Preservice elementary school teachers' fragmented understanding of mathematics is widely documented in the research literature. Their understanding of division by 0 is no exception. This article reports on two teacher education tasks and experiences designed to challenge and extend preservice teachers' understanding of division by 0. These tasks asked preservice teachers to investigate division by 0 in the context of responding to students' erroneous mathematical ideas and were respectively structured so that the question was investigated through discussion with peers and through independent investigation. Results revealed that preservice teachers gained new mathematical (what the answer is and why it is so) and pedagogical (how they might explain it to students) insights through both experiences. However, the quality of these insights were related to the participants' disposition to justify their thinking and (or) to investigate mathematics they did not understand. The study's results highlight the value of using teacher learning tasks that situate mathematical inquiry in teaching practice but also highlight the challenge for teacher educators to design experiences that help preservice teachers see the importance of, and develop the tools and inclination for, mathematical inquiry that is needed for teaching mathematics with understanding.

Today we showed our elementary preservice teachers a video where an interviewer was asking young students what the answer to $5 + 0$ was. The students in the video consistently responded that the answer was 0, and when asked how they knew this, many responded: "my teacher taught me that." We asked our preservice teachers to imagine that these were students in their 5th grade class, and to discuss in their groups how they would respond to and challenge such students' thinking. The response was a complete silence for what seemed to be an eternity, until one preservice teacher sheepishly asked: "Do you mean to say that the answer is not 0?" (Instructor journal, Fall '93).

Our interest in preservice teachers' understanding of division by 0 began over a decade ago while coteaching a mathematics methods course for elementary teacher candidates. The episode in the quote above relates what happened the day we came to class with the question of division by 0 as a context for exploring the challenges of listening and responding to students' mathematical ideas. We had planned to engage the class in thinking about how they might respond to children's mathematical ideas in ways that respect but still challenge and extend their reasoning. As the journal excerpt suggests, this particular class did not turn out at all as we had expected.

We had imagined that the preservice teachers would be challenged to think of ways of explaining to young students why the answer to dividing by 0 is not 0 using mathematical arguments rather than appeal to authority. We thought they could find it difficult to explain why division by 0 might be "undefined," "impossible," or "not allowed" without using language or mathematical tools or representations that were inaccessible to young students, but we did not expect them to not know that division by 0 is not 0. Instead, as the excerpt shows, the preservice teachers in our class were surprised to hear that division by 0 is not 0 and were instead challenged to think about what to do with students' ideas when they, themselves, did not know the mathematically correct answer.

Since that first incident we have continued to experiment with different ways of posing the question of division by 0 in our mathematics methods courses. This paper describes our investigations of preservice teachers' responses to the question of division by 0 when the question is presented in different ways—to be discussed in class and to be investigated outside of class. Our findings are discussed along with their
implications for designing opportunities for preservice teachers to wrestle with mathematical ideas that they do not quite understand. First, the theoretical perspectives and relevant research literature that framed and informed this study are discussed.

Challenges of Understanding Division by 0

In the research literature, the concept of 0 and the result and rationale of arithmetic operations on 0 have been explored historically (Kline, 1962; Seiffe, 2000), linguistically (Blake & Verhille, 1985), and cognitively, both in terms of young children’s misconceptions (Henry 1969; Reys & Grouws, 1975) and in terms of prospective and practicing teachers’ difficulties with this concept (Ball, 1990; Even & Tirosh, 1995; Simon, 1993; Tsamir, Sheffer, & Tirosh, 2000; Wheeler & Feghali, 1983). These researchers reported on the widespread confusion over 0 and division by 0 by students of all ages.

Through interviews with school students, Reys and Grouws (1975) for instance, found (as Piaget did) that the notion of 0 is not well developed in young children. They reported that the most common misconception they found was whether or not 0 is a number or “nothing.” An eighth grader in their study, for instance, explained that 0 ÷ 0 was 0, “because when you divide nothing by nothing you can’t get something” (p. 601). Another persistent problem reported was that most students tended to justify their answers by saying, “My teacher told me.”

Understanding the concept of 0 and operations with and on it also presents challenges for practicing and intending teachers. After interviewing 19 elementary and secondary teacher candidates, Ball (1990) concluded that the participants’ understanding of division including division by 0 seemed founded more on memorization than on conceptual understanding. Few teacher candidates in Ball’s study were able or disposed to provide mathematically legitimate explanations for their answers. Even and Tirosh (1995) reached a similar conclusion when they explored 33 Israeli secondary mathematics teachers’ conceptions of four undefined mathematical operations (4/0, 0/0, 0, (-8)^½). The researchers reported that, though the teachers in their study could state that 4 divided by 0 is undefined, many provided a rule-based argument—both to themselves and to students.

A decade earlier Wheeler and Feghali (1983) also noted elementary preservice teachers’ difficulties with the meaning of 0 and division by 0 and suggested that “explicit attention should be given to concepts of 0 in mathematics education courses” (p. 154). They also pointed to the lack of attention to the concept of 0 and division by 0 in instructional materials and urged educators to focus attention on developing such materials. Twenty years later, their findings and recommendations are far from outdated. A quick browse through any mathematics methods textbook reveals that the topic of division by 0, if mentioned at all, is covered briefly. On the topic of division by 0, one text, for instance, states,

Many children are simply told “Division by 0 is not allowed.” To avoid an arbitrary rule, pose problems to be modeled that involve 0: “Take thirty counters. How many seis of 0 can be made?” or “Put twelve blocks in 0 equal groups. How many in each group?” (Van de Walle, 2004, p. 150).

