A Robust Model of Bubbles with Multidimensional Uncertainty

Antonio Doblas-Madrid*
Department of Economics
Michigan State University
110 Marshall-Adams Hall
East Lansing, MI 48824
Phone (517) 355 83 20
Fax (517) 432 10 68
Email: doblasma@msu.edu

Abstract

Observers often interpret boom-bust episodes in asset markets as speculative frenzies where asymmetrically informed investors buy overvalued assets hoping to sell to a greater fool before the crash. Despite its intuitive appeal, however, this notion of speculative bubbles has proven difficult to reconcile with economic theory. Existing models have been criticized on the basis that they assume irrationality, that prices are somewhat unresponsive to sales, or that they depend on fragile, knife-edge restrictions. To address these issues, I construct a rational version of Abreu and Brunnermeier (2003), where agents invest growing endowments into an asset, fueling appreciation and eventual overvaluation. Riding bubbles is optimal as long as the growth rate of the bubble and the probability of selling before the crash are high enough. This probability increases with the amount of noise in the economy, as random short-term fluctuations make it difficult for agents to infer information from prices.

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1 Introduction

Over the last two decades, a series of dramatic boom-bust episodes in global asset markets have led many economists to question the long dominant efficient market hypothesis and to devote increasing attention to theories of asset price bubbles. Asset price bubbles are often referred to as speculative bubbles, a term that conjures the idea of a market timing game, in which investors buy overvalued assets hoping to sell to a greater fool before the crash. This idea comes up repeatedly, for example, in Kindleberger and Aliber (2005)’s famous chronicles of historical boom-bust episodes. There is also experimental work (Moinas and Pouget (2009)) documenting the emergence of bubbles in a design with asymmetrically informed participants who ride the bubble knowing that they may get ‘stuck’ with the asset at the end of the game. Despite its intuitive appeal, however, this notion of a speculative bubble has traditionally been difficult to reconcile with standard economic theory. As Tirole (1982) and Milgrom and Stokey (1982) show, bubbles are inconsistent with rational expectations equilibrium in a wide range of environments with finite numbers of rational agents, even under asymmetric information.

Some approaches that have been taken in order to circumvent these impossibility results include introducing some form of irrationality, assuming heterogeneous priors or marginal utilities, and assuming an infinite number of overlapping generations. For example, Harrison and Kreps (1978) and Scheinkman and Xiong (2003) consider agents who are ‘overconfident’ in the sense that they consider their own information to be superior to that of others, and fail to fully adjust their beliefs as they observe what others believe. In Abreu and Brunnermeier (2003), there are rational agents who ride the bubble—and make profits with a certain probability—along with behavioral agents who fuel bubble growth and who are doomed to suffer losses in the crash. In Allen et al. (1993) and Conlon (2004), agents are rational, but have either heterogenous priors, or heterogenous state-contingent marginal utilities that may give rise to gains from trade. This approach generates speculative bubbles, but has the drawback of relying on fragile, knife-edge parameter restrictions. Another strand of literature (Caballero and Krishnamurthy (2006), Fahri and Tirole (2009), and others) builds on Tirole’s (1985) work on rational bubbles with overlapping generations, where bubbles—like money in Samuelson (1958) —improve allocations by alleviating a shortage of stores of value. However, the bubbles in these models are
less reminiscent of speculation. Trades are typically driven by the lifecycle rather than beliefs, and the focus is often on steady states where bubbles grow slowly and never burst.\footnote{Further approaches to bubbles focus on agency problems (Allen and Gorton (1993), Allen and Gale (2000), Barlevy (2008)), solvency constraints (Kocherlakota (2008)), and others. For a survey, see Brunnermeier (2001).}

The aim of this paper is to contribute to the theory of speculation by developing a model of a greater fool’s bubble that circumvents some of the main critiques of previous studies. To this end, I construct a discrete-time version of Abreu and Brunnermeier (2003)—henceforth referred to as AB—where all agents are rational and prices reflect supply and demand at all times. The model inherits from AB the property of being robust to small changes in parameters, and is therefore not subject to the fragility critique of Allen et al. (1993) and Conlon (2004).

Following AB, I assume that rational agents hold a rapidly appreciating asset. For some time, rapid price growth is justified by fundamentals. However, a bubble emerges as appreciation continues to rise past the point where fundamental gains have been priced in. Asymmetric information is introduced in such way that, in equilibrium, rational agents (optimally) continue to buy the asset even after learning that it has become overvalued. At different times, different agents observe private ‘overvaluation’ signals revealing that a bubble has started to grow. The key uncertainty in the model is that agents do not know when others observe the signal. They do know, however, that if they are among those who observed the signal relatively early they can ride the bubble, sell before the crash and make profits. If the likelihood of being an ‘early-signal’ agent is high and the speed at which the bubble grows are high enough, investing in overvalued assets is optimal. To embed these ideas into a rational model, I depart from AB in the following ways.

In AB, bubble growth is fueled by behavioral agents who invest growing amounts into the risky asset and are willing to do so indefinitely. These agents are doomed to ‘get caught’ in the crash, which occurs when a critical mass of rational agents exit the market. By contrast, in this paper, rational agents themselves fuel bubble growth. I assume that agents receive growing endowments, which they invest in the bubble as long as they expect it to grow. Importantly, these endowments cannot be pledged as collateral, i.e., agents cannot borrow against their time-$t$ endowment at some earlier date $s < t$. Binding wealth constraints limit the amounts that agents can use to bid up the risky asset. Therefore, the price during the boom does not reflect agents’ estimates of the ultimate resale price, as it would in an interior solution. Instead, it reflects the maximum amount of resources the agents can invest into the bubble at each date. This also has
the important implication that the boom, instead of consisting of a one-time jump in the price, takes place gradually over time as investors access larger endowments.

A second new ingredient in the model is a preference shock, which forces a fraction $\theta_i$ of agents to sell for reasons—such as life events or liquidity needs—unrelated to price expectations. This ensures that a positive mass of shares is sold every period, even when nobody expects an imminent crash. Preference shocks also serve another function, adding noise to the economy. Because $\theta_i$ is subject to random variability, prices are noisy. If the variability of $\theta_i$ is high enough, prices can ‘hide’ sales, as late-signal agents cannot distinguish whether a price slowdown is due to sales by early-signal agents or a high realization of $\theta_i$. In other words, the likelihood that an agent can sell before the crash tends to increase with the variability of $\theta_i$.

To solve the model, I first consider the case with so little noise that, as soon as one type sells (a type includes all those who observe the overvaluation signal in the same period), all uncertainty is revealed, triggering a crash in the next period. Agents trade in an asset market modeled as a Shapley-Shubik trading post, where they submit orders to buy or sell in a first stage, and the price emerges once all orders are combined later in a second stage. In this ‘case without noise’, the effect of the first type’s sales on the price is always larger than the effect of any possible random fluctuation in $\theta_i$. I show that, in this effectively noiseless environment, the no-bubble equilibrium in which agents sell as soon as they observe the signal always exists. The no-bubble equilibrium is unique if $G/R$, the growth rate of the bubble net of the risk-free rate, is below a threshold $\Gamma$, where $\Gamma$ is a function of $\lambda$, the parameter governing the relative likelihoods of early versus late ‘overvaluation’ signals. As $G/R$ rises above $\Gamma$, the set of equilibria that can be supported expands to include, in addition to the no-bubble equilibrium, equilibria with bubbles. The duration of the bubbles ranges from zero to a maximum that is a function of $G/R$ and $\lambda$.

I continue the analysis by increasing the amount of noise so that it can conceal sales of one type, but not more. This allows multiple types to sell before the crash, since sales of the first can be confused with noise, and thus may fail to burst the bubble. In other words, while prices do reflect selling pressure monotonically, they reveal information only imperfectly. This helps

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2 Note that the preference shock is not a substitute for irrational agents, since it does not force agents to stay in the market during the crash. On the contrary, it forces some agents to sell before they otherwise would.
address another critique of AB, where it is assumed that, for a nontrivial time interval, a growing mass of agents are gradually leaving the market, but the price nevertheless continues to grow at the same rate as if nobody was selling. Under the assumption that noise can hide sales by one type, but not two, prices fall into one of three categories. High prices reveal with certainty that nobody has sold, medium prices reveal that sales may or may not have begun, and low prices reveal with certainty that sales have begun, thereby triggering the crash. If the number of types is large, the analysis of equilibria with Markov strategies is simple enough to be analytically tractable. The strategies I consider are Markovian, in the sense that agents’ sell-or-wait choices depend only on how much time has passed since observing the signal and on whether the last price observed was high, medium, or low. Restricting attention to this class of strategies, I show that there are two key ways in which noise helps generate bubbles. First, in the noisy case, it is possible to rule out equilibria without bubbles for high enough \( G/R \). Second, there exist parameters such that, with noise, (arbitrarily) long bubbles may arise, even if \( G/R \) is below \( \Gamma \). In other words, there exist parameters such that, in the noiseless case the only equilibrium is the one without bubbles, while in the noisy case, arbitrarily long bubbles may arise. Finally, I relax the assumption—made in the basic analysis for simplicity—that agents cannot reenter the market after selling, and show that, although some equilibria vanish, the overall picture remains unchanged, and bubbles with Markovian strategies still arise.

The paper is organized as follows. In sections 2 and 3, respectively, I describe the model and define equilibrium. In section 4, I illustrate how bubbles arise in the basic analysis. In section 5, I consider the extension where agents may reenter the market. Section 6 concludes.

2 The Model

2.1 The Environment

Time is discrete and infinite with periods labeled \( t = \ldots, -1, 0, 1, \ldots \). There are two assets, a risk-free asset with exogenous gross return \( R > 1 \), and a risky asset. The supply of the risky asset is fixed at 1, and its price at time \( t \) is \( p_t \) units of the risk-free asset. At any time, the risk-free asset can be turned into consumption at a one to one rate.

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3 AB do conjecture, in a remark, that adding noise to the price process would allow prices to respond to supply and demand at all times without revealing all private information.
While $t \leq 0$, the risky asset’s fundamental value $f_t$ and the price $p_t$ are equal and given, in expected value, by $\alpha R^t$, where $\alpha > 0$. Starting at $t = 1$, fundamental shocks cause $f_t$ to grow, on average, at the faster rate $G > R$. Both $f_t$ and $p_t$ grow on average at the faster rate $G$ until $t = t_0 - 1$. But starting at time $t_0 \geq 1$, the average $f_t / f_{t-1}$ falls back to $R$, and if $p_t$ continues to grow faster than $R$, a bubble arises. As in AB, the increase in fundamental value does is not due to an increase in current dividends (set equal to zero for convenience), but instead it reflects improving prospects about dividends to be paid in a distant future. The bubble inflates until period $T \geq t_0$ and bursts in period $T + 1$, at which point equality between price and fundamental value is restored. Thus, as in AB, bubbles arise as markets overreact to developments that are at first fundamental in nature. The first period of overvaluation $t_0$ is geometrically distributed with probability function $\varphi$ given by

$$\varphi(t_0) = (e^\lambda - 1)e^{-\lambda t_0} \quad \text{for all } t_0 = 1, 2, \ldots,$$

where $\lambda > 0$. The expected value of $t_0$ is given by $1/(1 - e^{-\lambda})$.

There is a unit mass of rational agents indexed by $i \in [0, 1]$. They do not observe $t_0$ perfectly. Instead, every period from $t_0$ to $t_0 + N - 1$, a mass $1/N$ of them observe a signal revealing that the risky asset is overvalued, i.e., that $f_t$ is no longer growing at the rate $G$. Signals give rise to $N$ types, $n = t_0, \ldots, t_0 + N - 1$. More formally, $\psi : [0, 1] \to \{t_0, \ldots, t_0 + N - 1\}$ assigns a type to each agent, where $\psi(i) = n$ denotes that agent $i$ is of type $n$, or in other words, that agent $i$ observes the signal at time $n$. As in AB, agents observe $n$, but not $t_0$. If an agent observes her signal at time $n$, she knows that $t_0$ may have been as early as $n - (N - 1)$, or as late as $n$. (Except for the special case with $t_0 < N$, where types with $n < N$ know that $t_0$ must be greater than $n - (N - 1)$, since $n - (N - 1) \leq 0$.) Conditional on $n$, the distribution of $t_0$ becomes

$$\varphi(t_0 | n) = \begin{cases} e^{-\lambda t_0} / \left( e^{-\lambda \max\{1, n-(N-1)\}} + \cdots + e^{-\lambda n} \right) & \text{if } \max\{1, n-(N-1)\} \leq t_0 \leq n \\ 0 & \text{otherwise.} \end{cases}$$

According to Kindleberger and Aliber (2005), bubbles typically follow major fundamental displacements, which cause large shifts in prices. Price movements that are justified by fundamentals for some time turn into bubbles if markets overshoot. In keeping with this idea, AB mention episodes in stock markets after the arrival of new technologies (e.g., the Internet in the 1990s, the radio in the 1920s) as examples of bubbles.
In words, sequential arrival of signals places agents along a line, but agents are uncertain about their relative order in the line. This plays a key role in generating bubbles, as all agents—even those late in the line—assign the same positive probability to the event that they could be early in the line. As we will see in Section 4, in equilibrium, signals are the key reference points on which agents condition their selling strategies. In the absence of noise, type-$n$ agents will plan to ride the bubble for $\tau_0^*$ periods and sell at time $n + \tau_0^*$. When prices are noisy, strategies will be augmented to allow agents to wait longer if they observe higher prices.

Figure 1 — Timeline of events.

Figure 1 summarizes the assumptions made thus far. The boom starting at $t = 1$ is at first fundamental, but turns into a bubble at the imperfectly observed time $t_0$, with signals arriving at $t = t_0, \ldots, t_0 + N - 1$. Bubble duration $T - t_0$ will be endogenously determined in equilibrium.

