( |\chi_t(B|i, \delta_p) - \chi_\infty(B|i, \delta_p)| \leq \varepsilon \) for all \( B \in \mathcal{B} = \{\{1\}, \{2\}, \ldots, \{\overline{N} - 1\}, \{\overline{N}, \ldots, \infty\}\} \) and \( i \in [0, 1] \). Assume, without loss, that there is \( t_2 \) such that \( \chi_t(B|i, \delta_p) = \chi_\infty(B|i, \delta_p) \) for all \( B \in \mathcal{B} \) and \( i \in [0, 1] \). Now observe that there is no aggregate uncertainty. Hence, if \( t \geq t_1 = \max\{t_1, t_2\} \), then the following holds for almost all families \( i \): either (i) \( \#_M d_t(h^\infty) \) is constant in \( t \), so that no members of \( i \) enter the market after \( t \); or (ii) \( \#_M d_t(h^\infty) \geq \overline{N} \). In other words, if \( t \in \mathbb{N}_1(\sigma) \cap \{t_1 + 1, \ldots\} \), then almost all agents who enter the market in \( t \) have at least \( \overline{N} \) previous members of his family who did the same.

To finish, let \( t \in \mathbb{N}_1(\sigma, m_L) \) be such that \( t > t_1 \). The patient money supplier’s lifetime payoff from sticking to its prescribed strategy from period \( t \) on is

\[
\sum_{k=1}^{\infty} \delta^k_p \mu_{t+k}(\delta_p) m_{t+k}(\delta_p) + \mu_t(\delta_p) m_L. \tag{D.5}
\]

If, instead, it does a one-shot deviation in \( t \), its lifetime payoff is at least

\[
\mu_t(\delta_p) m_H + \left(1 - \frac{1}{2} \mu(1 - \delta_p)(m_H - m_L)\right) \sum_{k=1}^{K} \delta^k_p \mu_{t+k}(\delta_p) m_{t+k}(\delta_p). \tag{D.6}
\]

Hence, the patient money supplier’s payoff gain from a one-shot deviation in \( t \) is no less than

\[
\mu_t(\delta_p)(m_H - m_L) - \frac{1}{2} \mu(1 - \delta_p)(m_H - m_L) \sum_{k=1}^{K} \delta^k_p \mu_{t+k}(\delta_p) m_{t+k}(\delta_p) - \sum_{k=K+1}^{\infty} \delta^k_p \mu_{t+k}(\delta_p) m_{t+k}(\delta_p) \geq \mu(m_H - m_L) - \frac{1}{2} \mu(1 - \delta_p)(m_H - m_L) \sum_{k=1}^{K} \delta^k_p - \sum_{k=K+1}^{\infty} \delta^k_p > 0, \tag{D.7}
\]

as \( \mu_t(\delta_p) m_t(\delta_p) < 1 \) for all \( t \in \mathbb{N} \). In other words, the patient money supplier has a profitable deviation, and so \( \sigma \) cannot be an equilibrium. \( \square \)

**Theorem 3.1 from [11].** *Let \( X \) be a regular Hausdorff space and suppose \( \mathcal{C} \) is a collection of open sets of \( X \) satisfying:

(I) If \( C_1, C_2 \in \mathcal{C} \), then \( C_1 \cap C_2 \in \mathcal{C} \).

(II) If \( C_1, C_2 \in \mathcal{C} \) and \( C^c_1 \cap C^c_2 = \emptyset \), then \( C_1 \cup C_2 \in \mathcal{C} \), where \( C^c \) denotes the closure of \( C \).

(III) If \( K \) is compact, \( U \) is open, and \( K \subseteq U \), then there are \( C_1, C_2 \in \mathcal{C} \) with \( K \subseteq C_1 \subseteq X \setminus C_2 \subseteq U \).