All of the cited studies have painted a discouraging picture of students’ understanding of 0, of division by 0, and of the likelihood that preservice teachers could teach this idea with understanding. Intrigued by these results and our own experiences as mathematics teacher educators, we set out to investigate ways to challenge and extend preservice teachers’ ideas about division by 0. We were interested in the questions raised by the studies of Ball (1990), Even and Tirosh (1995), and Wheeler and Feghali (1983) regarding the design of instructional tasks and learning opportunities for prospective teachers. Considering the range of difficulties reported in the previous studies, particularly Even and Tirosh’s findings of teachers’ offering rule-bound explanations to students, we thought that reading a brief paragraph in a textbook is not enough to help prospective teachers develop their own understanding of division by 0 and be in a better position to teach it conceptually to their students.

Understanding Division by 0—Why and How?

Teaching elementary school mathematics requires what Ma (1999) has called a “profound understanding of fundamental mathematics,” that is, as Ball (1988) suggested, knowledge of mathematics that is rooted in conceptual understanding and in the modes of inquiry of the discipline. Helping prospective teachers develop such understanding during teacher preparation is a challenge. Mathematics teacher educators are constantly faced with the question of how to help prospective teachers develop a deeper understanding of mathematics while also learning about teaching and learning. The typical 12-15 weeks of a mathematics methods course are not enough time to cover meaningfully the mathematical content that teacher candidates will face in elementary school. Choosing what gets
Division by Zero

A. C. Tirosh

Covered and in what depth is always a challenge.

In our courses we choose to incorporate investigations related to division by 0, not only because this is a question that presents a real mathematical challenge for intending teachers, but also because the question offers opportunities to engage in making sense of mathematics in the ways we hope intending teachers will come to value and promote in their future classrooms. Because division by 0 is unlike most elementary mathematics questions (it does not have a satisfactory arithmetic answer), it offers the opportunity to wrestle with an abstract mathematical idea that is accessible to everyone independent of mathematical background and can be investigated within a reasonable amount of time. It also provides opportunities to develop the "mathematical attitudes" Ma (1999) found in the Chinese teachers she studied; that is, a disposition toward generating their own explanations and exploring the mathematics they do not understand.

With the question of division by 0, we seek to stimulate our teacher education students' curiosity and disposition to raise and investigate other such questions of their taken-for-granted and unquestioned mathematical knowledge. Another goal is to promote the curiosity and sensitivity for students' thinking Even and Tirosh (1995) proposed. Investigations of how others make sense of division by 0 can also help prospective teachers attend to the quality of students' explanations and notice that these are not always justified with sound mathematical reasoning. Instead students' explanations might appeal to outside sources of authority (teacher taught me that; the book says so; calculator gave me the answer), appeal to a sense of consistency (we all got the same answer; we can get the result two ways), or appeal to an intuitive sense (it makes sense; it looks or feels right).

Good explanations, however, like well-reasoned arguments (Weston, 2000), can be justified by using (a) examples that confirm/contradict given arguments, (b) deductive logic that tests hypothetical and false premise, or (3) analogy or alternate representation that illustrates or proves a more general case. In the case of division by 0, students' well-reasoned explanations can include these three types. Examples of each type are included in the appendix. In addition to being well versed in these and other such explanations, it is important that intending teachers compare and notice important differences and similarities across different kinds of formal and intuitive explanations for division involving 0 (see Knifong & Burton, 1980; Watson, 1991). They could, for example, attend to how these different explanations draw on different meanings for division (measurement/quotative or sharing/partitive) and various models and representations (arrays, number line, story problem) of this operation. Another important distinction is the difference between rendering an operation "undefined" because the operation cannot be carried out or because it leads to an unending process (e.g., sharing 5 apples among 0 children vs. subtracting 0s from a basket until all items are gone). Similarly, 0/0 is undefined, not because there is no ordinary number that will satisfy the related multiplication number sentence, but because all too many numbers satisfy the equation.

Research Context and Design

We came to this study with many questions and assumptions about prospective teachers' difficulties with division by 0. We saw the question of division by 0 as a rich context for mathematical and pedagogical inquiry and had visions for the learning opportunities that such a mathematical question could provide in our courses. We designed two different instructional experiences, both aimed at challenging and extending the elementary prospective teachers' understanding of division by 0. In this study the participants' responses before and after these two different experiences are examined in order to explore what happened to their ideas but also to investigate what it takes to challenge prospective elementary teachers' understanding of mathematics.

The main questions guiding the design and implementation of the study included the following:

1. How do prospective elementary teachers respond to the question of division by 0 before they have opportunities to discuss their ideas or investigate the topic? (What kinds of explanations do they use to justify their answers? And what kinds of explanations do they use to explain to young students?)

2. How do prospective teachers participating in two different instructional interventions respond to the question of division by 0 after their explorations? (What can we learn from their responses about what might help prospective teachers of mathematics develop the "mathematical attitudes" and "sensitivity to students' thinking" needed in and for teaching?)

Participants and Setting

The participants in this study were prospective elementary teachers enrolled in two different mathematics methods course sections that were each taught by one of the authors. These courses are offered in two different teacher education programs. The two...
programs are similar in their overall structure of general and methods courses followed by the student teaching experience but are offered within different timelines. The two programs also require that all students complete a required mathematics for teaching course before they can take the mathematics methods course.

Course A was offered to intending teachers enrolled in a 12-month postbaccalaureate teacher education program in the second author’s institution. The mathematics methods course in this program is offered on campus and meets twice a week for 1.5 hours over 12 weeks. Course B was offered to intending teachers in a 5-year undergraduate elementary teacher education program at the first author’s institution. The mathematics methods course is located in the fourth year of this program, and it is scheduled for a 3-hour on-campus seminar and 2 hours of school fieldwork each week over 16 weeks.