Preferences are characterized by risk neutrality and preference shocks à la Diamond and Dybvig (1983), which force agents to liquidate assets and consume. At time $t$, a randomly chosen mass $\theta_t \in (0,1)$ of agents are hit by a shock that sets their discount factor $\delta_{t,t}$ equal to zero. The remaining mass $1 - \theta_t$ have $\delta_{t,t} = 1 / R$. Agent $i$’s expected utility is defined as

$$E_{i,t} \left[ U\left( \left\{ c_{i,t} \right\}_{t=1}^{\infty} \right) \right] = E_{i,t} \left[ c_{i,t} + \sum_{\tau=t+1}^{\infty} \left( \prod_{s=t}^{\tau-1} \delta_{s,t} \right) c_{i,s} \right],$$

where $c_{i,t}$ denotes agent $i$’s time-$t$ consumption, $U$ denotes utility, and $E_{i,t}$ expectation given information available to agent $i$ in period $t$. This information includes whether $\delta_{i,t}$ is zero, in
which case (3) reduces to \( E_{t,t} C_{t,t} \). Preference shocks are i.i.d., and thus, the probability that \( \delta_{t,t} = 0 \) does not depend on past values \( \ldots, \delta_{t-2,t}, \delta_{t-1,t} \). Shocks are also type-independent, which means that for all \( t \), and within any type, the fraction of agents hit by the shock is \( \theta_t \).

Since \( \theta_t \) is unobservable, agent \( i \) knows whether she has been hit by the shock, but not how many agents have been hit. Moreover, \( \theta_t \) varies over time as follows

\[
\theta_t = \bar{\theta} + \varepsilon_t, \tag{4}
\]

where \( \bar{\theta} \in (0,1) \) is a constant and \( \varepsilon_t \) an i.i.d. random variable which is uniformly distributed over \([ -\bar{\varepsilon}, \bar{\varepsilon} ]\), with \( 0 < \bar{\varepsilon} < \min\{\bar{\theta}, 1-\bar{\theta}\} \). The term \( \varepsilon_t \) serves an important function in the model by generating random price fluctuations. If \( \theta_t \) is constant, as soon as the first agents sell in anticipation of the crash, the price reveals these sales, precipitating a crash. In a noisy environment, by contrast, agents cannot distinguish whether a price deceleration is due to a high \( \varepsilon_t \) or the start of the crash. It is important to note that the role of preference shocks is precisely to generate a positive and noisy amount of sales. The role of the shock is not to make speculation a positive sum game by forcing some agents to stay in the market and get caught in the crash. On the contrary, the shock saves some agents from the crash by forcing them to sell.

The boom is fueled by agents investing endowments into the risky asset. Every period, agents receive \( e_t > 0 \) units of the risk-free asset. As long as they do not anticipate an impending crash and are not hit by the shock, they invest the endowment into the risky asset. Endowments cannot be capitalized, i.e., an agent receiving \( e_t \) at \( t \) cannot borrow against it at earlier dates \( s < t \). After time 0, endowment growth accelerates as follows:

\[
e_t = \begin{cases} R^t & \text{if } t \leq 0 \\ G^t & \text{if } t > 0. \end{cases} \tag{5}
\]

\( \text{Endowment growth captures the idea that, as a bubble grows the availability of funds that can be invested into it also grows. Two plausible interpretations of this increasing resource availability can be found in Kindleberger and Aliber (2005), who view the expansion of credit and the arrival of new investors as typical sources of 'bubble fuel'. Regarding the first interpretation, it is easy to see how bubbles and credit can reinforce each other. If the risky asset can be pledged as collateral, price growth loosens credit constraints, allowing investors to borrow more and bid prices even higher. The other suggested source of new funds, the gradual arrival of investors, may reflect liberalization of capital flows into a country or industry. Alternatively, it may reflect technological factors that prevent agents from investing all their wealth into the risky asset at time 1. For example, if some investments take time to mature, in the short run some wealth is 'tied up' in projects that can only be liquidated at a loss. In such a setting, funds would become progressively available as projects matured.} \)
Three remarks are in order. First, the assumption that $e_t$ grows at the rate $G$ forever shall not be interpreted literally. In the long run, endowment growth must eventually slow down. Limits to endowment growth are not modeled, however, because the focus of the paper is on endogenous crashes, where agents’ sales burst the bubble before its growth decelerates for exogenous reasons.\(^6\) Second, agents’ inability to borrow against future endowments is a key difference between this environment, and for example, Tirole (1982). In this model, prices during the boom do not reveal agents’ expectations of the ultimate resale price. Instead, prices reflect maximum resources available to agents each period. Another crucial implication of growing endowments and borrowing constraints is that prices grow gradually instead of posting a one-time jump. AB also assume that rational agents are constraint, or fully invested into the bubble. However, in AB the inflow of funds that fuels bubble growth is ‘dumb money’ from behavioral agents who are doomed to suffer losses. By contrast, in this model rational agents continue investing into the bubble only as long as it is optimal to do so. The third remark is that an alternative specification with a constant $\theta_t$ and a noisy aggregate endowment would also generate price fluctuations, although it would not generate fluctuations in trading volume.

The within-period timing of shocks and actions is as follows. Agent $i$ starts period $t$ with nonnegative holdings $b_{i,t}$ and $h_{i,t}$ of the risk-free and risky assets, respectively. The period proceeds in two steps. In Step 1, agent $i$ receives $e_t$, learns whether $\delta_{i,t}$ is zero or $1/R$, and, if $\nu(i) = t$, observes her signal. Also in Step 1, agent $i$—knowing $\delta_{i,t}$, $p^{t-1} = \{\ldots, p_{t-2}, p_{t-1}\}$, and if $\nu(i) \leq t$, the signal—chooses actions $a_{i,t} = (m_{i,t}, s_{i,t}, \chi_{i,t})$. The pair $(m_{i,t}, s_{i,t})$ captures the agent’s asset market choices, while $\chi_{i,t} \in [0,1]$ captures her consumption choice. (Although consumption takes place in Step 2, the decision whether to consume or not depends only on the preference shock, which is realized in Step 1.) In the asset market—which is modeled as a Shapley-Shubik trading post—agent $i$ bids $m_{i,t}$ units of the risk-free asset and offers $s_{i,t}$ shares of the risky asset for sale. Due to short sales constraints, agents’ choices must satisfy

$$0 \leq m_{i,t} \leq b_{i,t} + e_t \quad (6)$$

and

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\(^6\) A similar issue arises in AB, where behavioral agents are assumed able to purchase a given number of shares of the risky asset no matter how high the price becomes, but there is also an exogenous cap on bubble duration.
Agent $i$ chooses $(m_{i,t}, s_{i,t})$ before knowing the price $p_t$, which will be determined in Step 2 when all bids and offers are combined.\(^7\) Preference shocks and risk neutrality greatly simplify agents’ choices. Agents with $\delta_{i,t} = 0$ sell everything to consume as much as possible in Step 2, i.e., they set $(m_{i,t}, s_{i,t}) = (0, h_{i,t})$. Agents with $\delta_{i,t} = 1/R$ set $(m_{i,t}, s_{i,t}) = (0, h_{i,t})$ if they expect the risky asset’s return $p_{t+1}/p_t$ to fall below $R$; they invest as much as they can into the risky asset, setting $(m_{i,t}, s_{i,t}) = (b_{i,t} + e_t, 0)$ if this expected return exceeds $R$, and are indifferent between any linear combination of these two actions in the knife-edge case.\(^8\) Agent $i$ comes out of the asset market holding

$$h_{i,t+1} = h_{i,t} + \frac{m_{i,t}}{p_t} - s_{i,t}$$

and

$$\tilde{b}_{i,t} = b_{i,t} + e_t - m_{i,t} + p_t s_{i,t},$$

where $\tilde{b}_{i,t}$ denotes agent $i$’s within-period, or interim, risk-free asset holdings.

In Step 2, bids and offers are combined and the price is determined by market clearing

$$\int_{i \in [0,1]} h_{i,t} \, di = 1. \tag{10}$$

Substituting (8) into this expression and solving for the price yields

$$p_t = \frac{M_t}{S_t}, \tag{11}$$

where for all $t$,

$$M_t \equiv \int_{i \in [0,1]} m_{i,t} \, di \quad \text{and} \quad S_t \equiv \int_{i \in [0,1]} s_{i,t} \, di. \tag{12}$$

Since there is always a positive measure of shock-induced sellers, $S_t$ is always positive and (11) is well defined. Finally, agent $i$ consumes a fraction $\chi_{i,t} \in [0,1]$ of $\tilde{b}_{i,t}$.

\(^7\) The assumption that agents submit orders before knowing others’ orders or the price is similar to Kyle (1985), and also to models à la Cournot. In microstructure terms, agents are placing market orders, which they know will be executed, but they do not know at what price.

\(^8\) Since all individuals within a type observe the same signal and prices, they compute the same expected return for the risky asset. Combined with risk neutrality, this implies that, although they hold different portfolios due to different shock realization histories, they invest/disinvest into the risky asset in lockstep.
\[ c_{i,j} = \chi_{i,j} \tilde{b}_{i,j}, \]  

and saves the rest, so that next-period’s risk-free asset holdings \( b_{i,t+1} \) are given by

\[ b_{i,t+1} = R(1 - \chi_{i,j}) \tilde{b}_{i,j}. \]

Figure 2 summarizes within-period timing.

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**Figure 2 — Within-period timing.**

Having described market clearing, we can now fill in details about the pre-boom, boom and post-crash phases. But before proceeding, it will be useful to assume that—even if the boom is not anticipated—the risky asset is valuable enough to absorb agents’ entire wealth in the pre-boom phase. This makes it optimal for agents to hold \( b_{i,j} = 0 \) while \( t \leq 0 \), which in turn implies that at \( t = 1 \), the price growth rate increases, but there is no additional one-time jump in price.\(^9\)

With this assumption in place, consider now a pre-boom period \( t \leq 0 \), and let agents start with \( b_{i,j} = 0 \) units of the risk-free asset. Preference shocks force a mass \( \theta_j \) of agents to sell \( S_j = \theta_j \) shares of the risky asset, while all other agents use their endowments to bid for these shares. Since \( b_{i,j} \) is zero, \( M_{i,t} = (1 - \theta_j)e_{i,j} \), and thus

\(^9\) If the risky asset during the pre-boom phase is not valuable enough to absorb all of the agents’ wealth, agents accumulate shares of the risk-free asset before the boom starts. They pour these holdings into the risky asset once the boom begins, causing a price jump at time 1, followed by some time where the price grows at the rate \( R \). Once holdings of risk-free asset reach zero, wealth constraints become binding again, and the price grows at the rate \( G \). As long as the time it takes for the risk-free holdings to reach zero is not too long, relative to the expected duration of the boom, it is possible to incorporate these additional dynamics into the analysis without affecting results.
\[ p_t = \left( [\theta_t]^{-1} - 1 \right) e_t. \] (15)

Since the expected \( p_{t+1} / p_t \) is \( R \), agents who are not hit by the shock find it (weakly) optimal to continue investing only in the risky asset, letting \( b_{i,t+1} = 0 \). Agents who are hit by the shock consume \( c_{i,t} = \tilde{b}_{i,t} = e_t + p_t h_{i,t} \) and save nothing, setting \( (b_{i,t+1}, h_{i,t+1}) = (0,0) \). Given (15) and the assumption that \( p_t \approx \alpha e_t \) in the pre-boom and boom phases, it must be that

\[
\alpha = E \left[ \frac{1}{\theta_t} - 1 \right] = \frac{1}{2 \bar{\epsilon}} \int_{-\bar{\epsilon}}^{\bar{\epsilon}} \left[ \frac{1}{\bar{\theta} + \epsilon_t} - 1 \right] d\epsilon_t = \frac{\ln((\bar{\theta} + \bar{\epsilon})/(\bar{\theta} - \bar{\epsilon}))}{2 \bar{\epsilon}} - 1. \] (16)

At \( t = 1 \), endowment and price growth accelerate. For a while, the only sales are those forced by shocks, those who are not forced to sell remain fully invested in the risky asset, and \( p_t \) is given by (15) with \( e_t = G' \). Since shocks are type-independent, in the aggregate each type holds \( h_{n,t} = 1/N \) shares of the risky asset, where for all \( n \in \{t_0, \ldots, t_0 + N - 1\} \) and for all \( t \),

\[ h_{n,t} = \int_{\{i : \epsilon(i) = n\}} h_{i,t} di. \] (17)

All \( N \) types hold \( h_{n,t} = 1/N \) shares until the last few periods of the boom, when some start to sell in anticipation of the crash. When the first \( z_t > 0 \) types sell at \( t \), the number of shares for sale becomes \( S_t = z_t / N + \theta_t (1 - z_t / N) \), where a mass \( z_t / N \) of agents sell anticipating a crash and a mass \( \theta_t (1 - z_t / N) \) of agents sell forced by preference shocks. Aggregate bids \( M_t \) amount to \( (1 - \theta_t)(1 - z_t / N) G' \), as only agents who are not hit by the shock and are not of the exiting types wish to buy. Consequently, the price becomes

\[ p_t = \left( \frac{z_t}{N} + \theta_t \left(1 - \frac{z_t}{N}\right) \right) - 1)G'. \] (18)

After trade, \( h_{n,t+1} \) is 0 for the \( z_t \) types that have sold and \( 1/(1 - z_t / N) \) for other types. Agents hit by the shock consume the proceeds from selling the risky asset. Those who sell without being forced by the shock store their wealth in the risk-free asset, setting \( b_{i,t+1} = R \tilde{h}_{i,t} \). The likelihood that \( p_t \) reveals the exit of these \( z_t \) types depends on the relative magnitudes of \( \bar{\epsilon} \) and \( z_t / N \). If \( \bar{\epsilon} < (1 - \bar{\theta}) z_t / (2N - z_t) \), sales will surely be revealed, as \( ((\bar{\theta} + \bar{\epsilon})^{-1} - 1)G' \), the lowest possible
price if \( z_t = 0 \) exceeds \( ((z_t / N + (\bar{\theta} - \bar{\epsilon})(1 - z_t / N))^{-1} - 1)G' \), the highest possible price if \( z_t \) types have sold. However, if \( \bar{\epsilon} \geq (1 - \bar{\theta})z_t / (2N - z_t) \), \( p_t \) may be greater or equal than \( ((\bar{\theta} + \bar{\epsilon})^{-1} - 1))G' \), in which case the bubble will continue until period \( t+1 \).