(IV) If \( \{C_n\} \) and \( \{D_n\} \) are two sequences of elements of \( \mathcal{C} \) such that \( \{C_n\} \) is increasing, \( \{D_n\} \) is decreasing, and \( C_n \subseteq D_n \) for all \( n \in \mathbb{N} \), then there is \( C_0 \in \mathcal{C} \) such that \( C_n \subseteq C_0 \subseteq D_n \) for all \( n \in \mathbb{N} \). Now let \( \{\mu_n\} \) be a sequence of Borel probability measures on \( X \) such that if \( U \subseteq X \) is open, then \( \mu_n(U) = \sup\{\mu_n(K) : K \subseteq U \text{ is compact}\} \). If \( \{\mu_n(C)\} \) converges for all \( C \in \mathcal{C} \), then \( \{\mu_n(B)\} \) converges for all Borel subsets \( B \) of \( X \).*
Appendix E

Here we now show that if there is \( n_0 \in \mathbb{N} \) such that

\[
\eta(k, n) > \ln \left( \frac{1 - m_L}{1 - m_H} \right) \left[ 1 + \ln \left( \frac{m_H}{m_L} \cdot \frac{1 - m_L}{1 - m_H} \right) \right]^{-1}
\]

(E.1)

for all \( k \in \mathbb{N} \) when \( n > n_0 \), then \( \lim_{n \to \infty} \sup_{t,k\in\mathbb{N}} |\xi^{(k,n)}_{t,k}(m_H, \theta_0, n)| = 0. \)

**Proof.** Under \( \sigma^* \), autarky is absorbing when private histories are always passed from one generation to the next. Hence, \( \{\xi^{(k,n)}_{t,k}(m_H, \theta_0, n)\}_{t} \) is a non-increasing sequence for each \( k \) and \( n \), and so we just need to show that \( \lim_{n \to \infty} \sup_{t,k\in\mathbb{N}} |\xi^{(k,n)}_{t,k}(m_H, \theta_0, n)| = 0. \) Assume, without loss, that (E.1) is satisfied for all \( k, n \in \mathbb{N} \). Now let \( q \in \{0, 1\} \) denote the units of money received from the money supplier upon entering the market and let \( r \in \{0, \ldots, n\} \) denote the number of market meetings with money in the first period of life. Then

\[
\tilde{\xi}^{(1,n)}_{1,k}(m_H, \theta_0, n) = \text{Prob} \left\{ \left( \frac{m_H}{m_L} \right)^{q+r} \left( \frac{1 - m_H}{1 - m_L} \right)^{n+1-q-r} \leq \frac{\theta_0(1 - \theta_M)}{\theta_M(1 - \theta_0)} \right\}; \tag{E.2}
\]

where \( \text{Prob}\{q\} = m^q_H \left( 1 - m_H \right)^{1-q} \) and \( \text{Prob}\{r\} = \binom{n}{r} (1 - \eta(k,n))^r (1 - \eta(k,n))^{n-r}. \)

Notice that an upper bound for \( \tilde{\xi}^{(1,n)}_{1,k}(m_H, \theta_0, n) \) is

\[
= \text{Prob} \left\{ \frac{r}{n} \leq \frac{n + 1}{n} A + \frac{B}{n + 1} \right\}, \tag{E.3}
\]

where

\[
A = \left( \frac{1 - m_L}{1 - m_H} \right) \left[ 1 + \ln \left( \frac{m_H}{m_L} \cdot \frac{1 - m_L}{1 - m_H} \right) \right]^{-1}
\]

and

\[
B = \ln \left( \frac{\theta_0(1 - \theta_M)(1 - m_L)}{\theta_M(1 - \theta_0)(1 - m_H)} \right). \tag{E.4}
\]

By Chebyshev’s inequality, for all \( \varepsilon > 0, \)

\[
\text{Prob} \left\{ \left| \frac{r}{n} - \eta(k,n) \right| \geq \varepsilon \right\} \leq \frac{\eta(k,n)[1 - \eta(k,n)]}{n^2 \varepsilon^2} < \frac{1}{4n^2 \varepsilon^2}. \tag{E.5}
\]

Since \( \eta(k,n) > A \) for all \( k, n \in \mathbb{N}, \eta(k,n) > (n + 1)A/n \) for all \( k \in \mathbb{N} \) when \( n \) is sufficiently large. Therefore, (E.5) implies that \( \tilde{\xi}^{(1,k)}_{1,k}(m_H, \theta_0, n) \) converges to zero uniformly in \( k \), from which we can conclude that the desired result holds. \( \square \)

**References**