The participants included 18 of 28 prospective teachers enrolled in Course A and 14 of 20 enrolled in Course B. Prospective teachers in Course A, as is typical of that institution’s program, represented a diverse range of cultural backgrounds. In Course B as it is typical of this institution, the majority were Caucasian, and only a few were Hispanic or African American. Across the courses, participants ranged in age from the youngest being in their early 20s to the oldest being in their late 30s. Although the participants in Course A were ethnically more diverse than those of Course B, they shared similar mathematical and undergraduate course preparation and, as will be described later, had similar initial responses to the question of division by 0.

In each program the mathematics methods courses were designed to engage prospective teachers in both mathematical and pedagogical investigations. Class activities were offered to prospective teachers through a “pedagogy of inquiry” rather than a “pedagogy of presentation”; that is, as contexts for investigation rather than as methods and techniques to be accepted (Lampert & Ball, 1998). The case of division by 0 was presented to prospective teachers midway through the course at a time when class members felt comfortable sharing and challenging their own and each other’s ideas and subsequent to work on developing number sense and whole number operations. In both courses the work on the division by 0 task was part of the course activities. The students were expected to complete the task but were not assigned a grade.

Two Versions of the Division-by-0 Task

In these mathematics methods courses the instructors offered the question of division by 0 as a mathematical question situated in practice. The task is situated within the context of challenging children’s erroneous ideas rather than a straight mathematical question. This design draws on common practices in teacher education derived from studies of practicing and prospective teachers’ development of mathematical knowledge for teaching (e.g., Ball, 1988; Ma, 1999) and draws on theories of situated learning underlying much of the work and research on teacher learning (see Putnam & Borko, 2000). This design also fits case methods—a pedagogical approach to teacher education—which uses “cases” (images and stories of real classroom teaching) as problem solving devices (Doyle, 1990) to help prospective teachers develop and practice the problem solving and decision-making skills they will need and use in the practice setting.

We explored prospective teachers’ understanding of division by 0 using two different versions of the task. The two tasks had slight variations, and both aimed to challenge the preservice teachers’ ideas about division by 0 and how they might explain it to elementary school students. The instructional experiences associated with these tasks were designed for the whole class and took place during regular class time. Each version of the division-by-0 task took place in sequential rather than concurrent order, and the first version informed the design of the second one.

First version. The first version of the task was used in Course A and consisted of a short video showing young students answering an interviewer’s question about division by 0. In this video the children consistently answered that $5 \div 0$ is 0 and either “explained” with the rule that “anything divided by 0 is 0” or offered, “My teacher taught me that,” as an explanation. After watching this video the preservice teachers were instructed to first work individually on a set of focus questions and then to discuss their ideas within their group to later report back and discuss with the rest of the class. The focus questions included,

- How do you think the students in this video may have arrived at the conclusion that “anything divided by 0 equals 0”? If these were students in your class, how would you help them understand division by 0? When and how would you explain $0 \div 0$?

Second version. The second version was used in Course B, and it included a written prompt modeled after the video used in the first version. It was structured similarly to the first version—jot down your own ideas, discuss within your group, then debrief with whole class—but in addition, the prospective teachers were asked to investigate further beyond class and
write up their findings and new insights in their math journal. The task read as follows:

A student in your Grade 5 class thinks that $5 \div 0 = 0$. When asked to justify the answer, the student replies: "My teacher taught me that." What do you think of this student’s response? When did you learn about division by 0 and what do you remember about it? Is the answer really 0? Can you prove it right (or wrong)? Would the answer still be the same when you divide $0 \div 0$? What happens when you divide $0 \div 5$? Use words or pictures to prove your answers. How would you teach students about division by 0 so that they can make sense of it rather than memorize the answer?

Differences between the two task versions. As noted earlier, the design of the second version was informed by what happened with the first version of the task. One obvious difference between the two versions is that one used video footage of interviews with students and the other did not. We substituted the video with a written description of a child’s response because the video footage distracted the conversations toward examining what was going on in the video that could explain why students were giving such responses (e.g., intimidated by interviewer, scared to be on camera).

Another difference between the two task versions is that the second version asks prospective teachers to reflect first on their own understanding of division by 0 before attempting to devise an explanation for younger students. Notice that the first version does not explicitly ask the prospective teachers to engage with the mathematics question. Rather it indirectly engages them in doing mathematics through the challenge of figuring out a way to explain to students what the answer is in a way that makes sense. The second version, in contrast, prompts them to think about their own understanding of the question before moving on to figure out an explanation for students.

Data Collection and Analysis

The participants’ individual responses and the instructors’ observation notes were collected from two different sets of prospective elementary school teachers attending the authors’ mathematics methods courses. These data sets were used to analyze the preservice teachers’ initial ideas about division by 0 and how they might explain it to students. Further data was then collected from the two groups in order to investigate changes in the quality of preservice teachers’ explanations and to investigate the strengths and weaknesses of the design features of each instructional approach. In Course A whole and small group discussions were video and audio recorded, and for Course B the participants’ journal entries were collected.