If the bubble survives period \( t \) and another \( z_{t+1} \geq 0 \) types sell at \( t+1 \), the aggregate bid becomes \( M_{t+1} = (1 - \theta_{t+1})(1 - (z_t + z_{t+1}) / N)G^{t+1} \). On the selling side, \( z_{t+1} / N \) sellers anticipate a crash and \( \theta_{t+1}(1 - (z_t + z_{t+1}) / N) \) sellers are strictly shock-induced. Since risky-asset holdings across sellers average \( 1/(1 - z_t / N) \), the total mass of shares for sale equals

\[
S_{t+1} = \left[1 - \frac{z_t}{N}\right]^{-1}\left(\frac{z_{t+1}}{N} + \theta_{t+1}\left(1 - \frac{z_t + z_{t+1}}{N}\right)\right).
\]

Rearranging terms, the equilibrium price can be written as

\[
p_{t+1} = \left(1 - \frac{z_t}{N}\right)\left[\frac{1 - \frac{z_t}{N}}{\frac{z_{t+1}}{N} + \theta_{t+1}\left(1 - \frac{z_t + z_{t+1}}{N}\right)} - 1\right]G^{t+1}.
\]

The likelihood that \( p_{t+1} / G^{t+1} \) falls below \( ((\bar{\theta} + \bar{\epsilon})^{-1} - 1)) \) now depends on \( \bar{\epsilon} \), \( z_t \) and \( z_{t+1} \). If \( p_{t+1} / G^{t+1} \) falls below this threshold, sales will be revealed, causing a crash. Otherwise, the bubble will last until time \( t+2 \) or later. Equation (19) can be generalized to allow sales over more than two periods. However, since in the equilibria analyzed later, sales burst the bubble in one or two periods, (19) lays all the groundwork necessary for our purposes.

The post-crash phase starts at time \( T+1 \), where \( T \) is the first period in which \( p_t / G' \) falls below \( ((\bar{\theta} + \bar{\epsilon})^{-1} - 1)) \). Assuming that \( t_0 \) is revealed at \( T \), the expected fundamental value \( \alpha(G' / R)^{y-1} R' \) also becomes known. From time \( T+1 \) onward, agents who are not hit by the

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10 The market clearing condition is different if \( z_{t+1} \) is negative (i.e., if some types reenter the market after selling). I restrict attention to \( z_{t+1} \geq 0 \) because there are no equilibria with reentry in the analysis that follows.

11 If sales start at \( t \geq 0 \) and \( z_t, \ldots, z_{t+h} \geq 0 \) types sell at times \( t, \ldots, t+h \), the price \( p_{t+h} \) is given by (19), replacing \((\theta_{t+h}, G^{t+h}, z_{t+h})\) with \((\theta_{t+h}, G^{t+h}, z_{t+h})\) and \( z_t \) with \( z_t \).

12 In the equilibria presented later, prices \( p_t, \ldots, p_T \) often reveal \( t_0 \) exactly. However, in some instances, this will hold only approximately, and a few values of \( t_0 \) will be consistent with prices. Nevertheless, to avoid burdening the reader with inessential complications, I will assume that \( t_0 \) is exactly revealed. Generalizing the fundamental-value formula to take the latter cases into account adds complications in exchange for little or no insight.
shock invest a decreasing fraction \( (R / G)^{-(t_0-1)} \) of their endowments in the risky asset, and the rest in the risk-free asset. This choice is weakly optimal since, throughout the post-crash phase, the expected ratio \( p_{t+1} / p_t \) equals \( R \). Moreover, equality between \( p_t \) and \( f_t \) is preserved.

\[ \frac{\Delta}{t} \]

### 3 Equilibrium

I next define equilibrium under the restriction—to be relaxed in Section 5—that once a type has sold in anticipation of the crash, agents of that type stay out of the market until the bubble bursts. (Note that this does not preclude agents who are forced to sell by shocks from investing their endowments in the risky asset in later periods.)

**Restriction I - No Reentry:** For any \( i \) and any \( t \leq T \), if \( b_{it} > 0 \), \( h_{it} = 0 \), \( \forall \tau \in \{t + 1, \ldots, T\} \).

The equilibrium concept is Perfect Bayesian Equilibrium (PBE), consisting of strategies and beliefs \( \{a_i, \mu_i\}_{i \in [0,1]} \). Agent \( i \)'s strategy \( a_i \) is a sequence \( \{a_{i,t}\}_{t \in \mathbb{Z}} \), where \( a_{i,t} \) is a triplet \((m_{i,t}, s_{i,t}, \zeta_{i,t})\). Agent \( i \)'s belief \( \mu_{i,t}(t_0) \) is a probability distribution over values of \( t_0 \). Both \( a_{i,t} \) and \( \mu_{i,t}(t_0) \) are contingent on information available to agent \( i \) in Step 1 of date \( t \). This includes the discount factor \( \delta_{i,t} \), past prices \( p^{t-1} = \{p_{t-2}, p_{t-1}\} \), and if \( \nu(i) \geq t \), the signal \( \nu(i) \). Since \( \delta_{i,t} \) does not inform about \( t_0 \), and all agents within a type observe the same prices and signal, they have the same common belief, defined as \( \mu_{i,t}(t_0) \equiv \mu_{i,t}(t_0) \) for all \( i \) with \( \nu(i) = n \).

In equilibrium, for all \( i \), \( a_{i,t} \) is optimal given agent \( i \)'s shock realization \( \delta_{i,t} \) and the belief \( \mu_{i,t}(t_0) \), and \( \mu_{i,t}(t_0) \) is consistent with the equilibrium strategy profile. To be consistent with a strategy profile, a belief \( \mu_{i,t}(t_0) \) must assign positive probability only to values of \( t_0 \) that are not ruled out by strategies, given past prices and if \( t \geq \nu(i) \), the signal. The set of values of \( t_0 \) that are not ruled out is the support of \( t_0 \), denoted by \( \text{supp}_{i,t}(t_0) \). Since beliefs are the same for all agents within a type, we can define \( \text{supp}_{i,t}(t_0) \equiv \text{supp}_{i,t}(t_0) \) for all \( i \) with \( \nu(i) = n \). To see how \( \text{supp}_{i,t}(t_0) \) evolves in equilibrium, recall that the signal \( n \) implies that \( \text{supp}_{i,t}(t_0) \subseteq \{\max\{1, n - (N - 1)\}, \ldots, n\} \). Moreover, prices \( p^{t-1} \) and strategies rule out values of
$t_0$ as follows. If $t_0$ takes on the value $\tau_0$, given the price history $p^{t-1}$, there are—discount-factor contingent—implied values of $a_{i,\tau}$ for all $i$ and all $\tau < t$. These implied actions and the price $p_{\tau}$ can be substituted into (19) to compute the implied $\varepsilon_{\tau}$. The value $\tau_0$ is excluded from $\text{supp}_{a,\tau}(t_0)$ if it implies $|\varepsilon_{\tau}| > \varepsilon$ for some $\tau$. After discarding all the values of $t_0$ that are ruled out by this process, the probabilities that $\mu_{a,\tau}(t_0)$ assigns to each remaining values in $\text{supp}_{a,\tau}(t_0)$ are obtained using Bayes’ rule as follows:

$$
\mu_{a,\tau}(t_0) = \sum_{\tau_0 \in \text{supp}_{a,\tau}(t_0)} \frac{\varphi(t_0)}{\varphi(\tau_0)}.
$$

With equilibrium beliefs embedded in the expectations operator $E_{i,\tau}$, the equilibrium strategy $a_{i,\tau}$ solves the following recursive problem for all agents and at all times

$$
V(b_{i,\tau}, h_{i,\tau}) = \max_{m_{i,\tau}, x_{i,\tau}, \xi_{i,\tau}} E_{i,\tau}[c_{i,\tau}] + \delta_{i,\tau}E_{i,\tau}[V(h_{i,\tau+1}, h_{i,\tau+1})],
$$

subject to (6)-(9), (13), (14), and Restriction I. As previously stated, preference shocks and risk neutrality greatly simplify the program’s solution. Agents hit by the shock set $a_{i,\tau} = (0, h_{i,\tau}, 1)$, i.e., sell and consume everything. Agents with $\delta_{i,\tau} = 1/\delta$ set $\chi_{i,\tau} = 0$ and, depending on whether $E_{i,\tau}[p_{\tau+1} / p_{\tau}]$ is above, below, or equal to $\delta$, they are, respectively, fully invested in the risky asset, fully invested in the risk-free asset, or indifferent between any mix of the two.

## 4 Equilibria with Bubbles: Basic Analysis

For ease of exposition, I will carry out the analysis in Sections 4 and 5 assuming that $N$ is large, which implies that throughout the boom—including the last few periods—the price $p$, approximates $\alpha G'$. That is, I assume that price fluctuations matter because of their informational content, but are otherwise too small to have any sizable revenue effects. This assumption keeps

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13 In a more general expression, $\varphi$ would be multiplied by the likelihood of observed prices for each value of $t_0$. But in (20) this likelihood is simplified away, because in the coming analysis, it is equal for all values in $\text{supp}_{a,\tau}(t_0)$. This does hold exactly in most instances, although in a few cases it is only true under an approximating assumption. In Appendix D, I show how the analysis is modified when that assumption is removed.
formulas simple enough to convey intuition and tractable enough to allow for analytical
equilibrium characterization. In Appendix D, I derive the formulas for general values of $N$.

I will begin the analysis in subsection 4.1 with the case in which $\bar{\epsilon} < (1 - \bar{\theta}) / (2N - 1)$. In this case, which for brevity I will refer to as the noiseless case, the price is certain to reveal sales as soon as one type ($z_i = 1$) exits the market. I examine the possibility of bubbly equilibria with simple trigger strategies akin to those studied by AB. These strategies dictate that after observing the signal at $t = \nu(i)$, agent $i$ shall—unless forced to sell by the preference shock—ride the bubble for $\tau_0^*$ periods and sell at $t = \nu(i) + \tau_0^*$. Although noise cannot hide sales of the first type, the discreteness of the model, together with the within-period timing, allows a mass $1/N$ of agents to succeed in riding the bubble. Since there is positive probability of being among these sellers, there are equilibria with bubbles if $G/R$ is high enough. More precisely, in Proposition 1, I will show that to support equilibria with $\tau_0^* = 1$, $G/R$ must surpass a threshold $\Gamma = \Gamma(\lambda)$. More generally, I derive an (increasing) relationship between the highest $\tau_0^*$ that can be supported and $G/R$. The Proposition also establishes that $\tau_0^* = 0$ is always an equilibrium, no matter how high we set $G/R$. This is because, if an agent knows that others of her same type are selling, she knows that sales will be revealed for sure, and therefore sells.

In subsection 4.2, I increase the amount of noise so that it can hide sales by one type. In this case, which I will refer to as the noisy case, the inference from prices is often ambiguous, with investors unable to distinguish the beginning of the crash from a temporary price dip due to noise. For tractability, I restrict attention to the case where noise can hide sales of one type, but not two, so that prices can be categorized as high, medium, or low. High prices reveal that no types have left the market, medium prices are consistent with either no sales or with sales by one type, and low prices reveal with certainty that sales have begun. While restrictive, this assumption simplifies the analysis and makes it possible to focus on Markovian strategies, which condition behavior—besides the signal and preference shock—only on the most recent price. In

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14 In AB, since time and types are continuous, in the benchmark case, sales begin gradually and the mass of agents selling at any given instant is zero. If the price reflected sales, any positive mass of sales would be detected, and thus the probability of selling before the crash would be zero. To avoid this, AB assume that as long as sales do not surpass a threshold $\kappa$, the price simply does not reveal these sales. The price continues to grow as if sales had not started, and it reacts only when total sales reach a threshold $\kappa$. As far as the ability of agents to exit the market is concerned, assuming discrete periods and types is in a sense similar to having $\kappa = 1/N$ in a continuous model.
Proposition 2, I establish conditions under which equilibrium without bubbles can be ruled out, and conditions under which there exist equilibria with long bubbles. Finally, in Proposition 3, I contrast Propositions 1 and 2 to compare bubble formation with and without noise.

4.1 The Noiseless Case

In the noiseless case, the amount of noise $\varepsilon$ is so low that, as soon as the first type sells, the price is certain to reveal these sales, as $([\bar{\theta} + \varepsilon]^{-1} - 1)G'$, the lowest possible price while all types are still in the market, exceeds $([1/ N + (\bar{\theta} - \varepsilon)(1-1/ N)]^{-1} - 1)G'$, the highest possible price when one type sells. That is, in the noiseless case $\varepsilon$ is below a threshold $\varepsilon_{0,1}$ given by

$$\varepsilon_{0,1} \equiv (1 - \bar{\theta}) / (2 N - 1),$$

where the subscript 0,1 denotes zero types selling before time $t$ and one type selling at $t$.

The strategy of agent $i$ is the following. If hit by the shock, she sells and consumes. Otherwise, she does not consume. Pre-crash, she invests into the bubble before period $\nu(i) + \tau_0^*$ and exits the market at time $\nu(i) + \tau_0^*$. Post-crash, she invests a fraction $(R / G)^{-(6 - 1)}$ of her endowment into the risky asset and the rest into the risk-free asset. Put more formally, we have:

**Strategy Profile 1:** For any $i \in [0,1]$, agent $i$ follows $\{a_{i,t}\}_{t \in \mathbb{Z}} = \{m_{i,t}, s_{i,t}, \chi_{i,t}\}_{t \in \mathbb{Z}}$ given by:

- If $\delta_{i,t} = 0$, $a_{i,t} = (0, h_{i,t}, 1)$ for any $t$.
- If $\delta_{i,t} = 1 / R$, $\chi_{i,t} = 0$ for all $t$, and the choice of $(m_{i,t}, s_{i,t})$, is as follows:
  - If $t \leq T$,
    $$\begin{cases} (m_{i,t}, s_{i,t}) = (h_{i,t} + e_i, 0) & \text{if } t < \nu(i) + \tau_0^* \\ (0, h_{i,t}) & \text{if } t \geq \nu(i) + \tau_0^* \end{cases}$$
  - with $\tau_0^* \geq 0$.
  - If $t \geq T + 1$, $(m_{i,t}, s_{i,t}) = ((R / G)^{-(6 - 1)} e_i, 0)$.

When agents follow these strategies, only type-$t_0$ agents succeed in riding the bubble. They sell at $t_0 + \tau_0^*$, $p_{t_0 + \tau_0^*}$ reveals their sales, and the crash happens at $T + 1 = t_0 + \tau_0^* + 1$. In Proposition 1, I characterize the set of possible equilibria in different regions of the parameter
space. I show that, if $e^\lambda < G / R < \Gamma$, where $\Gamma = e^\lambda (1 + \sqrt{1 + 4e^{-\lambda}}) / 2$, agents follow (23) if and only if $\tau_0^* = 0$. That is, if $e^\lambda < G / R < \Gamma$, there is a unique no-bubble equilibrium where agents sell as soon as they observe the signal. We will later use this case as a benchmark for comparison with the noisy case. If $\Gamma \leq G / R < 1 + e^\lambda$, equilibrium can be supported for any $\tau_0^*$ between zero and a positive upper bound. And if $G / R \geq 1 + e^\lambda$, any integer $\tau_0^* \geq 0$ can be supported in equilibrium.\footnote{While infinite bubbles are technically possible for some parameter values, this possibility is not of interest, since as remarked in Section 2, the focus of the paper is on bubbles that burst endogenously in finite time.} Note that, even when bubbly equilibria exist, $\tau_0^* = 0$ is always an equilibrium.

