We begin with a theorem which is of fundamental importance.

**The Fundamental Theorem of Calculus (FTC)**

If \( F'(t) \) is continuous for \( a \leq t \leq b \), then

\[
\int_a^b F'(t) \, dt = F(b) - F(a).
\]

Moreover the antiderivative \( F \) is guaranteed to exist.

In other words, the definite integral of the *derivative of a function* over an interval \([a, b]\) gives the change in the *function* from \( a \) to \( b \). Equivalently, the definite integral of a function over an interval gives the change in the function’s antiderivative over that interval.

1. The rate of change of a stock, in \$/day can be modeled by the function \( S'(t) \) given below, where \( t \) is days after the start of the month.

(a) Approximately how much did the value of the stock change over the 30 days?

(b) At the start of the month the stock was worth $50. Approximately how much is the stock worth at the end of the 30 days?
2. Ice is forming on a pond at a rate given by \( \frac{dy}{dt} = \frac{\sqrt{t}}{2} \) inches per hour, where \( y \) is the thickness of the ice in inches (in) at time \( t \) measured in hours since the ice started forming.

   (a) Estimate the thickness of the ice after 8 hours given that at 2 hours there is roughly 1 in of ice and that from the 2\textsuperscript{nd} to the 8\textsuperscript{th} hour roughly 6.5 in of ice has formed.

   (b) At what rate is the thickness of the ice increasing after 8 hours?

3. The graph of a derivative \( f'(x) \) is shown in the following figure. Use the FTOC to fill in the table of values for \( f(x) \) given that \( f(0) = 2 \).

The FTC connects the two main ideas we have developed over the course of the semester.

The integral of a function over an interval gives the total change of the function’s antiderivative over that interval.
A more direct example of the FTC is given below.

**Example 1** Determine \( \int_{0}^{1} (1 - x^2) \, dx \) using the FTC.

First we find the indefinite integral \( F(x) \). According to our rules this gives us that \( F(x) = x - \frac{1}{3} x^3 + C \) since \((x - \frac{1}{3} x^3 + 3)' = 1 - x^2\). So by the FTC

\[
\int_{0}^{1} (1 - x^2) \, dx = F(1) - F(0), \\
= ((1) - \frac{1}{3}(1)^3 + C) - ((0) - \frac{1}{3}(0)^3 + C), \\
= 2/3.
\]

**Total Area**

We must be careful to make the distinction between change in area and total or absolute area.

4. Suppose a delivery biker’s velocity is described by the function

\[ v(t) = -15t^4 + 101.25t^3 - 215.625t^2 + 140.625t \]

in miles per hour where \( t \) is time in hours. The plot of the bikers velocity is given below.

(a) Shade the area that represents \( \int_{0}^{3} v(t) \, dt \).

(b) Determine \( \int_{0}^{3} v(t) \, dt \) using your calculator.

(c) Explain the meaning of your answer above.
(d) Does the answer you found in part (b) give the total distance the biker traveled? Why or why not?

After the first 3 hours the biker traveled a total of _________ miles.

After the first 3 hours the biker was _________ miles from their starting point.

Practice Problems

5. Determine the following integrals, without a calculator, using the FTC.

(a) \( \int_0^1 (1 - x^2) \, dx \).

(b) \( \int_{-1}^1 (x^2 + 2) \, dx \)

(c) \( \int_{-2}^2 (t^3 - 2t) \, dt \)
(d) \[ \int_{0}^{1} 3^x \, dx \]

(e) \[ \int_{1}^{3} \left( \frac{2}{t^2} + 3t \right) \, dt \]

6. Your deep ocean oil rig has suffered a catastrophic failure! Oil is leaking from the ocean floor wellhead at a rate of

\[ v(t) = 0.08t^2 - 4t + 60 \text{ thousand barrels per day} \quad (0 \leq t \leq 20), \]

where \( t \) is time in days since the failure. Compute the volume of oil released during the first 20 days\(^1\).

\(^1\)The BP Oil spill released 4.9 million barrels of oil which amounts to an oil slick covering over 68,000 sq mi. To put this in perspective 68,000 sq mi is enough oil to completely cover the state of Florida.
7. The marginal cost, \( C'(x) \), in dollars per unit of your business is given by

\[
C'(x) = -0.025x^2 + 2.5x + 140,
\]

where \( x \) is the number of units produced.

(a) Calculate \( \int_0^{50} C'(x) \, dx \). Explain the meaning of your answer in the context of the problem.

(b) Given that you have a fixed cost of $5,000 determine the total cost of producing 50 items. The fixed cost is the value of \( C(0) \).

8. A company’s marginal profit is given by in millions of dollars per month is given by

\[
\pi'(t) = -t^4 + 18t^3 - 89t^2 + 72t + 180,
\]

where \( t \) is months since the beginning of 2015.

(a) What is the change in profit from \( t = 1 \) to \( t = 6 \)?

(b) Given that the company’s profit at \( t = 4 \) was 150 million dollars determine \( \pi(7) \). Interpret your answer.
The questions below ask you to determine the area of an enclosed region. This means that we are trying to find the total area between the function and the horizontal axis over a specified interval.

Questions that ask for an enclosed area, total area, or area between two curves, are all referring to the total absolute area.

9. Use your calculator to determine the area of the region enclosed by the graph of \( y = xe^{x^2} \), the \( x \)-axis, and the vertical lines \( x = -1 \) and \( x = 1 \). \textit{Hint: Sketch the region!}

10. Use your calculator to determine the area of the region enclosed by the graph of \( g(x) = -x^3 \), the \( x \)-axis, and the vertical lines \( x = -3 \) and \( x = 4 \).