The analysis of preservice teachers’ initial ideas about division by 0 began by grouping and coding responses, first by their correctness, and second by type of explanation or justification, such as appeal to authority or reasoned explanation (as discussed earlier). The analysis also focused on the quality of these arguments in terms of the language, representations, and details used. The analysis of the data generated from the two versions of the task focused on how preservice teachers’ initial ideas were (or not) challenged and (or) extended in each context. In the context of in-class discussions, for instance, we looked for new insights, ideas, or questions that were generated in conversations with peers. In the context of further investigation we contrasted preservice teachers’ explanations before and after investigation by looking for similarities and differences between the two and paying close attention to the sources of new insights.

Results: Preservice Teachers’ Responses

Prior to Class Explorations

This section reports the kinds of responses the research participants tended to give to the question of division by 0 across the two course sections before having any opportunity to explore the question through the two instructional experiences investigated in this study. The explanations of correct and incorrect respondents are examined and discussed in relation to the explanations they might offer to students.

Responses to the Question of Division by 0

I think the students don’t understand how the problem works. They just know what the teacher told them. I don’t remember a specific time when I learned about division by 0. I do remember discussing when the answer is 0 and when it is undefined, but I can’t remember the explanation. (Preservice teacher in Course A).

When I think about it, I’m not really sure what the answer is. I should know what $5 + 0$ is. It seems that it would be 5 because when you divide 5 into nothing, the answer should stay the same. But then when you write out the answer $0 \div 5$, ’0 doesn’t go into 5 at all. I then think that the answer is 0. I’m a little confused now and I’m not sure what I would say to teach this to my students. (Preservice teacher in Course B).

These responses show the confusion and puzzle-
Division by Zero

Responses to How to Teach Division by 0

I think this student's response is sad, but all too typical of many math classrooms. I cringe when I hear a student regurgitate something their teacher says is a right answer instead of thinking for themselves. I do not remember when or if I ever learned the properties of 0. My more recent memories were of high school algebra and 0's significance there, but I can not say as though I ever really learned that 5 divided by 0 is equal to 0. Most memories of early math were memorizing certain properties, procedures, and formulas which seems to still be the case with many math teachers who place too much emphasis on finding the right answers and not enough on the processes. Still to this day I look at this problem and have no idea what the right answer is. (Preservice teacher in Course B)

The question of division by 0 raised multiple issues for the preservice teachers in this study. As this response shows, it brought to the foreground preservice teachers' commitment to empower their students to think for themselves and make sense of mathematical ideas. At the same time it brought about painful realizations of their own lack of knowledge and powerlessness to make sense of what seemed to be a simple mathematical idea. Prospective teachers in both groups had limited ideas about how they might explain to their future students what the answer to 5 ÷ 0 is and why.

Looking more closely at their responses (see Table 2) it is interesting to note that only 2 of the 32 participants resorted to a rule to explain division by 0 to students. This result is different from Even and Tirosh's (1995) findings (their participants, who were secondary school teachers, resorted to rules to explain to students). However, 21 of our research participants did not attempt an explanation; many stated they were "not sure what I would say to teach this to my students." Of the 9 who attempted a reasoned explanation, 6 had the correct answer and 3 were incorrect. Both correct and incorrect respondents had broad and unspecific ideas, such as, "I would use pictures of groups to teach my students," and they also had specific ideas, such as contextualizing the operation in a story problem.

An interesting aspect of the attempted explanations to students is that only 2 of the 32 preservice teachers related the idea of division by 0 to how division works with other numbers. Prospective teachers in this study seemed to think of division by 0 as an isolated
### Table 1

**Preservice Teachers’ Responses Prior to Class Explorations**

<table>
<thead>
<tr>
<th>Answer is ...</th>
<th>Rule Bound Explanation</th>
<th>CourseA</th>
<th>CourseB</th>
<th>Reasoned Explanation</th>
<th>CourseA</th>
<th>CourseB</th>
</tr>
</thead>
<tbody>
<tr>
<td>“0”</td>
<td>Can’t remember</td>
<td>3</td>
<td>1</td>
<td>Division as sharing and flawed logic</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>I learned that anything divided by 0 is 0. That’s all I remember.</td>
<td></td>
<td></td>
<td>Dividing a number by 0 gives you 0 because you are taking a number and putting it into piles with 0 in each group.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Can’t divide by nothing</td>
<td>3</td>
<td>2</td>
<td>Division as grouping and flawed logic</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Since 0 is nothing we can’t divide five into anything. So we end up with nothing.</td>
<td></td>
<td></td>
<td>If you divide by 0, you are then creating 0 groups, which gives you 0 in each group.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Overgeneralizing rule</td>
<td>3</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>You cannot do 0÷5 because one of the rules in division is that you have to put the larger number first.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>“5”</td>
<td>Can’t divide by nothing</td>
<td>3</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>I feel like 5÷0 should equal 5. This is because you are basically saying that you are dividing it by nothing so it remains the same.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>“Undefined”</td>
<td>Outside authority</td>
<td>1</td>
<td>1</td>
<td>Division as inverse of multiplication</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>It’s not possible because the calculator says it’s an error, or teacher said.</td>
<td></td>
<td></td>
<td>The trick I used was the answer times the number of groups the number is being divided equals that original number. Wow, that sounds confusing but it works! 5÷0=? Therefore 0 x ? = 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Can’t remember rule</td>
<td>3</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>I am not sure how to prove it right or wrong. I remember learning about what the answers are, but I don’t remember learning why.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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**Summary of Responses Prior to Explorations**

To summarize, in both course sections the preservice teachers’ initial responses to the division by 0 scenarios revealed multiple misunderstandings—about 0, division, and about justifying and explaining in mathematics. Even those who correctly stated the answer or could provide a reasonable explanation also harbored misunderstandings regarding $0 \div 0$. Few preservice teachers across the two groups initially could see beyond their intuitive notion of “0 as nothing,” connect division by 0 to how division works with other numbers, or could think of $5 \div 0$ in a problem context, such as...
Table 2

| Frequency and Type of Answers and Explanations for Students Prior to Class Explorations |
|---------------------------------|---|---|---|
| No Explanation                  | 0 | 5 | Undefined |
| Example:                        | I am a little confused now and I’m not sure what I would say to teach this to my students. |
| Course A                        | 6 | 3 | 3 |
| Course B                        | 4 | N/A | 5 |

Rule

Example: I would teach them about how 0 has no value, just like 0 + 0, the answer would be 0.