Before proceeding to the proposition and proof, it may be useful to sketch the main ideas behind the results. In an equilibrium with $\tau_0^* \geq 0$, type-$n$ agents must be willing to (i) sell at time $n + \tau_0^*$ and (ii) not sell before $n + \tau_0^*$. For any $\tau_0^* \geq 0$, a type-$n$ agent will always sell at $n + \tau_0^*$, since other type-$n$ agents are selling and $p_{n+\tau_0^*}$ will reveal the sales, causing a crash. The key to (ii) is to focus on the time when a type-$n$ agent is most tempted to deviate from the strategy by selling early. This crucial time is $t = n + \tau_0^* - 1$, one period before she is supposed to sell. Since the bubble has not burst, she knows that type $n$ must have been either first or second to observe the signal. Thus, $\text{supp}_{n,t}(t_0) = \{n-1, n\}$ with $\mu_{n,t}(n-1) = 1 / (1 + e^{-\lambda})$ and $\mu_{n,t}(n) = e^{-\lambda} / (1 + e^{-\lambda})$.

If she waits, she will receive the discounted post-crash price $\alpha G^{n-2} R^\tau_{n+1}$ if $t_0 = n-1$, and if $t_0 = n$, she will ride the bubble for one more period and earn the discounted price $\alpha G^{n+\tau_0^*} / R$. If she sells, she will earn the expected time-$t$ price $\alpha G^{n+\tau_0^*-1}$. In sum, waiting is preferable if

$$1 \leq \frac{1}{1 + e^{-\lambda}} \left( \frac{G}{R} \right)^{-\tau_0^*} + \frac{e^{-\lambda} G}{1 + e^{-\lambda} R}.$$  \hspace{1cm} (24)

Since the right hand side of (24) is decreasing in $\tau_0^*$, if (24) fails for $\tau_0^* = 1$, it fails for all $\tau_0^* > 1$. In Appendix A, I show that $1 + e^{-\lambda} > (G / R)^{-2} + e^{-\lambda} G / R$ holds if $1 < G / R < \Gamma$. That is, if $G / R \leq \Gamma$, there are no equilibria with $\tau_0^* \geq 1$. If $G / R \geq \Gamma$, equilibrium can be sustained for any integer below an upper bound $-1 - \ln(1 + (1 - G / R) e^{-\lambda}) / \ln(G / R)$, which is obtained solving (24) for $\tau_0^*$. Finally, note that if $G / R \geq 1 + e^\lambda$, (24) holds for any $\tau_0^* \geq 0$.\footnote{While infinite bubbles are technically possible for some parameter values, this possibility is not of interest, since as remarked in Section 2, the focus of the paper is on bubbles that burst endogenously in finite time.}
To see why type-n agents are most tempted to sell at $t = n + \tau_0^* - 1$, consider for instance period $n + \tau_0^* - 2$. Two periods before agents are supposed to sell, $\text{supp}_{n,t}(0) = \{n-2, n-1, n\}$ and the crash probability is $\mu_{n,t}(n-2) = 1/(1 + e^{-\lambda} + e^{-2\lambda})$, less than in (24). The crash probability only falls further as we consider type-n agents’ choices at times $n + \tau_0^* - s$, for $s > 2$.

**PROPOSITION 1:** Assuming that $1/N \approx 0$, and letting $\Gamma \equiv e^\lambda (1 + \sqrt{1 + 4e^{-\lambda}}) / 2$, the values of $\tau_0^*$ that can be supported in equilibrium depend on parameters as follows:

a) If $e^\lambda < G / R < \Gamma$, only $\tau_0^* = 0$ can be supported in equilibrium.

b) If $\Gamma \leq G / R < 1 + e^\lambda$, equilibrium can be supported for any integer $\tau_0^*$ between zero and an upper bound $-1 - \ln(1 + (1 - G / R)e^{-\lambda}) / \ln(G / R)$.

c) If $1 + e^\lambda \leq G / R$, any integer $\tau_0^* \geq 0$ can be supported in equilibrium.

**PROOF:** Choices by agents who are hit by the shock, as well as the choices of agents who are not hit in pre-boom and post-crash periods, are rather trivial. Agents hit by the shock do not value the future, and thus willingly sell and consume everything. Pre-boom and post-crash, agents have no reason to deviate from strategies, since the expected price growth rate is $R$. Thus, for the rest of the proof, we will focus on the choices of agents who have not been hit by the shock in boom periods $t \in \{1, \ldots, T\}$. Moreover, the assumption that $1/N$ is small will allow us to neglect revenue effects of the sales by the first type and make the convenient approximation $p_t \approx \alpha G^\tau$.

To support a given $\tau_0^* \geq 0$ in equilibrium, it must be that for any $n$ and any boom period $t \in \{1, \ldots, T\}$, type-n agents are willing to (i) sell at time $n + \tau_0^*$ and (ii) not sell before time $n + \tau_0^*$. Condition (i) holds for any $n$ and $\tau_0^* \geq 0$. To see why, note that at $t = n + \tau_0^*$, a type-n agent knows that $t_0 = n$, that other type-n agents are selling and that $p_{n+\tau_0^*}$ will reveal these sales, causing a crash at $n + \tau_0^* + 1$. Selling is optimal as the expected time-$t$ price $\alpha G^{n+\tau_0^*}$ exceeds the payoff from waiting, given by the expected discounted post-crash price $\alpha G^{n-1} / R$.\(^{16}\)

\(^{16}\) Note that the expected payoff to the agent is she waits is $\alpha G^{n+1}$, regardless of whether she is hit by the shock at $t+1$ or not. If she is hit, she will have a strict preference for selling at $t+1$. If not, she will be indifferent between selling and not selling. In either case, the utility she expects from 1 unit of the risky asset is $\alpha G^{n-1}$.\]
Regarding \((ii)\), consider the sell-or-wait choice of a type-\(n\) agent at time \(t = n + \tau_0^* - s\), with \(s > 0\). First, focus on cases with \(s \leq \tau_0^*\), so that the agent has observed the signal as of time \(t\). The agent may be motivated to deviate from the strategy and sell preemptively if it is possible that \(t_0 = n - s\), in which case type-\(t_0\) agents will sell at \(t\) and precipitate a crash at \(t+1\). Clearly, the greater the probability that \(t_0 = n - s\), denoted by \(\mu_{n,t}(n-s)\), the greater the agent’s incentive to deviate. Since \(t_0\) must be positive and greater than \(n - N\), the probability \(\mu_{n,t}(n-s)\) is zero if \(s \geq n\), or \(s \geq N\). If \(s < \min\{n,N\}\), however, \(\text{supp}_{n,t}(t_0)\) is given by \(\{n-s,\ldots,n\}\), and the likelihood of a crash at \(t+1\) is \(\mu_{n,t}(n-s) = 1/(1 + \cdots + e^{-\lambda})\). Clearly, this likelihood is highest for \(s = 1\), as in (24), and therefore, if agents choose not to sell preemptively when \(s = 1\), they will not choose to do so either when \(s > 1\).

Next, consider cases with \(s > \tau_0^*\), which implies that \(t < n\). Before observing the signal, the type-\(n\) agent only knows that \(t_0\) must be positive, greater than \(t - (N-1)\), since otherwise she would have observed the signal, and greater than \(t - \tau_0^* - 1\), since otherwise sales would have started. Thus, \(\text{supp}_{n,t}(t_0) = \{\tau_0 \in \mathbb{Z} | \tau_0 \geq \max\{1,t-N+2,t-\tau_0^*\}\}\) and the likelihood of a crash at \(t+1\) is \(\mu_{n,t}(t-\tau_0^*)\). If \(t - \tau_0^* \geq \max\{1,t-N+2\}\), as is the case in the equilibrium with \(\tau_0^* = 0\), \(\mu_{n,t}(t-\tau_0^*) = 1 - e^{-\lambda}\). (If \(t - \tau_0^* < \max\{1,t-N+2\}\), \(\mu_{n,t}(t-\tau_0^*) = 0\).) An agent in this situation can sell preemptively at a price \(\alpha G^t\), or wait, in which case with probability \(e^{-\lambda}\) she will sell at \(t+1\) at a higher (discounted) price \(\alpha G^{t+1}/R\) and with probability \(1 - e^{-\lambda}\) obtain the post-crash price. Even if the post-crash price is zero, if \(1 < e^{-\lambda}G/R\), waiting is optimal. Thus, the mild condition \(e^{-\lambda} < G/R\) suffices to rule out preemptive sales if \(t < n\). Note that, since \(1 - e^{-\lambda} < 1/(1 + e^{-\lambda})\), agents are less tempted to sell preemptively before observing the signal than in the situation captured by (24).

We have now established that, of all situations a type-\(n\) agent may face before \(n + \tau_0^*\), the situation considered in (24), with \(t = n + \tau_0^* - 1\), \(n \geq 2\) and \(\tau_0^* \geq 1\), is the one where she is most inclined to sell preemptively. Thus, if (24) precludes preemptive sales when they are most tempting, it also precludes such sales in all possible cases with \(t < n + \tau_0^*\). Parts (a)-(c) follow.
directly from (24). As I show in Appendix A, if \( G / R < \Gamma \), agents are not even willing to wait for \( \tau_0^* = 1 \) period after observing the signal. Part (a) follows from here. If \( G / R > \Gamma \), equilibrium can be sustained for any integer \( \tau_0^* \) between zero and an upper bound \( -1 - \ln(1 + (1 - G / R)e^{-\lambda}) / \ln(G / R) \), which is obtained solving (24) for \( \tau_0^* \). Part (b) follows from here. To establish (c), simply note that if \( G / R \geq 1 + e^\lambda \), (24) holds for any \( \tau_0^* \), including \( \tau_0^* = \infty \). Q.E.D.

4.2 The Noisy Case

In this section, I raise \( \varepsilon \) above \( \varepsilon_{0,1} \), so that noise can hide sales by one type. For simplicity, I will focus on the case where noise can hide sales of one type, but not more. This assumption, while restrictive, yields a large payoff in terms of tractability, making it possible to construct analytically solvable equilibria in which agents follow simple Markov strategies.\(^{17}\)

4.2.1 Preliminaries: Admissible Levels of Noise and Informational Content of Prices

To determine the values of \( \overline{\varepsilon} \) for which noise can hide sales by one type, but not two, one must keep in mind that prices respond more strongly to simultaneous than gradual sales. That is, for any \( \varepsilon_{t+1} \), \( p_{t+1} \) as given by (19) is lower when two types sell at once, i.e., when \( z_t = 0 \) and \( z_{t+1} = 2 \), than when one type sells at \( t \), and another sells at \( t + 1 \), i.e., when \( z_t = z_{t+1} = 1 \). If two types sell at once, \( p_{t+1} / G^{t+1} \) is given by \( [2 / N + (\overline{\theta} + \varepsilon_{t+1})(1 - 2 / N)]^{-1} - 1 \). The sales are revealed with certainty if the highest possible value of this ratio (obtained for \( \varepsilon_t = -\overline{\varepsilon} \)) is below \( 1 / (\theta + \overline{\varepsilon}) - 1 \), the lowest possible ratio before sales begin. This is the case when \( \overline{\varepsilon} < \overline{\varepsilon}_{0,2} \), where

\[
\overline{\varepsilon}_{0,2} \equiv (1 - \overline{\theta}) / (N - 1).
\]

On the other hand, if one type sells at \( t \), the bubble does not burst, and a second type sells at \( t + 1 \),\(^{18}\) the sales by the second type will be revealed with certainty if

\(^{17}\)In a previous version of the paper, which featured behavioral agents, I considered a case in which noise was able to conceal sales by multiple types. In that case, the strategies played by agents were more complex, and conditioned behavior on a variable that was affected by the whole price history. While some insights were similar, equilibria could not be characterized analytically. Instead, the characterization relied on numerical simulation results. That version of the paper is available upon request.

\(^{18}\)Although this does not happen in the analysis that follows, the formula for \( p_{t+1} \) would be the same if the first type sold before time \( t \), and there were no more sales until the sales of the second type in period \( t + 1 \).
The cutoff $\varepsilon_{1,1}$ is the level of $\varepsilon$ for which the above holds with equality. To find this threshold, rearrange terms to transform the above into the quadratic equation

$$\varepsilon^2 - 2\varepsilon \left( N + \frac{1}{N - 2} \right) + \frac{N - 2\bar{\theta}}{N - 2} - \bar{\theta}^2 = 0,$$

and let $\varepsilon_{1,1}$ be the positive root

$$\varepsilon_{1,1} \equiv \left( N + \frac{1}{N - 2} \right) - \sqrt{\left( N + \frac{1}{N - 2} \right)^2 - \frac{N - 2\bar{\theta}}{N - 2} + \bar{\theta}^2}. \tag{27}$$

While (27) is rather unwieldy, if $N$ is large, one can replace it with the approximation\(^1^9\)

$$\varepsilon_{1,1} \approx \frac{1 - \bar{\theta}^2}{2N}. \tag{28}$$

Since $\varepsilon_{1,1} < \varepsilon_{0,2}$, the restriction which ensures that, regardless of timing, sales by two types will be revealed is $\varepsilon < \varepsilon_{1,1}$. In sum, if $\varepsilon_{0,1} < \varepsilon < \varepsilon_{1,1}$, noise may hide sales by one type, but not two.

With noise $\varepsilon$ in the ($\varepsilon_{0,1}, \varepsilon_{1,1}$) range, prices (for $t \leq T$) can be categorized as high, medium, or low. High prices exceed $\left((1/(N + (\bar{\theta} - \varepsilon)(1-1/N))^{-1} - 1)e_\varepsilon, \right.$ the highest possible price when one type sells, and thus reveal that all types are still in the market.\(^2^0\) Medium prices range between $\left((1/(N + (\bar{\theta} - \varepsilon)(1-1/N))^{-1} - 1)e_\varepsilon \text{ and } ((\bar{\theta} + \varepsilon)^{-1} - 1)e_\varepsilon, \right.$ and are thus consistent both with one type having sold and with zero types having sold. Finally, low prices are those under $\left((\bar{\theta} + \varepsilon)^{-1} - 1)e_\varepsilon. \right.$ Low prices unambiguously reveal that sales have begun.

