Learning Objectives

Procedural Skills

1. Apply the definition of the derivative to determine the derivative of a function.
2. Evaluate a derivative at a point.
3. Identify derivative notation \( f'(x), \frac{dh}{dx}, g'(4), \frac{dB}{dt} \big|_{t=2} \)
4. Given a context involving a function \( f(x) \), determine the correct units and interpretation of \( f'(x) \)

Interpretation Skills

1. Compare and contrast AROC and the derivative.
2. Demonstrate the difference between a secant line and a tangent line graphically and verbally.
3. List context clues that indicate the use of the derivative and how to calculate it (as an estimate, or exactly).
4. Provide examples or problems where we can determine derivatives and examples where we cannot.

Derivative of a Function at a Point

A function is said to be \textit{differentiable} at the point \( x = a \) if the limit,

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h},
\]

exists. If the limit exists, we denote the derivative using the following notation

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

Now that we have a general understanding of the limit and difference quotient we can calculate the derivative of a function using the definition. Let us consider a few examples to whet our appetite.

\textbf{Example 1} Using the definition of the derivative determine the derivative \( f'(3) \) given that \( f(x) = x^2 \).
Example 2 Determine if \( f(x) = |x| \) is differentiable at the point \( x = 1 \). If so, determine the derivative there, otherwise explain why.

Using the definition above we have that \( f(x) \) is differentiable at \( x = 1 \) if the following limit exists.

\[
\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}
\]

Since \( f(x) = |x| \) this limit is equivalent to

\[
\lim_{h \to 0} \frac{|1 + h| - |1|}{h}
\]

Recall a limit exists if limit from the left and the limit from the right agree. We can determine the limits graphically or by using a table. Notice that as \( h \) becomes small the left and right limits both approach 1 and we have that

\[
\lim_{h \to 0} \frac{|1 + h| - |1|}{h} = 1.
\]

So we see that since the limit exists, \( f(x) \) is differentiable at \( x = 1 \), and the derivative is 1. Put another way, \( f'(1) = 1 \).

In general we have that the derivative of \( f(x) = |x| \) is

\[
f'(x) = \frac{|x|}{x},
\]

for \( x \neq 0 \). This is the first of several derivative rules we will encounter.

A non-differentiable example

The limit definition of the derivative requires a limit which many not always exist given a function and a point. In such a case we say the function is not differentiable at that point. An example of this is \( f(x) = |x| \) at \( x = 0 \). See the textbook for more details.

Other Derivative Rules

The derivative rule above gives us the derivative of \( f(x) = |x| \) for all values of \( x \) that are not zero. This means if \( f(x) = |x| \) and we’d like to know \( f'(3) \) we can simply do the evaluation

\[
f'(3) = \frac{|3|}{3} = 1,
\]

which is much quicker than deriving the derivative from the definition as we have previously done.

It is natural to wonder if there are derivative rules for other functions. As we will see in the following lectures we do in fact have derivative rules for all the functions we will encounter in this course.
1. (a) Using the definition of the derivative determine the derivative of \( f(x) = mx + b \) where \( m \) and \( b \) are constants.

By using general constants \( m \) and \( b \), you have derived a derivative rule for any function of the form \( f(x) = mx + b \). Put another way this gives us a derivative rule for \textbf{any} linear function.

(b) Use your derivative rule from part (a) to determine the derivative of \( f(x) = 2x + 4 \).\(^1\)

(c) Use the derivative rule you determined in part (a) to determine \( g'(20) \) where \( g(t) = 500t - 23 \).

\(^1\)You do not need the definition to determine this derivative. Use your result from part (a)!
Practice determining the following derivatives using the derivative definition, or using one of the rules we determined above. The definition of the derivative of a function $f(x)$ is provided below.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

2. (a) Determine the derivative of $p(x) = \frac{1}{x}$.

(b) Using your answer from the previous question to determine $p'(2.1)$.

3. If a stone is dropped from a height of 400 feet, its height after $t$ seconds is given by $s(t) = 400 - 16t^2$. Find the instantaneous velocity of the stone at $t = 4$. Note that velocity is by definition space units per time units, such as $\text{ft/sec}$. 

4. A maritime research group determines a model, \( S(t) \), for the population of sardines in a region of ocean where \( t \) years after 2010. Recent data has shown that \( S(5) = 20 \) and \( S'(5) = -3 \). This means that in the year 2015

(a) There were 3 million sardines and this number was decreasing at a rate of 20 million per year.

(b) There were -3 million sardines and this number was increasing at a rate of 20 million per year.

(c) The number of sardines had dropped by 3 million since the previous year, but was now increasing at a rate of 20 million per year.

(d) There were 20 million sardines and this number was decreasing at a rate of 3 million per year.

(e) There were 20 million members and the population had dropped by 3 million since the previous year.

5. Your friend Taylor just got his exam back and he can’t understand why his teacher marked his answer wrong. Explain what is wrong with Taylor’s answer.

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) + h - f(x)}{h} = \lim_{h \to 0} f(x) - f(x) = 0
\]