Course A 1 0 0
Course B 0 N/A 1

Reasoned

Example: I would show them division by 1 as comparison, and say 5 ÷ 1 = 5 that means 5 things made into groups of 1 is 5 groups of 1. So when you divide 5 ÷ 0 you make no groups.

Course A 3 0 2
Course B 0 N/A 4

Watson’s (1991) measurement and partition division story problems: (a) Given a book with 10 pages in it, if you read 0 pages per minute, how long will it take you to read it? Or (b) If you have six apples and no friends to share them with, how many apples will your friends get? These issues were also revealed in their explanations to students. Preservice teachers’ explanations to students revealed their inexperience with, and lack of sensitivity to, students’ thinking; that is, their responses to students were aimed at correcting more than understanding students’ thinking and revealed an unawareness of the relative or comparative benefits and costs of the explanation they devised for students.

Results: Preservice Teachers’ Responses After Instructional Explorations

The responses reported in the previous sections are similar to the kinds of responses we have collected over the years of using this particular task in our courses. These kinds of responses to the question of division by 0 are what led us to more systematically explore ways to challenge, extend, and connect prospective teachers’ mathematical and pedagogical ideas on this topic. This section reports the study participants’ responses to the question of division by 0 after they had opportunities to engage in two instructional experiences: in-class discussions and investigating beyond class. First, preservice teachers’ ideas in the context of in-class discussions are examined. Then we look at what happened to preservice teachers’ ideas when they were required and encouraged to further investigate their questions and misunderstandings.

In-Class Discussions

In this version preservice teachers watched the short video mentioned earlier and were asked to first write their individual responses to the prompt (how might you help these students understand that the answer to division by 0 is not 0) and then to discuss their ideas with their group to later share their collective understanding with the whole class. The group we examined had “Carmen,” “Fiona,” and “Pete,” who thought the answer to 5 ÷ 0 is 0; “Tom” initially thought the answer was 0 but then concluded that the answer could be 5; and “Dana” thought division by 0 was undefined and proved it using the multiplication as inverse of division argument. Their discussion was very animated and intense, as illustrated in the following excerpts of their conversation.

Tom: But what is the answer? Like 5 divided into 0 groups is that 5 or zip?
Dana: How can you put 5 into 0 groups?
Tom: Well, you don’t, but so that would still leave the number, if you’re not performing an action, then that still leaves the number there.
Fiona: Sure, turn it into a fraction. Turn it into 5/1 divided by 0/0. Okay, now you want to turn it into a multiplication, let’s do the reciprocal 5/1 times...
Tom: [interrupts] But those are still rules, I want the conceptual [laughs]...
Tom: I still wanna know how to explain this. [Calls the instructor.] Is 0 inert or can it have, it has action, it can cancel other things?
Instructor: Like what?
Tom: Like 5 times 0 is 0, right. So it is almost as if the 5 gets sucked into the 0
Pete: Oh! Like a black hole.
Carmen: That’s a good visual!
Tom: I’m working with that but I am not sure you can do that mathematically, or say that 0 goes “ssssuck” and sucks up the 5, right.
Instructor: Well, I wouldn’t recommend that [laughs].
Tom: Children with nightmares, ohh! Zero is going to get me!
Instructor: Because then you have 5 plus 0, it doesn’t have any effect on the 5, right? So then the
idea that 0 sucks up everything doesn’t work….

Instructor: Think about division, what are you doing when you divide?

Dana: You have to tell them to do the multiplication of it, and tell them you know 2 times 5 equals 10, and 10 divided by 2 equals 5. Then you got, then you should have that idea that these are connected, okay?

Instructor: That’s interesting. You’re saying use the division as the reverse or the opposite of multiplication, so what would happen with 0?

Dana: Then they know, 0 times 5, you know, you can’t put 5 groups into nothing. So if you go through the multiplication way, then maybe you can see 5 divided by 0.

Tom: I can see it in terms of rules and abstract concepts. I can still see that, and that’s fine. But I wanna show them in blocks. I mean, okay, 5 groups of nothing is still nothing, but for 0, no groups of 5, again, where does the 5 go? ...

Dana: [Grabs textbook] You wanna know how the 0 works in multiplication with 0? This is what the book says. “Rob got a baseball bat for his birthday. He was up at bat 4 times and he got struck out each time, how many hits does he have?”

Fiona: Okay, again because we’re talking about something different, he was up to bat 4 times. We’re talking about hits, he got 0 hits; but here we’re saying we have 5 blocks, so you actually have ...

Tom: [interrupts] You want to do something to those 5.

Fiona: It’s like saying he has 4 hits, divide these 4 hits by 0, he got 0 hits, so what happened to his 4 hits?

Tom: Is it dangerous to actually have them [kids] convert these things into sentences?

Pete: Nobody ever did these with me in math, never!

Fiona: You never get to this point because they’ll get frustrated, they’ll say forget it.