While all types are still in the market, $p_t$ is high with probability $\pi_{H0}$ given by

$$\pi_{H0} \equiv \Pr\left[\left((\bar{\theta} + \varepsilon_t)^{-1} - 1\right)e_\varepsilon \text{ is high}\right] = \Pr\left[e_\varepsilon < \frac{1 - \bar{\theta}}{N} \right. - \left(1 - \frac{1}{N}\right)\varepsilon \left. \right] = \frac{1 - \bar{\theta}}{2N\varepsilon} + \frac{1}{2N}. \tag{29}$$

And $p_t$ is medium probability $1 - \pi_{H0}$. Since $\varepsilon < (1 - \bar{\theta})/(N - 1)$, it must be that $\pi_{H0} > 1/2$.

When the first type sells, the price is low with probability

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\(^{1^9}\) To see why, note that in (27), as $N$ grows and $\varepsilon$ approaches zero, $\varepsilon^2$ and $\varepsilon/(N - 2)$ approach zero, while $(N - 2\bar{\theta})/(N - 2)$ approaches one. In the limit (27) becomes $-2\varepsilon N + 1 - \bar{\theta}^2 = 0$. Solving for $\varepsilon_{1,1}$ yields (28).

\(^{2^0}\) Since it is irrelevant for the analysis that follows, I do not discuss the possibility that, after the first type sells, there are some periods in which no further types sell, and sales continue later.
\[ \pi_{L^3} \equiv \Pr \left[ \frac{N}{1 + (1 + \epsilon_t) (N - 1)} - 1 \right] \text{, is low} = \Pr \left[ \epsilon_t > \frac{\bar{\varepsilon} - (1 - \bar{\theta})}{1 - 1/N} \right] = \frac{1 - \bar{\theta}}{2\bar{\varepsilon}(N - 1)} - \frac{1}{2(N - 1)}, \quad (30) \]

and medium with probability \(1 - \pi_{L^3}\). Since \(\bar{\varepsilon} < e_{i,1} < (1 - \bar{\theta}) / N\), it follows that \(\pi_{L^3} > 1/2\).

Once more, a large-\(N\) allows us to work with a very useful approximation

\[ \pi \equiv \frac{1 - \bar{\theta}}{2\bar{\varepsilon}N} \approx \pi_{H|0} \approx \pi_{L^3}. \quad (31) \]

Finally, since \(\bar{\varepsilon}\) cannot exceed \(e_{i,1}\), it follows that in the large-\(N\) case \(\pi\) is bounded below by

\[ \pi \geq \frac{1}{1 + \bar{\theta}} > \frac{1}{2}. \quad (32) \]

### 4.2.2 Equilibrium Strategies

For simplicity and continuity with the noiseless case, I will focus on strategies similar to those in Strategy Profile 1. Agents who are hit by the shock sell and consume everything, while those who are not hit plan to ride the bubble for some time and then sell. What is new here is that the length of time for which agents plan to ride the bubble depends on prices. Setting shock-induced sales aside, a type-\(n\) agent’s strategy during the boom is as follows: Do not sell before observing the signal. After the signal, wait for \(\tau^*\) periods.\(^{21}\) Then, if the last price \(p_{n+\tau^* -1}\) is medium, sell and if it is high, wait. Follow this sell-if-medium/wait-if-high rule for \(d = \tau^{**} - \tau^*\) periods, and if prices remain high for \(d\) consecutive periods, sell at \(n + \tau^{**}\), regardless of whether \(p_{n+\tau^{**} -1}\) is high or medium. Do not reenter the market after selling, and post-crash, invest a fraction \((R / G)^{\gamma (n - 1)}\) of the endowment into the risky asset. Put more formally, we have:

**Strategy Profile 2:** For any \(i \in [0,1]\), agent \(i\) follows \(\{a_{i,t}\}_{t=Z} = \{m_{i,t}, s_{i,t}, \chi_{i,t}\}_{t\in Z}\) given by:

- If \(\delta_{i,t} = 0\), \(a_{i,t} = (0, h_{i,t}, 1)\) for any \(t\).
- If \(\delta_{i,t} = 1/R\), \(\chi_{i,t} = 0\) for all \(t\), and the choice of \((m_{i,t}, s_{i,t})\), is as follows:
  - If \(t \leq T\), and letting \(n = \nu(i)\),

\(^{21}\) Except for the special case where \(t_0 = n = 1\), in which case agents wait for \(\tau^* + 1\) periods. This exception is made to simplify the analysis of equilibrium conditions, but it does not change the overall gist of the analysis.
\((m_{ij}, s_{ij}) = \begin{cases} 
(b_{ij}, e_{ij}, 0) & \text{if } t < \min\{t^*(n), t^{**}(n)\} \\
(0, h_{ij}) & \text{if } t \geq \min\{t^*(n), t^{**}(n)\},
\end{cases} \tag{33}\)

where \(\tau^{**} \geq \tau^* \geq 0, \ t^{**}(n) = \min\{t \mid t \geq n + \tau^{**} \text{ and } p_{t-1} \text{ is high}\} \) and

\[
t^*(n) = \begin{cases} 
\min\{t \mid t \geq n + \tau^* \text{ and } p_{t-1} \text{ is medium}\} & \text{if } n > 1 \\
\min\{t \mid t \geq n + 1 + \tau^* \text{ and } p_{t-1} \text{ is medium}\} & \text{if } n = 1.
\end{cases}
\]

- If \(t \geq T + 1, \ (m_{ij}, s_{ij}) = ((R / G)^{-(n-1)}e_{ij}, 0)\).

Figure 4 depicts an example of a bubble where agents follow these strategies. In an equilibrium with a given \((\tau^*, \tau^{**})\) pair, depending on the realizations of \(\varepsilon_t\), type-0 agents may sell as early as period \(t_0 + \tau^*\) and as late as \(t_0 + \tau^{**}\), with the number of types who manage to sell before the crash ranging from 1 to \(d + 2\). The bubble is shortest if \(p_{t_0 + \tau^* - 1}\) is medium, type \(t_0\) sells at \(t_0 + \tau^*\), and \(p_{t_0 + \tau^*}\) is low. The bubble is longest if—as in Figure 4—\(p_t\) is high for all \(t \in \{t_0 + \tau^*-1, \ldots, t_0 + \tau^{**}-1\}\), type \(t_0\) sells at \(t_0 + \tau^{**}\), \(p_{t_0 + \tau^{**}}\) is medium and \(d + 1\) types sell at \(t_0 + \tau^{**} + 1\). In intermediate cases, \(p_{t_0 + \tau^* - 1}\) is high, but some price before \(t_0 + \tau^{**}\) is medium.

**Figure 4**—At \(t = t_0\), \(p_t\) and \(f_t\) begin to diverge. Signals are observed from \(t_0\) to \(t_0 + N - 1\). (Bars above these periods, which decrease in height, denote conditional probabilities for type \(n = t_0 + N - 1\).) In this example, \(d = \tau^{**} - \tau^* = 6\). Since \(\tau^* > N - 1\), sales begin after all signals are observed. For these realizations of \(\varepsilon_t\), \(p_t\) is high \(\forall t \in \{t_0 + \tau^*-1, \ldots, t_0 + \tau^{**}-1\}\), sales start at \(t = t_0 + \tau^{**}\), \(p_{t_0 + \tau^{**}}\) is medium, and types \(t_0 + 1, \ldots, t_0 + 7\) sell at \(T = t_0 + \tau^{**} + 1\), causing a crash. If \(p_t\) had been medium for some \(\tau \in \{t_0 + \tau^*-1, \ldots, t_0 + \tau^{**}-2\}\) sales would have started at \(\tau + 1 < t_0 + \tau^{**}\).
Since different prices elicit different selling behavior, the equilibria analyzed exhibit—in the terminology of Kai and Conlon (2007)—informational leakage. A medium $p_{t-1}$ followed by a high $p_t$ reveals that $t_0$ exceeds $t - \tau^*$, for otherwise the medium $p_{t-1}$ would have triggered sales and $p_t$ could not be high. On the other hand, if $p_{t-1}$ and $p_t$ are both high, the possibility that $t_0 = t - \tau^*$ cannot be ruled out. By the same token, a series of consecutive high prices \ldots, $p_{t-2}, p_{t-1}, p_t$ is consistent with $t_0$ being several periods before $t - \tau^*$, in which case there would be multiple types ready to sell as soon as the price turns medium. Since medium prices provide opportunities to learn about $t_0$, confidence in the bubble is strongest after prices bounce back from a medium price, and weakest when there is a slowdown after a string of high prices.\footnote{This pattern of information revelation also appears in the last section of AB, where prices may exogenously dip at random times.}

4.2.3 Characterizing Equilibria

With bubble duration $T - t_0$ ranging between $\tau^*$ and $\tau^{**+1}$ periods, the task at hand is to characterize which $(\tau^*, \tau^{**})$ pairs can be supported in equilibrium in different regions of the parameter space. A pair $(\tau^*, \tau^{**})$ can be supported if and only if agents are willing to follow equilibrium strategies at all times and under all contingencies. In some instances, verifying this is trivial. Such is the case for agents hit by the preference shock, for whom selling and consuming everything is optimal because their discount factor is zero. Similarly in pre-boom ($t \leq 0$) and post-crash ($t \geq T + 1$) periods, agents who are not hit by the shock view the risky and riskless assets as perfect substitutes since both assets return $R$ in expectation. However, the remaining choices—made by agents who are not hit by the shock at $t = 1, \ldots, T$—require a more complex analysis, which I organize into Lemmas 1-4. In Lemma 1, I establish conditions under which type-$n$ agents choose to sell if $t = n + \tau^{**}$ and $p_{t-1}$ is high. In Lemma 2, conditions under which they wait if $t < n + \tau^{**}$ and $p_{t-1}$ is high, and in Lemmas 3 and 4, respectively, conditions under which type-$n$ agents willingly sell if $t \geq n + \tau^*$ (or $t \geq n + \tau^{**+1}$ if $n = 1$) and $p_{t-1}$ is medium, and willingly wait if $t < n + \tau^*$ and $p_{t-1}$ is medium.
Broadly speaking, the message of Lemmas 1 and 3 is that, for high enough levels of $G/R$, there exist no equilibria with $\tau^{**}$ and $\tau^*$ below certain thresholds, because agents would not be willing to sell according to strategies. For example, Lemmas 1 and 3 respectively establish that if $G/R > \pi / (1 - \pi)$, equilibria with $\tau^{**} = 0$ can be ruled out and that, if $G/R$ is greater than a threshold $\Gamma^*$—which depends on $\lambda$ and $\pi$—equilibria with $\tau^* = 0$ can be ruled out. By contrast, Lemmas 2 and 4 establish that, for low enough $G/R$, there exist no equilibria with $\tau^{**}$ and $\tau^*$ above certain thresholds, since agents would not be willing to wait according to strategies. More precisely, Lemma 2 finds that equilibria with high values of $\tau^{**}$ only exist if $G/R$ is above, or just below $(1 + e^{-\lambda}) / (1 + e^{-\lambda} - \pi)$ and Lemma 4 finds that equilibria with high $\tau^*$ exist only if $G/R > e^{\lambda} / [\pi(1 - \bar{\theta})]$. The results from all lemmas are combined in Proposition 2, which compiles the full set of conditions needed to generate bubbles of different durations. Finally, these conditions are contrasted to those from the noiseless case in Proposition 3.

As we proceed, let us recall two assumptions that are in place throughout Lemmas 1-4. The first is that period $t$ is a boom period, i.e., that $t \in \{1, \ldots, T\}$. The second assumption is that the choices analyzed are the choices of agents who have not been hit by shock at time $t$.

**Lemma 1:** Assume that $(d + 2) / N$ is small and consider a type-$n$ agent at $t = n + \tau^{**}$, with $\tau^{**} \geq 0$ and high $p_{t-1}$. Then:

(a) If $G/R < \pi / (1 - \pi)$, the agent is willing to sell at $t$ for any $\tau^{**} \geq 0$.

(b) If $\pi / (1 - \pi) \leq G/R < 1 / (1 - \pi)$, the agent is only willing to sell at $t$ if $\tau^{**} > \overline{\tau}^{**}$, where $\overline{\tau}^{**} = -1 + \ln[\pi / (1 - (1 - \pi)G/R)] / \ln(G/R)$.

(c) If $1 / (1 - \pi) \leq G/R$, the agent is not willing to sell at $t$ for any $\tau^{**} \geq 0$.

**Proof:** If $t = n + \tau^{**}$ and $p_{t-1}$ is high, the agent knows that $t_0 = n$. If this was not the case, i.e., if $t_0 < n$, $p_{t-1}$ could not possibly be high. The agent also knows that, since other type-$t_0$ agents will sell at $t$, $p_t$ will be low—causing a crash—with probability $\pi$, and medium—allowing the bubble to last one more period—with probability $1 - \pi$. If the type-$t_0$ agent sells at $t$, she will

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23 In two cases, the agent would have already known that $t_0 = n$ before observing $p_{t-1}$. The first case is the one with $n = t_0 = 1$, where the agent learns that $t_0 = n$ at time 1. The second case is the one where the price at $t_0 + \tau^* = 2$ is medium. Then, she would have learned that $t_0 = n$ upon observing a high price (in Step 2 of) period $t_0 + \tau^* - 1$. 

25
obtain an expected price approximately equal to \( \alpha G_{n+\tau^{**}} \). If she waits, she will earn the post-crash price \( \alpha(G/R)^{(d-1)} R_{n+\tau^{**}+1} \) if \( p_t \) is low and if \( p_t \) is medium, she will sell at \( t+1 \), with \( d+1 \) types, at an expected price—since \( (d+2)/N \) is small—close to \( \alpha G_{n+\tau^{**}+1} \). In sum, if

\[
1 \geq \pi \left( \frac{G}{R} \right)^{-(\tau^{**}+1)} + (1-\pi) \frac{G}{R},
\]

selling is optimal. Note that the agent’s expected payoff from waiting does not depend on her preference shock realization at \( t+1 \). If the bubble does not burst, the agent will sell at \( t+1 \), regardless of whether she is hit by the shock or not. If the bubble bursts, she will sell at \( t+1 \) if she is hit, and will otherwise be indifferent between selling and waiting. In any case, as discussed in footnote 16, once the bubble bursts, the utility an agent can expect from her holdings of the risky asset does not depend on when the shock forces her to consume.

