Learning Objectives

Procedural Skills
1. Determine limits graphically or numerically using your calculator and/or a table.
2. Use graphical and numerical techniques to determine limits at ±∞.

Interpretation Skills
1. Give examples of contexts where limits approaching infinity are meaningful.
2. Compare and contrast speed and velocity.

We now explore the idea of the limit as a mathematical tool. In the context of our study of Calculus the motivation for understanding limits comes from our desire to fully describe the rate of change of a function at a point. For example many fitness apps make use of a phone’s gps to determine an individuals speed in the direction they are traveling. As we move these apps are able to estimate our instantaneous speed. But how do they go about doing so?

Velocity and Speed

Velocity is speed with a direction, indicated by the sign of the number. Speed only refers to the magnitude of the rate of change but gives us no directional information. Throughout this course, we will interpret 2 mi/hr as a velocity. Put another way this means “2 mi/hr in the forward direction”. Similarly, −3 ft/sec is the velocity “3 ft/sec in the backward direction”.

When we read 0.2 miles per minute we mean \[ \frac{0.2 \text{ miles}}{\text{minute}}. \]

This implies that to determine velocity we need some change in distance over the change in time it took to cover that distance. In other words

\[
\text{Velocity} = \frac{\text{Change in Distance}}{\text{Change in Time}} = \frac{\Delta f(t)}{\Delta t},
\]

where \( f(t) \) represents our distance at time \( t \). This answer is somehow not yet complete. If we have run for 30 minutes, and covered 4 miles total we get an average velocity of

\[
\frac{\Delta f(t)}{\Delta t} = \frac{4 - 0}{30 - 0} \approx 0.13 \text{ mi/min},
\]

but our app reports differing speeds the entire time we’re running. This has to do with the interval of time we choose. To get 0.13 we chose the data points at the beginning of and end of our run, (0, 0) and (30, 4) respectively. This rate is visualized by the dashed line in the figure below. We can see from the plot that the instantaneous rate of change, or speed, of our run was not constant.

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1. Although such apps report our “speed”, they are actually using AROC over very small intervals. This gives us a good enough approximation!

2. This is an average because the data comes from our change in distance over a finite interval.
By now you may have likely realized our previous calculation was the **average rate of change (AROC)** over the interval \([0,30]\). In other words we’re using the slope of the **secant** line corresponding to the interval \([0,30]\) to estimate the rate of change at any point in time contained in the interval.

Since slope is well understood and easy to calculate we can use it to approximate the rate of change of a function at a point. This is done by taking the AROC over an interval containing the point of interest. As we see in our example this estimate is only as good as our interval of choice. However we could do better by taking smaller time intervals, at every 6 minutes for example. Suppose our phone recorded the following data.

<table>
<thead>
<tr>
<th>Time(min)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>22</th>
<th>24</th>
<th>26</th>
<th>28</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance(mi)</td>
<td>0</td>
<td>0.2</td>
<td>0.6</td>
<td>0.6</td>
<td>1</td>
<td>1.4</td>
<td>1.8</td>
<td>1.8</td>
<td>2</td>
<td>2</td>
<td>2.4</td>
<td>2.8</td>
<td>2.8</td>
<td>3.2</td>
<td>3.6</td>
<td>4</td>
</tr>
</tbody>
</table>

Data from running app.

We could estimate our speed at 8 minutes using the smaller interval \([6,8]\) to get

\[
\frac{1 - 0.6}{8 - 6} = 0.2 \text{ mi/min}
\]

What we’re calculating is \(\frac{f(t_2) - f(t_1)}{t_2 - t_1}\), where our change in \(t\) is \(\Delta t = t_2 - t_1\). As we discussed in the previous lecture, another way to write this is

\[
\frac{f(t + \Delta t) - f(t)}{\Delta t}.
\]

The figure below shows the same data but at different scales. We can see that our original approximation doesn’t do a very good job. Here the function in green represents the mathematical model of our run which fits the data in the table.
By taking a smaller time interval about the point $t = 8$ we can get a better estimate of our velocity at that moment in time. We can do this by taking $\Delta t$, the width of our interval, to be increasingly smaller. An example is illustrated can be seen in the table below.