In these short segments of the group’s discussion it is evident that an agreement was not reached about what the answer was or how to explain it to students. The group aired in the open their mathematical and pedagogical ideas. Tom questioned why the answer was 0, Dana challenged, “How can you put 5 into 0 groups?” and Tom challenged back, pushing the group to think conceptually (which he seemed to mean concretely or contextually) by refusing to consider as such the explanations others in the group were offering. Several meanings and representations of division were offered. Fiona offered to think about division in terms of fractions. Tom offered the measurement interpretation of division “5 divided into 0 groups” and the rule of “0 cancels everything.” Dana, in turn, offered the idea of using multiplication in order to help students understand division and attempted to contextualize the question in a real life setting.

The group searched for two outside sources to help further their thinking: their instructor and a mathematics textbook. The instructor’s interventions seemed to help settle the argument as to whether or not 0 has canceling properties and helped put forth Dana’s ideas, but it did not seem to help settle the argument about whether 5 could be an answer to the question. A possible intervention that could aid in that direction is to redirect the students to think about explaining 12 ÷ 3 and use this situation to help them think about how to explain 5 ÷ 0.

The fifth-grade mathematics textbook that was provided to each group as a resource, on the other hand, proved less helpful. Dana found a story problem whose context helped make sense of 4 x 0: “Rob had 4 times at bat, struck out each time, how many hits?” However, the text did not seem to have a similar idea for representing division by 0 and the group was not able to translate or use this idea to help them think about division by 0.

There is much in this discussion that is encouraging. The group was seriously engaged in “substantive mathematical discussion” (Pirie & Schwarzenberger, 1988). Students were on topic, seemed to follow and respond to each other’s contributions, and generated ideas beyond what any one individual had generated on their own. The conversation moved seamlessly between the mathematical and pedagogical aspects of the question raised by the task, and it offered multiple insights to the participants. However, the conversation also raised serious concerns. These preservice teachers did not seek to understand their peer’s thinking or use it to refute their explanations. No one, for instance, said to Tom, “Okay let’s say that 5 ÷ 0 = 5. How would you explain that 5 ÷ 1 also equals 5?”

Further, note how the group moved quickly to explore how to explain to students without spending much time exploring their own understanding of division by 0. This was observed in all of the other group conversations of this particular class. None of the groups spent much time clarifying their own ideas before moving on to discussing how to explain to
students. As noted earlier, this is one of the issues that led to the slight variation in the second version of the task. Although we acknowledge that this issue relates to the wording in this task, the preservice teachers’ lack of mathematical exploration before designing a pedagogical response is concerning. In the context of teaching practice, mathematical challenges arise in much the same way as in this first version of the task; that is with no invitation or prompt to first explore and clarify one’s own mathematical understanding.

We have no further data on these preservice teachers’ thinking beyond this class, but our observations of this and later similar classes offer much insight and concerns. A major insight to be drawn from this data is that group discussions are good contexts for creating dissonance in preservice teachers’ thinking and generating multiple meanings, interpretations, and representations but may not be enough to help clear up their confusion and misunderstandings. In this case, while the group’s discussion brought about ideas and questions beyond what individual people in the group had generated on their own, the set-up of the activity, instructor’s interventions, length and structure of the discussion, and available resources are just a few of the factors that also can play a role in extending the participants’ initial ideas.

Our assessment of the preservice teachers’ reactions to this class also worried us. Although the majority of preservice teachers seemed to be challenged to rethink their ideas and were left with many more questions than answers, no one came back to the next class having investigated or gained further insights beyond what had been discussed in class. This is an important issue because it relates to the mathematical attitudes (Ma, 1999) referenced earlier but also is consistent with research reports about similar kinds of behaviors exhibited by prospective teachers, not only in their teacher education courses, but also in the context of their student teaching experience. Borko and colleagues (1992) for instance reported on the case of Ms. Daniels, a student teacher who, after failing to provide a conceptual explanation during class to a student’s why question, simply moved on without ever returning for a second attempt at explaining.

Another concern this class raised for us was that even after whole class debriefing and after all sorts of explanations were discussed, there were still some prospective teachers who remained unconvinced by any evidence presented in this class and continued to cite their past teachers, textbooks, or their own flawed reasoning to justify their answers. Some preservice teachers seemed to conclude that if division by 0 is too hard for them to understand, then trying to explain it to young students is pointless. We can sense the development of this argument at the end of the excerpt shared earlier. Notice how Tom and Fiona expressed some problematic views about children’s ability to reason mathematically when they implied that it is “dangerous to ask children” to try to make sense of such mathematical ideas and that most likely “they will get frustrated and say forget it.” Still others seemed resentful that their instructor had raised questions that they were expected to answer on their own rather than being shown or told how to answer. These were the concerns that led to the redesign of the task and to the idea of including a follow-up investigation beyond class.

Investigating Beyond Class

In this version of the division by 0 task, as with the first version, preservice teachers were asked to think about division by 0 first on their own, then within their group, and then with the whole class. Different from the first version of the task, they were then asked to continue to investigate their questions about division by 0 outside of class and to write up their new insights in their math journal. This second task version was meant to address the concerns noted that the preservice teachers seemed to be blindly justifying their thinking with rules and citing outside sources of authority and looking up to the instructors to validate their ideas or tell them who had the right answer or justification. It was meant to encourage preservice teachers to adopt an empowered stance toward their mathematical gaps and misunderstandings.