Parts (a), (b) and (c) of the Lemma follow directly from (34). The threshold \( \pi / (1-\pi) \) is derived evaluating (34) at \( \tau^{**} = 0 \), solving for \( G/R \) and choosing the root that above 1. The lower bound \( \tau^{**} \) is simply (34) solved for \( \tau^{**} \). Finally, we can see directly from (34) that, if \( 1/(1-\pi) \geq G/R \), selling is never optimal, no matter how large \( \tau^{**} \) becomes. Q.E.D.

For any \( \tau^{**} \geq 0 \), Lemma 1 establishes that equilibria with \( \tau^{**} < \tau^{**} \) can be ruled out for high enough \( G/R \). As we will see in Lemma 3, similar results hold for \( \tau^{*} \). The fact that it is possible to rule out equilibria without bubbles in the economy with noise marks a qualitative difference vis-à-vis the noiseless case, where the equilibrium with \( \tau^{*}_0 = 0 \) cannot be ruled out for any \( G/R \). The reason for this difference is that, without noise, if an agent knows that her type is selling, the bubble is sure to burst that period, whereas with noise, the bubble may withstand the sale and continue to grow for one more period.

Lemma 1 considers type-\( n \) agents selling at \( n+\tau^{**} \) after a high \( p_{n+\tau^{**}+1} \). At this point, they know that they were the first to observe the signal and that they have successfully ridden the bubble. However, in earlier periods \( n+\tau^{**} - j \) (with \( j \geq 1 \)), they did not know that they were first. Following (33) by waiting was risky, since \( t_0 \) could have been \( n-j \), in which case type \( n-j \) would have sold at \( n+\tau^{**} - j \), causing a crash with probability \( \pi \). Under what conditions was it optimal for them to take this risk? As we will see in Lemma 2, the key lies in
the sell-or-wait trade-off of a type-$n$ agent at $t = n + \tau^{**} - 1$, after $d + 1$ high prices $p_{n+\tau^{*}-2}, \ldots, p_{n+\tau^{**}-2}$. Just one period before the agent is supposed to sell, she does not know whether $t_0$ is $n - 1$, in which case her type was second to observe the signal, or $n$, in which case her type was first. If type $n$ turns out to be second in line—which is the case with probability $1/(1 + e^{-\lambda})$—the first type will sell at $t$, making $p_t$ low with probability $\pi$ and medium with probability $1 - \pi$. If $p_t$ is low, the bubble will burst at $t$ and if it is medium, the bubble will survive until period $t + 1$, and the agent will be able to sell, along with $d + 1$ types at a price that, since $(d + 2)/N \approx 0$, is close to $\alpha G^{t+1}$. If type $n$ turns out to be the first—which is the case with probability $e^{-\lambda}/(1 + e^{-\lambda})$—no types sell at $t$, and the agent can sell at $t + 1$ at a price close to $\alpha G^{t+1}$. In sum, selling at $t$ yields $\alpha G^t$, and waiting yields the post-cash price $\alpha(G/R)^{n+1}R'$ with probability $\pi/(1 + e^{-\lambda})$ and $\alpha G^{t+1}$ with probability $1 - \pi/(1 + e^{-\lambda})$. (Once more, the payoff from waiting does not depend on the preference shock realization at $t + 1$.) Waiting is optimal if

$$1 \leq \frac{1}{1 + e^{-\lambda}} \left( \pi \left( \frac{G}{R} \right)^{(\tau^{**+1})} + (1 - \pi) \frac{G}{R} \right) + \frac{e^{-\lambda} G}{1 + e^{-\lambda} R}. \tag{35}$$

Lemma 2, which tells us how long agents are willing to wait depending on $G/R$, is based on this inequality. If $G/R$ is below a threshold $\Gamma^{**} \equiv \pi / [2(1 + e^{-\lambda} - \pi)] \cdot (1 + \sqrt{4(1 + e^{-\lambda}) / \pi - 3})$, agents are not even willing to wait for one period after observing the signal, and thus, equilibria with $\tau^{**} > 0$ do not exist. This minimum threshold $\Gamma^{**}$ is derived by evaluating (35) at $\tau^{**} = 1$ and solving for $G/R$. (See Appendix A for details.) If $G/R < \Gamma^{**}$, there may be equilibria with $\tau^{**} = 0$, where selling preemptively means selling before observing the signal. As discussed in the proof of Proposition 1, the mild condition $e^{\lambda} < G/R$ rules out such sales. If $\Gamma^{**} < G/R < (1 + e^{-\lambda})/(1 + e^{-\lambda} - \pi)$, waiting is optimal only if $\tau^{**}$ does not exceed

$$\tau^{**} = \frac{\ln \left( \frac{\pi}{(1 + e^{-\lambda}) - (1 + e^{-\lambda} - \pi)G/R} \right)}{\ln (G/R)} - 1. \tag{36}$$

And finally, if $G/R \geq (1 + e^{-\lambda})/(1 + e^{-\lambda} - \pi)$, (35) holds for any $\tau^{**}$, since type-$t_0 + 1$ agents are willing to wait even if the post-cash price is zero.
While (35) is based on the tradeoff of a type-$n$ agent at $n+\tau**-1$ after $d+1$ high prices, in the proof of Lemma 2, I show that, of all possible situations with $t<n+\tau**$ and high $p_{t-1}$, this is precisely the one where type-$n$ agents are most tempted to sell.

**LEMMA 2:** Suppose that $(d+2)/N$ is small, that $t<n+\tau**$ and that $p_{t-1}$ is high. Moreover, let $\Gamma** \equiv \pi/[2(1+e^{-\lambda}-\pi)] \cdot \left[1 + \sqrt{4(1+e^{-\lambda})/\pi - 3}\right]$. Then,

(a) If $e^\lambda < G/R < \Gamma**$, there exist no equilibria with $\tau**>0$, because type-$n$ agents may not be willing to wait at all times $t<n+\tau**$.

(b) If $\Gamma** < G/R < (1+e^{-\lambda})/(1+e^{-\lambda}-\pi)$, there exist no equilibria with $\tau**>\bar{\tau}**$, with $\bar{\tau}**$ given by (36), as type-$n$ agents may not be willing to wait at all times $t<n+\tau**$.

(c) If $(1+e^{-\lambda})/(1+e^{-\lambda}-\pi) < G/R$, type-$n$ agents are willing to wait at $t$ for any $\tau** \geq 0$.

**PROOF:** See Appendix B.

While the proof considers all possible situations with $t<n+\tau**$ and high $p_{t-1}$, the reason why (35) captures the situation where preemptive selling (after high prices) is most tempting can be sketched as follows. Consider type-$n$ agents at time $n+\tau**-2$ after $d+1$ high prices. In this situation, type-$n$ agents believe that they may have been first, second, or third to observe the signal and hence assign a probability $\pi/(1+e^{-\lambda}+e^{-2\lambda})$ ($\pi$ times the probability of being third) to the event of a crash at $t$. Given that probability is clearly below $\pi/(1+e^{-\lambda})$, the crash probability in (35), the incentive to sell is weaker. By the same logic, the incentive to sell only weakens further as we consider even earlier dates $t=n+\tau**-s$ for $s \geq 3$. Moreover, in some cases with $t<n+\tau**$ and high $p_{t-1}$, type-$n$ agents have absolutely no incentive to sell, since the crash probability is nil. For example, if $p_{t-s}$ is medium for some $s \in \{2,\ldots,d+1\}$, all types know that nobody will sell at $t$, and hence that a crash at $t+1$ is impossible. This is due to the fact that, if $t_0$ was $t-\tau**$, type-$t_0$ agents would have sold at $t-s+1$ after the medium $p_{t-s}$.

However, if that was the case $p_{t-1}$ could not possibly be high.

Inequality (35) is similar to (24), the condition that caps bubble duration in the noiseless case. In fact, (35) is (24) with crash probability $\pi/(1+e^{-\lambda})$ instead of $1/(1+e^{-\lambda})$. With a lower crash probability, the returns needed to entice agents to ride the bubble are also lower in the
Consequently, whenever (24) allows for equilibria with positive $\tau_0^*$, (35) holds for even greater values of $\tau^{**}$.

In equilibria with large values of $\tau^{**}$, the assumption that $(d+2)/N$ is small can only be satisfied if $\tau^*$ is also large. If $(d+2)/N$ was a substantial fraction of the total number of agents, as $t$ neared $t_0 + \tau^{**}$, the observation of a medium price would add so many sellers and take away so many buyers, that the approximation $p_i \approx \alpha G^t$ used in (35), would no longer be valid.24 Thus, to complete the set of conditions needed to support large values of $\tau^{**}$, we must turn to Lemmas 3 and 4 in order to determine conditions needed to support large values of $\tau^*$.

Lemma 3 considers type-$n$ agents’ willingness to sell at $t \geq n + \tau^*$ (or $t \geq n + \tau^* + 1$ if $n = 1$) after medium $p_{t-1}$. As in previous lemmas, the analysis proceeds by focusing on the case—of all possible cases with $t \geq n + \tau^*$ and medium $p_{t-1}$—where type-$n$ agents are most inclined to deviate from equilibrium strategies, and then by establishing conditions under which agents choose to follow the strategies even in that worst case scenario. Among cases with $t \geq n + \tau^*$ and medium $p_{t-1}$, one can immediately see that if $t > n + \tau^*$, type-$n$ agents will surely sell at $t$. They know that at least two types, $n$ and $n+1$, are selling and this will precipitate a crash.25 However, if $t = n + \tau^*$, with $n \geq 2$, the bubble need not burst at $t$, and thus, for high enough $G / R$ waiting may be optimal.26 Type-$n$ agents know that, if $t_0 < n$, two or more types will sell at $t$, inevitably bursting the bubble. They also know that, if $t_0 = n$, just one type will sell, bursting the bubble with probability $\pi$, and allowing it to grow for one more period (at a rate close to $G$ since $2/N \approx 0$) with probability $1 - \pi$. Thus, the likelihood that the bubble survives period $t$ is $(1 - \pi)\mu_{n,t}(n)$, where $\mu_{n,t}(n)$ is the probability that $t_0 = n$. This probability is highest when supp$_{n,t}(t_0)$ has only two elements $\{n-1,n\}$. As discussed above, confidence in the bubble is strongest after it bounces back from a medium price. Thus, supp$_{n,t}(t_0) = \{n-1,n\}$ when

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24 To see how the analysis is expanded in that case, see Appendix D.
25 Type-$n$ agents find themselves in this situation if $n \geq 2$ and $p_{n+\tau^*}$ happens to be high, or if $n = 1$.
26 Recall that, in the special case $t_0 = n = 1$, type-1 agents are not supposed to sell until $2 + \tau^{**}$ at the earliest. While one could characterize equilibria where type-1 agents sell at $1 + \tau^{**}$ if $p_{t_0}$ is medium, doing so is complicated by the need to discuss additional subcases, as type-1 agents always know that they are ‘first in line’.
is medium, or when \( p_{t-2} \) is high and \( p_{t-3} \) is medium.\(^{27}\) With \( \text{supp}_{n,t}(t_0) = \{n-1,n\} \), the probabilities that \( t_0 = n-1 \) and \( t_0 = n \) are given by 
\[
\mu_{n,t}(n-1) = \frac{1}{1 + e^{-\lambda}}, \quad \mu_{n,t}(n) = \frac{e^{-\lambda}}{1 + e^{-\lambda}},
\]
and selling at \( t \) is optimal if
\[
1 \geq \frac{1}{1 + e^{-\lambda}} \left( G \right)^{-1(t^{*}+2)} + \frac{e^{-\lambda}}{1 + e^{-\lambda}} \left[ \pi \left( G \right)^{-1(t^{*}+1)} + (1 - \pi) \frac{G}{R} \right]. \tag{37}
\]
Since (37) applies to the case where agents are least inclined to sell, it also suffices to guarantee willingness to sell whenever \( \text{supp}_{n,t}(t_0) \) has more elements.

Most of Lemma 3 follows from (37). Type-\( n \) agents are willing to sell for any \( \tau^{*} \), including \( \tau^{*} = 0 \), if \( G / R < \Gamma^{*} \equiv (e^{\lambda} + \pi) / [2(1 - \pi)] \cdot \left[ 1 + \sqrt{1 + 4e^{-\lambda}(1 - \pi) / (1 + e^{-\lambda} \pi)^2} \right] \). As shown in Appendix A, \( \Gamma^{*} \) is derived by evaluating (37) at \( \tau^{*} = 0 \), solving for \( G / R \) and choosing the root above 1. As \( G / R \) rises above \( \Gamma^{*} \), agents become unwilling to sell if \( \tau^{*} \) is less than
\[
\tau^{*} = \frac{\ln(e^{\lambda} R / G + \pi) - \ln(1 + e^{\lambda} - (1 - \pi)G / R) - 1}{\ln(G / R)}. \tag{38}
\]
Finally, if \( G / R > (1 + e^{\lambda}) / (1 - \pi) \), type-\( n \) agents are unwilling to sell at \( t \), for any \( \tau^{*} \geq 0 \).

**Lemma 3:** Assume that \( \Gamma^{*} = (e^{\lambda} + \pi) / [2(1 - \pi)] \cdot \left[ 1 + \sqrt{1 + 4e^{-\lambda}(1 - \pi) / (1 + e^{-\lambda} \pi)^2} \right] \), \( 2 / N \) is small, \( \tau^{*} \geq 0 \), \( p_{t-1} \) is medium and \( \tau^{*} \) is given by (38). Then:

(a) If \( t > n + \tau^{*} \) and \( n \geq 1 \), selling at time \( t \) is optimal for any \( \tau^{*} \geq 0 \).

(b) If \( G / R < \Gamma^{*} \), type-\( n \) agents are willing to sell at \( t \) for any \( \tau^{*} \geq 0 \).

(c) If \( \Gamma^{*} < G / R < (1 + e^{\lambda}) / (1 - \pi) \), there exist no equilibria with \( \tau^{*} < \tau^{*} \), because type-\( n \) agents are in some instances unwilling to sell at time \( t \).

(d) If \( (1 + e^{\lambda}) / (1 - \pi) \leq G / R \), there exists no equilibrium for any \( \tau^{*} \geq 0 \), because type-\( n \) agents are in some instances unwilling to sell at time \( t \).