<table>
<thead>
<tr>
<th>Interval</th>
<th>$\Delta t$</th>
<th>AROC(mi/min) of running data</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0,30]</td>
<td>30</td>
<td>0.13</td>
</tr>
<tr>
<td>[4,20]</td>
<td>16</td>
<td>0.14</td>
</tr>
<tr>
<td>[6,12]</td>
<td>6</td>
<td>0.20</td>
</tr>
<tr>
<td>[6,8]</td>
<td>2</td>
<td>0.20</td>
</tr>
</tbody>
</table>

If our distance is given as a function, $f(t) = t^2$ miles for example, then we can see from the table below that the average rate of change of seems to be approaching 4 $mi/min$ as we take smaller intervals about the point $t = 2$.

<table>
<thead>
<tr>
<th>Interval</th>
<th>$\Delta t$</th>
<th>AROC(mi/min) of $f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2,10]</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>[2,5]</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>[2,3]</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>[2,2.1]</td>
<td>0.1</td>
<td>4.1</td>
</tr>
<tr>
<td>[2,2.01]</td>
<td>0.01</td>
<td>4.01</td>
</tr>
</tbody>
</table>

We could keep calculating smaller and smaller intervals but what we’re really asking is what happens to the value of the average rate of change(AROC) as $\Delta t$ becomes infinitely small.

Mathematically we write this as

$$\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

This expression represents the value of our average rate of change approaches as $\Delta t$ becomes infinitely small. The value of the expression above is known as the *instantaneous rate of change of $f(t)$ at $t$.*

As we have explored in the work above we would like to be able to calculate such limits to determine instantaneous rate of change at a point. This will allow us determine exactly how that function in changing in time or space.  

### Understanding Limits

As an example suppose $g(x) = 3x + 6$, and we want to determine $\lim_{x \to 2} g(x)$. Mathematically this is expressed as

$$\lim_{x \to 2} 3x + 6.$$  

This expression is asking us to determine what value the expression, $3x + 6$ approaches as we get closer and closer to $x = 2$.

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6Interestingly this leads to all sorts of unexpected results. Following these principles we’ll learn how to minimize, maximize and optimize functions.
One way to understand the answer to this question is to create a table of values of \( g(x) \) as \( x \) becomes closer and closer to \( x = 2 \). From the table supplied on the right we can better understand what \( g(x) \) approaches as \( x \) approaches 2. As we continue taking values of \( x \) closer and closer to \( x = 2 \) we see \( g(x) \) gets even closer to 12. So it follows that

\[
\lim_{x \to 2} 3x + 6 = 12.
\]

A very natural question to wonder is “why can’t we just plug \( x = 2 \) into \( g(x) \)?”. After all \( g(2) = 12 \) which is what the table implies. The answer to this question is that this method of plugging in the limiting value only works for certain types of functions but not others. What we’ll see as we begin to understand limits is that this is not what the question is truly asking. So in general we must be very careful.

Determine the value of the limits below graphically or numerically by using your calculator to plot the expressions or by making a table.

1. \( \lim_{x \to 0} \frac{1}{x^2} = \)

2. \( \lim_{x \to -2} \frac{1}{x - 2} = \)

3. Given \( p(x) \), plotted below, determine the limit of \( p(x) \) as \( x \) approaches 0.

\[
\lim_{x \to 0} p(x) =
\]

4. \( \lim_{x \to 0} \frac{x^2}{1 + x} \)

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In fact by choosing a point close enough to \( x = 2 \) we can get as close to 12 as we’d like.
Practice limits by answering the following questions.

1. Use the figure below to answer the following questions.
   
   (a) \( \lim_{t \to 2} q(t) = \)

   (b) \( \lim_{t \to -2} q(t) = \)

2. \( \lim_{t \to \infty} 15.2e^{-0.2t} = \)

3. The percentage of research articles in the prominent journal \textit{Physical Review} written by researchers in the United States can be modeled by
   
   \[ A(t) = 25 + \frac{36}{1 + 0.6(0.7)^{-t}}, \]

   where \( t \) is time in years since 1983. Determine \( \lim_{t \to \infty} A(t) \) and interpret your answer.

The examples above illustrate the following:

- In general \( \lim_{x \to a} f(x) \neq f(a) \)
- Limits don’t always exist.
- Limits can approach \( \infty \) or \(-\infty\).
- The limit should be the same coming from both directions. Put another way, the left and right limit should be equal, otherwise we say the limit doesn’t exist.
- Use tables and graphs to help determine limits, if they exist.