A summary of the data is presented in Table 3. The responses are organized in columns that indicate the correctness and quality of the preservice teachers’ responses before and after investigation and the type of investigation and source of new insights as reported by the participants. This data revealed that after further investigation only 2 preservice teachers continued to incorrectly state that the answer to division by 0 is 0 and that 11 of the 14 participants wrote multiple explanations that drew on both meanings and more than one representation of division. Everyone seemed to want to teach this idea conceptually rather than with rules, and no one questioned whether students (fifth was the grade suggested in the prompt) could understand or be asked to explore this question. The participants’ ideas about teaching varied in detail and specificity from simply stating, “teach it much differently than the way I was taught,” or “explain with multiple explanations,” to more elaborate statements, such as,
Division by Zero

If I had to teach students about division by 0 I would supply them with different ideas. One idea is that division is the reverse of multiplication. By giving them the example of 12 divided by 6 equals 3, and 6 times 3 equals 12 the students will see that division is the reverse of multiplication. I would then show the question of 12 divided by 0 equals ?. With this being the case, ? times 0 would have to equal 12. The other idea is to present the students with the problem of 5 divided by 0 and ask them how many groups of 0 could you separate these 5 blocks into. The students would most likely come up with the ideas that it does not matter how many groups of 0 you have because it will always add up to 0 (rather than 5), and therefore the problem is undefined.

(Marta)

An important pattern illustrated in Table 3 is that the most successful respondents—successful in terms of gaining new insights, expanding and clarifying their ideas, and deriving appropriate and multiple explanations for school age students—were preservice teachers who, regardless of whether their initial answer was correct or incorrect, exhibited an inclination or disposition toward proving and justifying their answers, and/or a disposition to investigate the mathematics they did not know. Consider “Angela” who is one of the preservice teachers in the last row of Table 3. Angela’s explanation is interesting because she originally derived her own explanation for why the answer is undefined. In addition, Angela investigated further in a Web site and expanded her repertoire of explanations to include well-reasoned explanations, such as the multiplication as inverse operation and the patterned examples explanation—the quotient gets further away as the divisor gets closer to 0. She also noted the difference between 5 ÷ 0 and 0 ÷ 5 and the importance of explaining 5 ÷ 0 in relation to its reciprocal. The following is a small sample of what Angela wrote:

When I learned about division by 0, I remember learning that it did not work. I do not remember learning why it did not work. The easiest way for me to understand this concept is to explain it in context with objects. For example, 5 cupcakes divided by 0 kids is not possible. Nothing will be divided because there are no people to take the cupcakes. One cannot come up with an answer to that problem because the action of dividing them among 0 people is impossible. A student may argue that the answer is 5 saying: “No one got the cupcakes, so there are 5 left.” I would argue that the problem was never actually completed. The action is impossible, so the problem cannot actually have an answer. I would go on to explain the difference between 5/0 and 0/5. If there are 0 cupcakes divided by 5 kids, how many will each kid get? The action can be attempted, and the outcome will always equal 0.

This particular task version and instructional experience, therefore, seems to be a promising design to promote dissonance and further insight in preservice teachers’ ideas. The quality of preservice teachers’ insights, however, seems to depend on their “mathematical attitudes” or disposition toward generating their own explanations and exploring the mathematics they do not understand. This point is particularly salient when we look at the 3 preservice teachers who gained little insight. They did not attempt an explanation and/or did not investigate beyond class or beyond asking others who, like them, could not explain their answers. They continued to harbor incorrect ideas about division by 0 and had no ideas for how they might teach it.

Conclusion and Implications

The data presented here suggest that preservice teachers’ ideas about division by 0 made extensive progress across the two different instructional experiences investigated in this study. Initially, like Ball (1990), Even and Tirosh (1995), and Tsamir et al. (2000), we found that the preservice teachers’ initial understandings of 0 and division by 0 were founded more on rule-based and flawed reasoning than on well-reasoned mathematical explanations and that they lacked the experience and inclination to understand or appreciate different ideas and approaches to this topic. For many of the participants their later explanations became more conceptually based than rule bound. Participants also extended their initial ideas by gaining insight into appropriate and multiple kinds of explanations.

The contrast between the two versions of the task and the change of participants’ understanding revealed important differences in each task’s ability to engage and elicit the preservice teachers’ mathematical and pedagogical understandings. While group discussions challenged and extended the participants’ own mathematical ideas, these were not enough to engage the “mathematical attitudes” and “sensitivity to students’ thinking” that researchers suggest enable mathematical knowledge to become useful in the context of mathematics teaching. The preservice teachers’ mathematical attitudes seemed most engaged and revealed in the second task version and instructional experience. In other words the explicit directive to clarify their own

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### Table 3
**Summary of Course B Participants’ Responses Before and After Further Investigation**