\(^{27}\) If \( p_{t-2} \) is medium, the fact that \( p_{t-1} \) is also medium reveals that \( t_0 \) cannot be less than \( n - 1 \). Otherwise, two or more types would have sold at \( t - 1 \), and \( p_{t-1} \) would be low. Similarly, if \( p_{t-3} \) is medium and \( p_{t-2} \) is high, it must be that \( t_0 \geq n - 1 \). In all other cases, i.e., whenever \( p_{t-1} \) is the first medium price after \( k \geq 2 \) consecutive high prices, \( \text{supp}_{n,t}(t_0) \) is given by \( \{\max\{1,n-k\}, \ldots, n\} \). In general, as \( k \) increases, so does the crash probability.
**Proof:** Part (a) follows from the fact that type-\(n\) agents know that at least two types, \(n\) and \(n+1\), are selling in the current period, which will certainly cause a crash. Parts (b)-(d) follow from (37): \(\Gamma^*\) is derived setting \(\tau^*=0\), solving for \(G/R\) and choosing the root that is above 1 (see Appendix A). The expression for the lower bound on \(\tau^*\) is obtained solving (37) for \(\tau^*\). This lower bound on \(\tau^*\) becomes infinite if \((1+e^\lambda)/(1-\pi)\leq G/R\). Since (37) implies willingness to sell when agents are least inclined to sell, it also suffices to imply willingness to sell in all other situations, i.e., in cases where \(\text{supp}_{n,t}(t_0)\) has more elements. \(Q.E.D.\)

Given that (37) is akin to (34) with higher crash probability, Lemmas 1 and 3 are similar. Although it takes higher levels of \(G/R\) to make agents unwilling to sell in Lemma 3, it is still the case that, for any \(\tau > 0\), one can rule out equilibria with \(\tau^*<\tau\) if \(G/R\) is high enough.

Lemma 4 analyzes the choices of type-\(n\) agents in situations where \(t<n+\tau^*\) and the most recent price is medium. Once more, the analysis proceeds by focusing on the situation where deviating—in this case by selling preemptively—is most tempting, and finding conditions under which, even in that least favorable situation, agents wait. While full analysis of sell-or-wait choices under all possible contingencies is in the proof of Lemma 4 (see Appendix B), here, I will sketch the main argument. Consider an agent of type \(n=t_0+d+2\) at \(t=t_0+\tau^{**}+1\). (Note that \(t=t_0+\tau^{**}+1=n+\tau^*-1\), i.e., type \(n\) is the lowest type of those staying in the market at \(t\).)

At this point, confidence in the bubble is at its lowest, since a medium price comes after a string of consecutive high prices. In the example of Figure 4, after \(d+1\) high prices, type \(t_0\) sold at time \(t-1\), \(p_{t-1}\) is medium, and now \(d+1\) more types \(t_0+1,\ldots,t_0+d+1\) will sell at \(t\), while types \(n\) and higher wait. Clearly, the type-\(n\) agent would never wait if she knew the value of \(t_0\). But given her information, \(\text{supp}_{n,t}(t_0)\) contains the \(d+3\) values \(n-(d+2),\ldots,n\). The last value holds the key to her decision, since it is her hopes of being ‘first in line’ that entice her to wait. The agent assigns a probability \(\mu_{n,t}(n)=e^{-\lambda(d+2)}/(1+\cdots+e^{-\lambda(d+2)})\) to the event that \(t_0=n\). Selling in this case would be a costly mistake, as it would imply foregoing a large expected return \(W_d\), which may compound for up to \(d+1\) periods. Specifically, if \(t_0=n\), every period from \(t\) to \(t+d+1=n+\tau^{**}\), there is a probability \(\pi(1-\overline{\theta})\)—the probability that the price is high and the agent is not hit by the shock—that the agent will continue to ride the bubble, and a
probability $1 - \pi(1 - \overline{\theta})$ that she will sell, realizing capital gains accrued up to that point. As I show in Appendix B, if $(d+2)/N$ is small,\textsuperscript{28} the expected return $W_d$ is approximately given by:

$$W_d = (1 - \pi(1 - \overline{\theta})) \left( \frac{G}{R} \right) \left( \frac{\pi(1 - \overline{\theta})G}{R} \right)^{d+1} - 1 + \left( \frac{\pi(1 - \overline{\theta})G}{R} \right)^{d+1}. \quad (39)$$

If $\pi(1 - \overline{\theta})G/R > e^\lambda$, as $d$ grows, $W_d$ rises faster than $e^{-\lambda(d+2)} / (1 + \cdots + e^{-\lambda(d+2)})$ falls, and thus

$$1 < \frac{e^{-\lambda(d+2)}}{1 + \cdots + e^{-\lambda(d+2)}} W_d \quad (40)$$

holds for large enough $d$. In other words, if $\pi(1 - \overline{\theta})G/R > e^\lambda$, there exists a $\overline{d} > 0$, such that if $d \geq \overline{d}$, $W_d$ is large enough to entice type-$n$ agents to wait at time $t$.

**Lemma 4:** Let $t < n + \tau^*$, and let $p_{t-1}$ be medium. If $\pi(1 - \overline{\theta})G/R > e^\lambda$, and $(d+2)/N$ is small, there exists $\overline{d} > 0$ such that if $d \geq \overline{d}$, type-$n$ agents find it optimal not to sell at time $t$.

**Proof:** See Appendix B.

While full details are in the proof, the reason why the case discussed above is the one where preemptive selling is most tempting can be sketched as follows. First, if less than $d+1$ high prices precede the medium $p_{t-1}$, some of the lower values of $t_0$ can be ruled out, making $t_0 = n$ more likely. Second, consider type $n+1$ at time $t$. Just like for type $n$, for type $n+1$, the support of $t_0$ contains $d+2$ ‘bad’ values with $t_0 < n$. But for type $n+1$, there are two ‘good’ values in the support, $t_0 = n$ and $t_0 = n+1$, with expected return $W_d$ or higher. This makes type-$n+1$ agents less eager to sell than type-$n$ agents. Similarly, agents of types $n+2$ and higher are even more willing to wait.

Proposition 2 draws on Lemmas 1-4 to establish conditions for nonexistence of equilibria without bubbles, and conditions for existence of equilibria with long bubbles.

\textsuperscript{28} For the purposes of Lemma 4, assuming a small $1/N$ would be sufficient, since this implies that the last summand on the right hand side of (39), $(\pi(1 - \overline{\theta})G/R)^{d+1}$, closely approximates its equivalent in Appendix D. However, assuming small $(d+2)/N$ is necessary in Lemma 2 and helpful in Lemma 4, as it implies that the first summand on the right of (39) is also a close approximation to its equivalent term in Appendix D.
**PROPOSITION 2:** Assume that \((d+2)/N\) is small, \(\pi = (1 - \bar{\theta}) / (2\bar{e}N)\), \(1/(1 + \bar{\theta}) < \pi < 1\), and agents follow Strategy Profile 2. Then:

(a) **NONEXISTENCE OF EQUILIBRIA WITHOUT BUBBLES:** If \(G/R > \pi / (1 - \pi)\), there exist no equilibria with \(\tau^{**} = 0\). If \(G/R > \Gamma^*\), there exist no equilibria with \(\tau^* = 0\).

(b) **EXISTENCE OF EQUILIBRIA WITH LONG BUBBLES:** If \(G/R < 1/(1 - \pi)\), \(G/R > (1 + e^{-\lambda}) / (1 + e^{-\lambda} - \pi)\) and \(G/R > e^\lambda / (\pi(1 - \bar{\theta}))\), there exist equilibria with arbitrarily large \(\tau^*\) and \(\tau^{**}\).

**PROOF:** Part (a) follows from Lemmas 1 and 3. If \(G/R\) exceeds \(\pi / (1 - \pi)\), equilibria with \(\tau^{**} = 0\) do not exist, because—even if all other type-\(n\) agents type were selling—an individual type-\(n\) agent would not sell at time \(n\) after a *high* price. And if \(G/R\) exceeds \(\Gamma^*\), equilibria with \(\tau^* = 0\) do not exist, because the type-\(n\) would not sell at time \(n\) after a *medium* price.

Part (b) follows from the fact that, in any given equilibrium, agents must willingly follow strategies at all times. Thus, long bubbles may arise only if all the conditions from Lemmas 1-4 are simultaneously satisfied for large values of \(\tau^*\) and \(\tau^{**}\). Lemmas 1 and 2 allow \(\tau^{**}\) to be arbitrarily large if, respectively, \(G/R < 1/(1 - \pi)\) and \(G/R \geq (1 + e^{-\lambda}) / (1 + e^{-\lambda} - \pi)\). Note that, since \(G/R < 1/(1 - \pi)\) implies \(G/R < (1 + e^\lambda) / (1 - \pi)\), whenever Lemma 1 allows for high values of \(\tau^{**}\), Lemma 3 also allows for large values of \(\tau^*\). Finally, \(G/R > e^\lambda / (\pi(1 - \bar{\theta}))\), is required by Lemma 4 to support large values of \(\tau^*\).

It remains to verify that there exist parameters for which all conditions hold. This can be done by noting that, for any \(\lambda > 0\) and \(\bar{\theta} \in (0,1)\), one can always choose \(\pi\) close enough to 1 to satisfy \(1/(1 + \bar{\theta}) < \pi < 1\), and to ensure that the upper bound on \(G/R\) given by \(1/(1 - \pi)\) is greater than the lower bound \(\max\{e^\lambda / (\pi(1 - \bar{\theta})),(1 + e^{-\lambda}) / (1 + e^{-\lambda} - \pi)\}\). Q.E.D.

Note that some parameter values for which long bubbles may arise also admit the possibility of short or no bubbles. For instance, if \(G/R < \pi / (1 - \pi)\), the bubble-free equilibrium with \((\tau^*,\tau^{**}) = (0,0)\) exists. However, as \(G/R\) increases towards \(1/(1 - \pi)\), the minimum
equilibrium $\tau^{**}$ also increases. And within the class of equilibria with small $(d+2)/N$, as the minimum $\tau^{**}$ rises, the minimum admissible $\tau^* = \tau^{**} - d$ must also increase.\(^{29}\)

Contrasting Propositions 1 and 2, one can see that the noisy economy is more conducive to bubbles than the noiseless economy. This is best illustrated by the following two comparisons. First, in Proposition 1, the no-bubble equilibrium $\tau^*_0 = 0$ can never be ruled out in the noiseless economy, whereas Proposition 2 rules out equilibria with $\tau^{**} = 0$ and $\tau^* = 0$ for high enough $G/R$. This is the first part of Proposition 3. The second part establishes existence of parameters for which one can completely rule out bubbles without noise while still supporting arbitrarily long bubbles with noise. That is, there exist parameters such that $G/R < \Gamma$, so that $\tau^*_0 = 0$ is the only equilibrium without noise, $\pi \geq 1/(1+\overline{\theta})$, so that there is enough noise to hide sales by only one type, and $\max\{e^\lambda / (\pi(1-\overline{\theta})),(1+e^{-\lambda})/(1+e^{-\lambda} - \pi)\} < G/R < 1/(1-\pi)$, so that long bubbles can arise with noise.\(^{30}\)

**PROPOSITION 3:** Assume that $N$ is large. Further, assume that agents follow Strategy Profile 1 in the noiseless economy and Strategy Profile 2 in the noisy economy. Then:

(a) In the noiseless economy, the no-bubble equilibrium $\tau^*_0 = 0$ can never be ruled out, regardless of how high $G/R$ becomes, while in the noisy economy, no-bubble equilibria with $\tau^{**} = 0$ and $\tau^* = 0$ can be ruled for high enough values of $G/R$.

(b) There exists parameter values $(G, R, \lambda, \overline{\theta}, \overline{\tau}, N)$ and $\pi = (1-\overline{\theta})/(2N\overline{\tau})$, such that in the noiseless economy bubbles cannot arise, but in the noisy economy arbitrarily long bubbles can arise. That is, there exist parameter values such that $1/(1+\overline{\theta}) < \pi < 1$, $G/R < \Gamma$, and $\max\{e^\lambda / (\pi(1-\overline{\theta})),(1+e^{-\lambda})/(1+e^{-\lambda} - \pi)\} < G/R < 1/(1-\pi)$.

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\(^{29}\) Given the fact that $\Gamma^* > 1/(1-\pi)$, to actually construct equilibria with $\Gamma^* < G/R$, one must depart from the assumption that $(d+2)/N$ is small and refer to equations (AD.4)-(AD.10) in Appendix D to see how Lemmas 1-4 can be modified for large $d/N$. As $d/N$ grows, (AD.5) in a modified Lemma 1 would imply a much higher upper bound than $1/(1-\pi)$, while (AD.7)-(AD.8), the key to a modified Lemma 3, would not imply a $\Gamma^*$ that is increasing in $d/N$. The other necessary conditions for bubbles (from modified lemmas 2 and 4) can be met, since if (AD.7)-(AD.8) hold for large $\tau^*$, (AD.6) will hold for large $\tau^{**}$, and even if $d/N$ is not small, (AD.9) will hold for large enough $d$, as long as $1/N$ is small.

\(^{30}\) As discussed after Lemma 2, the opposite cannot be done because, whenever $G/R > \Gamma$ allows for bubbly equilibria without noise, the highest $\tau^*$ that can be supported in equilibrium must also above zero.
PROOF: Part (a) follows directly from Propositions 1 and 2.

To analyze part (b), it is helpful to depict all parameter restrictions in a graph with $\pi$ on the horizontal and $G/R$ on the vertical axis, as in Figure 5. The admissible $(\pi,G/R)$ pairs lie above (i.e., northwest of) the curve $G/R = (1 + e^{-\lambda})/(1 + e^{-\lambda} - \pi)$, and above (i.e., northeast of) the curve $G/R = e^\lambda/(\pi(1 - \bar{\theta}))$. These two curves intersect at the point $(\pi_i, [G/R]_i)$, given by

$$\pi_i = (1 + e^{-\lambda})/(1 + (1 - \bar{\theta})(e^{-\lambda} + e^{-2\lambda})) \quad \text{and} \quad [G/R]_i = 1 + 1/((1 - \bar{\theta})(e^{-\lambda} + e^{-2\lambda})).$$

At this point, $G/R$ is as low as it can be without falling below one (or both) of the two curves. Moreover, admissible $(\pi,G/R)$ points must also lie to the right of the vertical line $\pi = 1/(1 + \bar{\theta})$, and thus, one must consider three cases, depending whether $\pi_i$ is above, equal to, or below $1/(1 + \bar{\theta})$.

