In previous courses you have learned a variety of mathematical techniques to solve different types of problems. We will use these techniques to develop an understanding of Calculus. In particular, one of the problems Calculus is concerned with answering is given below.

**Given an arbitrary function of position, how can we determine how this function changing at any point in time?**

At first glance this statement may look rather unassuming. As we will shortly see, there are several layers of framework we will need to fully answer to this statement. By the end of the semester we will have developed the machinery to answer this question.\(^1\)

### Re-inventing The Wheel

To answer questions about the rate of change of a function at a point we will begin by first approximating rate of change. This will allow us to use tools we have already developed before we develop the tools to determine an exact rate of change at a point. As the title of this section suggests, there is no need to re-invent the wheel.

At this point the only tool we have developed that is related to rate of change has to do with linear functions. The slope of a linear function gives us its rate of change. We want to determine the rate of change of any function at a given point. However, slope only gives us the rate of change at any point of a linear function. Fortunately this doesn’t mean that we must abandon slope in our quest to define the rate of change of a function at a point.

\(^1\)What we now learn in a semester of Calculus took hundreds of years and many brilliant minds to fully develop and refine.
As an example let’s consider the data given by the following table. The data below gives the number new cases of influenza in Michigan \( t \) days after the start of the month.

<table>
<thead>
<tr>
<th>( t ) days</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of new cases</td>
<td>1,200</td>
<td>1,300</td>
<td>1,400</td>
<td>1,400</td>
<td>900</td>
<td>850</td>
<td>700</td>
</tr>
</tbody>
</table>

1. (a) Use the figure below to plot the points above and connect the data points with a smooth function labeled \( S(t) \).

(b) Sketch the line that corresponds to line which intersects \((5, 1200)\) and \((9, 900)\) and determine the slope of the line you drew in the plot above.

(c) Interpret the meaning of your answer above. How does this relate to the rate of change of \( S(t) \) at \( t = 6 \)?

\[2\text{The slope of this line is known as the average rate of change AROC over [5, 9].}\]
(d) At the beginning of the month does the rate of change of the new cases of influenza appear to be positive or negative? Explain.

(e) Calculate the average rate of change (AROC) of $S(t)$ over $[5, 6]$.

(f) Considering your answers above, which gives us a better understanding of how the number of new cases of influenza changing on the fifth day? Explain

In the space below write down the formula, or average rate of change (AROC), for the slope of the line that intersects the points $(a, f(a))$ and $(b, f(b))$, where $f(x)$ is any function and $[a, b]$ an arbitrary interval.  

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**Average Rate of Change**

The *average rate of change (AROC)* of a function $f(x)$ over an interval $[a, b]$ is given by:

$$\frac{f(b) - f(a)}{b - a}$$

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3Notice that we can apply this tool to any function.

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3If you can’t do this using variables choose values for $a$ and $b$ and any function for $f(x)$.  

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An Alternative Formula

What we’ve noticed from our investigation above is that by taking smaller/larger (circle one) intervals about a point, the AROC gives us a better estimate of the instantaneous rate of change of a function at that point.

Put another way, if we want to improve our estimate of how $f$ is changing at a point, then we can do so by taking a smaller interval in our AROC calculation. With this in mind, let’s rewrite our equation so we can easily modify the width to calculate AROC. An alternative formula for AROC is given by

The average rate of change of the function $f(x)$ over the interval $[x, x+h]$ where $h$ is some positive constant is given by

$$\frac{f(x+h) - f(x)}{(x+h) - x}$$

which simplifies to

$$\frac{f(x+h) - f(x)}{h}.$$

We typically use $h$ or $\Delta x$ to denote interval width.

**Example 1** Determine the AROC of $g(t) = e^{0.2t}$ over the interval $[4, 4+h]$ where $h = 0.01$.

**Example 2** The price per barrel of crude oil in dollars $t$ years after 1980 is given by $P(t) = 0.45t^2 - 12t + 105$.

(a) Use AROC to estimate the instantaneous rate of change at the start of 2008.

(b) Interpret your answer from part (a).

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If a problem doesn’t specify an interval for AROC, use an interval smaller than 1 that contains the point of interest.
2. Consider the following function

(a) Find the AROC of $f(x)$ over the interval $[2, 3]$. For the moment we can use this to estimate the instantaneous rate of change of $f(x)$ at $x = 3$.

(b) Sketch the secant line corresponding to part (a).

3. The quantity (in mg) of a drug in the blood at the time $t$ (in minutes) is given by $Q(t) = 30(0.7)^t$. Use AROC to estimate the instantaneous rate of change of the quantity at $t = 2$ and interpret the meaning of your answer.
4. The total acreage of farms from 1980 and 2000 is given by the following table.

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Farm land (million acres)</td>
<td>1039</td>
<td>1012</td>
<td>987</td>
<td>963</td>
<td>945</td>
</tr>
</tbody>
</table>

Estimate the instantaneous rate of change of the area of farm land in the year 1990. Explain your estimation and why it is a good approximation of the instantaneous rate of change. Could you determine a better estimate? Why or why not?

Summary

1. Suppose we have determined that the AROC of a function $f(x)$ over the interval $[2, 4]$ is 17. Does this necessarily mean the function is increasing over the $[2, 4]$?

2. Given a context how can we determine the units of the AROC?

3. What are important related formulas to AROC?

4. How is the rate of change of a function at a point related to AROC of the function over an interval containing that point?

5. In what kind of contexts is AROC useful?

6. How are slope and AROC similar? How are their formulas related?

7. What is the difference between slope and AROC? How are these terms related to the secant line?