<table>
<thead>
<tr>
<th>Before</th>
<th>Type of Investigation</th>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incorrect Answers - 4</td>
<td>Investigate by asking other people who did not know either and did not investigate further.</td>
<td>Not sure what the answer is. Both stated they had no idea how to teach it.</td>
</tr>
<tr>
<td>2 — justified with rules and made no attempt to provide a reasoned explanation.</td>
<td>Investigate by asking other people who did not know either and did not investigate further.</td>
<td>Not sure what the answer is. Both stated they had no idea how to teach it.</td>
</tr>
<tr>
<td>1 — justified with rules and made no attempt to provide a reasoned explanation.</td>
<td>Cite insights from class discussions and investigated in Web site and with other people whose answers were different from their own.</td>
<td>Both reported insights into correct answer and ways of justifying using more than one explanation that draw on both meanings of division and multiple representations. Both wrote very specific ideas for explaining to students.</td>
</tr>
<tr>
<td>1 — justified with flawed logic</td>
<td>Cite insights from class discussions and investigated in Web site and with other people whose answers were different from their own.</td>
<td>Both reported insights into correct answer and ways of justifying using more than one explanation that draw on both meanings of division and multiple representations. Both wrote very specific ideas for explaining to students.</td>
</tr>
<tr>
<td>Correct Answers - 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 — Justified with rule “teacher told me.”</td>
<td>Did not investigate further.</td>
<td>Reported no insights into ways of explaining what it means to divide by 0. Made no attempt to construct an explanation for students.</td>
</tr>
<tr>
<td>3 — Justified with rule but attempted to construct a reasoned explanation by thinking about limits, slopes, or graphing—then gave up.</td>
<td>Cited insights from their group discussions.</td>
<td>Reported insights into their difficulties with division by 0. Explained via multiplication as inverse of division and with pattern of divisors approaching 0. Reported gaining insight into difference between 5/0 and 0/0 and some of the difficulties with “language” and “meanings.” State they would use multiple examples and explanations to explain to students.</td>
</tr>
<tr>
<td>2 — Justified with rules and made no attempt to construct a reasoned explanation.</td>
<td>Insights from class discussion and investigated with other people who answered 0 or undefined but could not justify other than with rules.</td>
<td>Elaborated original explanations and gained insight into others such as sharing in story context. One speculates difficulty is due to fact that “undefined” is not a typical answer to arithmetic problems. The other reported using several counter challenges to a subject’s 0/0 = 1.</td>
</tr>
<tr>
<td>4 — Although were not taught why, they reasoned and justified with one or two explanations.</td>
<td>Investigated in Web sites and with other people who had explanations that were different from their own.</td>
<td>Expanded and clarified original explanations. Gained insight into multiple explanations that drew on two meanings of division. Clarified previous misconceptions such as difference between 5/0 and 0/0, 2 noted they would explain difference between 0 ÷ 5 and 5 ÷ 0 to students. Two anticipated students’ confusion with their explanations and ways of responding.</td>
</tr>
</tbody>
</table>

Understanding and to then investigate beyond class seemed to challenge the participants to seek out and develop a richer understanding for their explanations. Even so this latter version of the task did not seem to work for everyone, and it did not help the preservice teachers attend to the many pedagogical aspects of their explanations. Note that the preservice teachers’ ideas for teaching this topic were untested (no one, for instance, sought to test their explanations with a real fifth grader) and not many considered students’ possible difficulties with various kinds of explanations. Many preservice teachers seemed to assume that students would unproblematically see or understand their explanations without accounting or preparing for potential difficulties.
roadblocks, as Angela did. No one, for example, entertained the possibility that for young students the word “undefined” might mean “not in the dictionary.”

These results point to the challenges teacher educators face when designing opportunities for preservice teachers to extend their mathematical understandings during teacher preparation courses. Although Ma (1999) proposed that teachers develop a profound understanding of elementary mathematics by doing mathematics themselves, learning from colleagues and examining curricular materials, our study suggests that providing such experiences during beginning teacher preparation is not that simple. Note that the two instructional versions of the division by 0 task used in this study included all of Ma’s proposed contexts for teacher learning. We, however, found that the structure and support for the division by 0 task and that the instructional designs that supported these tasks mattered. The results also suggest that one experience alone with these kinds of teacher preparation tasks is not enough. They point to the need to study the possible effects of multiple investigative experiences on the development of prospective teachers’ mathematical understandings and attitudes.

Finally, the results of this study also suggest the need to further investigate preservice teachers’ dispositions to justify and (or) investigate further their mathematical ideas and how these are related to their teaching of mathematics. Our findings suggest that preservice teachers with such a disposition may be well (and perhaps even better) suited to teaching mathematics in ways that promote students’ understanding. They indicate that preservice teachers’ ability to teach mathematics well is perhaps more related to the ways they react to and deal with the mathematics they do not understand than to the mathematics they do understand. Duckworth (1987) called these “the virtues of not knowing,” which she says is what matters most in teaching and learning. Ma (1999) called them “mathematical attitudes,” which determines how a teacher approaches a new topic or responds to new mathematical ideas. These elusive dispositions seem important to understand, both in terms of how these might be promoted and developed during and beyond teacher preparation.

References


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Appendix

Three Types of Students' Explanations Regarding Division By 0

Explaining with Examples.
In the sample explanation below, testing examples with divisors that get closer to 0 disproves the claim that division by 0 is 0. This reveals a pattern that shows the answer cannot be 0.

5 ÷ 0 ≠ 0 because when you divide 5 by numbers closer and closer to 0 the answer gets farther rather than closer to 0.

5 ÷ 10 = 0.5
5 ÷ 5 = 1
5 ÷ 2 = 2.5
5 ÷ 0.5 = 10
5 ÷ 0.25 = 20
5 ÷ 0.05 = 50

Explaining with Deductive Logic.
The next explanation is a deductive argument that leads to the conclusion that there is no number that satisfies the given premise.

If 5 ÷ 0 = 0, then 0 x 0 = 5 which we know is not true. This is true for any number a ≠ 0, so there is no number that multiplied by 0 will equal a. This means that there is no number that satisfies this conditions, and therefore the operation is undefined.

Explaining with Analogy.
The following explanation uses an analogy or alternative representation that helps see the impossibility of division by 0.

Let us consider division as a problem in repeated subtraction. That is, 6/0 can be regarded as a problem of asking how many times we must reach into a basket to empty it if it contains 6 eggs and if each time we reach in we are to remove 0 eggs. We are now asking how many 0s must be subtracted from 6 to get a remainder of 0. (Duncan, 1971)