Panel (a) of Figure 5 depicts the first case, with $\pi_i > 1/(1 + \bar{\theta})$. In this case, one can decrease $[G/R]_i$ and $\pi_i$ by lowering the value of $\bar{\theta}$ without violating the constraint $\pi \geq 1/(1 + \bar{\theta})$. The constraint becomes binding when, as shown in panel (b), $\pi_i = 1/(1 + \bar{\theta})$, or, equivalently, when $\bar{\theta} = e^{-2\lambda}/(1 + e^{-\lambda})^2$. Finally, panel (c) depicts the situation where $\bar{\theta} < e^{-2\lambda}/(1 + e^{-\lambda})^2$, and the point $(\pi_i, [G/R]_i)$ is inadmissible because it lies to the left of the line $\pi = 1/(1 + \bar{\theta})$. In this case, further decreases in the value of $\bar{\theta}$ lead to increases in the lowest admissible $G/R$, which is now given by $(1 + e^{-\lambda})/(1 + e^{-\lambda} - 1/(1 - \bar{\theta}))$ instead of $[G/R]_i$.

In sum, the lowest admissible $G/R$ is $[G/R]_i$ evaluated at $\bar{\theta} = e^{-2\lambda}/(1 + e^{-\lambda})^2$, which equals $G/R = e^\lambda + 2/(e^\lambda + 2)$. At this level of $\bar{\theta}$, $\pi_i = (1 + e^{-\lambda})^2/[(1 + e^{-\lambda})^2 + e^{-2\lambda}]$. It remains to verify that, at this point, $G/R < 1/(1 - \pi)$ and $G/R < \Gamma$. It is straightforward to verify that $G/R < 1/(1 - \pi)$ holds for all $\lambda \geq 0$. To see whether $e^\lambda + 2/(e^\lambda + 2) < \Gamma$, by means of algebraic manipulations,\(^{31}\) one can show that this inequality holds if and only if $4 < e^{3\lambda} + 2e^{2\lambda}$. This cubic expression has one real root $e^\lambda \approx 1.1304$, which implies $\lambda \approx 0.1226$. For $\lambda$ above this threshold, there exist parameters for which all conditions in part b of the Proposition hold. Q.E.D.

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\(^{31}\) Start with $e^\lambda + 2/(e^\lambda + 2) < e^\lambda(1 + (1 + 4e^{-\lambda})^{1/2})/2$. Divide both sides by $(e^\lambda/2)$ and subtract 1 to obtain $1 + 4/(e^{2\lambda} + 2e^\lambda) < (1 + 4e^{-\lambda})^{1/2}$. Squaring and subtracting 1 yields $8/((e^{2\lambda} + 2e^\lambda) + 16/((e^{2\lambda} + 2e^\lambda)^2 < 4e^{-\lambda}$. Multiply both sides by $(e^{2\lambda} + 2e^\lambda)^2/4$ and rearrange terms to obtain $4 < e^{3\lambda} + 2e^{2\lambda}$.\hfill\(\square\)
Figure 5. Parameters for which long bubbles exist with noise and no bubbles exist without noise, with $\lambda > 0.1304$. If $\lambda < 0.1304$, shaded areas would be empty, since $\Gamma < e^\lambda + 2 / (2 + e^\lambda)$.
Using Proposition 3, one can construct examples in which there exist long bubbles with noise and no bubbles without it. To do so, choose any $\lambda > 0.1226$, let $\overline{\theta} = e^{-2\lambda} / (1 + e^{-\lambda})^2$ and pick a $(\pi, G / R)$ pair from the shaded area of Figure 5, Panel b. Then, choose some large value of $\tau^{**}$, and let $\tau^* = \tau^{**} - \overline{d}$, where $\overline{d}$ is the smallest integer for which (40) holds. Finally, choose $N$ large enough to make $(\overline{d} + 2) / N$ small, and set $\overline{e} = 1 / (2\pi N)$.

Two remarks are in order. First, it is only necessary to impose $\lambda > 0.1226$ under the strict interpretation that ‘no bubbles without noise’ literally means that $\tau^*_0 = 0$ is the only equilibrium without noise. If one relaxes this slightly and raises $G / R$ to also allow $\tau^*_0 = 1$, but not $\tau^*_0 > 1$, to be a noiseless equilibrium, one can generate arbitrarily long bubbles with noise for any positive $\lambda$.\(^{32}\) The second remark is that the sole purpose of the above discussion of parameter values is to show that the conditions listed in part b of Proposition 3 are compatible with each other. The parameter values discussed are not meant to be suggestive of a plausible calibration. Given the assumptions that noise can hide sales by one type, but not two, and that $(\overline{d} + 2) / N$ must be small, it is unsurprising that very large values of $G / R$ and $N$ are needed to generate long bubbles. Given the observations in footnote 17, however, it is reasonable to conjecture that a model in which noise could hide sales by multiple types would allow long bubbles to arise for lower values of $G / R$.

5 Extension: Reentering the market after selling

The no-reentry assumption greatly simplifies the analysis, and may be defensible for high-transaction-cost assets such as real estate. However, it is highly implausible in many other asset markets, where transaction costs are low. Furthermore, it is not obvious that the assumption is innocuous, since the option to reenter after selling changes agents’ sell or wait tradeoffs. In this subsection, I reexamine agents’ choices in Lemmas 1-4 allowing for reentry, and show that one

\(^{32}\) To show this, one needs to show that (24) fails for $\tau^*_0 = 2$ and $G / R = e^x + 2 / (e^x + 2)$. To do this, substitute $x = G / R$ and $\tau^*_0 = 2$ in (24) and rearrange terms to obtain $x^3 - 1 \leq e^{-x} (x^3 - x^3)$. Dividing both sides by $(x - 1)$ yields $x^2 + x + 1 \leq e^{-x} x^3$. Substituting $x = e^x + 2 / (e^x + 2)$ and carefully rearranging terms, one arrives at $e^x + 1 + 1 \leq 3e^x / (2 + e^x) + 8 / (2 + e^x)^2 + 8e^{-x} / (2 + e^x)^2$. From here, we can see that the inequality fails for all $\lambda > 0$, since $e^x > 3e^x / (2 + e^x)$, $1 > 8 / (2 + e^x)^2$ and $1 > 8e^{-x} / (2 + e^x)^2$. 

37
can still obtain bubbly equilibria in which agents follow Strategy Profile 2. That is, I consider equilibria where agents could reenter the market after selling, but choose not to.\footnote{I do not construct equilibria in which reentries actually occur. To construct equilibria in which reentry did happen, one would need to specify an entirely different strategy profile and completely redo the analysis of section 4.2.}

In Lemmas 1 and 3, it is not difficult to see why agents would not choose to reenter the market after selling. Along the equilibrium path, whenever an agent sells, she knows that the crash will arrive within one or, at most, two periods. Reentry would thus have to take place either at time $T$, i.e., when the price is about to collapse, or at time $T + 1$, i.e., after the crash. Neither does the reentry option make a difference in the situations considered in Lemma 2. Roughly (see Appendix C for details), the reason for this is the following. The reentry option makes preemptive sales more desirable by reducing their potential opportunity cost. After selling, if the bubble does not burst, reentering agents forego just one, instead of many, periods of appreciation. However, as I show in the proof of Lemma 2, of all situations with $t < n + \tau^{**}$ and high $p_{r-1}$, the decisive situation for type-$n$ agents is the one captured by (35), just one period before they are supposed to sell. Thus, even if the bubble does not burst at $t$, type-$n$ agents will sell at $t + 1$. This means that, even with forbidden reentry, the opportunity cost incurred by early sellers is already given by just one period of forgone appreciation. Since reentry cannot reduce this opportunity cost, it cannot tilt the balance in favor of preemptive selling.

It is in the situations covered by Lemma 4 where the reentry option matters most. In Lemma 4, if $\pi(1 - \theta)G / R > e^{\lambda}$ and $d \geq \bar{d}$, type-$n$ agents wait at $t = n + \tau^* - 1$ with medium $p_{r-1}$ (preceded by $d + 1$ high prices) because they may forego a large return $W_d$ in the event that $t_0 = n$. The return $W_d$ accrues over up to $d + 1$ periods. Allowing reentry lowers this opportunity cost, since an agent who sells at $t$ and then observes a high $p_t$ can reenter at $t + 1$, foregoing only part of $W_d$. Thus, with allowed reentry, the sell-or-wait choice is no longer governed by (40), since the return from selling (on the left) now includes a reentry return $W_{d-1}$ with probability $\pi(1 - \theta)e^{-\lambda(d+2)} / (1 + \cdots + e^{-\lambda(d+2)})$—the probability that $t_0 = n$, and $p_t$ is high, and the agent is not hit by the shock. Hence, waiting is now optimal if

$$1 + \pi(1 - \theta)\frac{e^{-\lambda(d+2)}}{1 + \cdots + e^{-\lambda(d+2)}}(W_{d-1} - 1) < \frac{e^{-\lambda(d+2)}}{1 + \cdots + e^{-\lambda(d+2)}}W_d.$$  

(41)
Since $W_{d-1} > 1$ for all $d \geq 1$, agents would rather sell, planning to reenter if $p_1$ is high, than sell without the option to reenter. Nevertheless, if $d$—and hence $W_d$—is large enough, not selling is still preferable to selling and reentering. This is because agents who sell and reenter forego one period of growth, and thus, part of $W_d$. For large enough $d$, this opportunity cost—which depends on the difference between $W_d$ and $W_{d-1}$—is important enough to preclude sales.

To see this more, use (39) and rearrange terms to rewrite (41) as

$$
\frac{1 - e^{-\lambda(d+1)}}{1 - e^{-2}} - e^{-\lambda(d+2)} \left[ \frac{\pi(1-\bar{\theta})}{\pi(1-\bar{\theta}) G R - \pi(1-\bar{\theta}) G R - 1} \right]^2 < \frac{\pi(1-\bar{\theta})}{\pi(1-\bar{\theta}) G R - 1} e^{-\lambda(d+2)} \left( \frac{G}{R} \right)^d. \tag{42}
$$

Note that the left hand side increases with $d$, but approaches $1/(1-e^{-\lambda})$ as $d \to \infty$. The right hand side, since $\frac{\pi(1-\bar{\theta})G}{R} > e^\lambda$, grows exponentially with $d$. Thus, there is a positive $\bar{d} > \bar{d}$ such that (41) holds if $d \geq \bar{d}$. However, once reentry is allowed, equilibria with $\bar{d} \leq d < \bar{d}$ vanish. While (41) captures the situation faced by type-$n$ agents at $t=n+\tau*1$ with medium $p_{t-1}$ preceded by $d+1$ high prices, it also suffices to rule out preemptive sales if $t < n+\tau*1$ and/or if fewer than $d+1$ high prices precede the medium $p_{t-1}$. As in the proof of Lemma 4, these changes increase the likelihood that $t_0 = n$, and hence the likelihood of earning $W_d$ by waiting. It is not difficult to verify that, as the probability that $t_0 = n$ increases, inequalities (41) and (42) hold by a wider margin than when the probability is $e^{-\lambda(d+2)} / (1 + \cdots + e^{-\lambda(d+2)})$.

Aside from the fact that the minimum $d$ is lengthened from $\bar{d}$ to $\bar{d}$, there are no new requirements that equilibria with bubbles must satisfy once reentry is allowed. Hence, within the class of equilibria where agents follow Strategy Profile 2, the possibility of reentry makes only a quantitative difference, but not a qualitative one. The mechanisms protecting bubbles from preemptive sales remain the same, and long bubbles still arise.

6 Conclusion

This paper presents a model of speculative bubbles where rational agents buy overvalued assets since, given their beliefs, they have a good chance to resell them at a profit to a greater fool. The model builds on existing theories of speculation under asymmetric information, especially Abreu
and Brunnermeier (2003), but introduces two key changes. Instead of assuming that some agents are irrational and that prices are sometimes unresponsive to sales, I assume that all agents are rational and that prices are market-clearing at all times. Even in the absence of noise, for appropriate parameter values, bubbly equilibria coexist along with non-bubbly equilibria. The introduction of noise only facilitates bubble formation, and makes it possible to rule out equilibria without bubbles for some parameter values. The emergence of bubbles in this environment shows that the key mechanism at work in AB does not hinge on irrationality or on prices being unresponsive to sales.

For future work, it would be interesting to extend the present model along a number of dimensions. Perhaps the most important extension would be to explicitly model the growing endowments. The expansion of credit seems like a natural candidate for this. If the bubbly asset can be posted as collateral, price increases relax credit constraints allowing agents to borrow more and to bid prices even higher. Alternatively, gradual arrival of resources into the bubble could arise from a model where, in the short run, many resources are tied up in long-term projects that take time to mature and can only be liquidated at a loss. In such a model, as projects matured, resources could be directed towards the bubble. Finally, as Brunnermeier (2001) points out, in models bubbles burst typically burst abruptly, while in reality, they often deflate gradually. It may be interesting to explore whether a version of the current model where the noisy component was not bounded could generate a slower crash.
**APPENDIX A:** Derivation of $\Gamma, \Gamma^{**}$ and $\Gamma^*$

To find $\Gamma$, let $x = G/R$ and note that (24) evaluated at $\tau_0^{**} = 1$ is an equation of the form:

$$1 = \sigma_1 x^2 + \sigma_2 x^{-1} + (1 - \sigma_1 - \sigma_2)x,$$  \hspace{1cm} (43)

with $\sigma_1 = (1 + e^{-\lambda})^{-1}$ and $\sigma_2 = 0$. This equation can be solved as follows:

\[
1 = \sigma_1 x^2 + \sigma_2 x^{-1} + (1 - \sigma_1 - \sigma_2)x \iff x^2[1 - \sigma_1 - \sigma_2 + \sigma_1 + \sigma_2] = \sigma_1 + \sigma_2 x + (1 - \sigma_1 - \sigma_2)x^3 \iff
\]

\[
\sigma_1(x^2 - 1) + \sigma_2(x^2 - x) = (1 - \sigma_1 - \sigma_2)(x^3 - x^2) \iff [\sigma_1(x + 1) + \sigma_2x](x - 1) = (1 - \sigma_1 - \sigma_2)x^2(x - 1).
\]

Clearly, $x = 1$ is a root. For $x \neq 1$, we have, $\sigma_1(x + 1) + \sigma_2x = (1 - \sigma_1 - \sigma_2)x^2$, a quadratic equation with one negative root and one positive root given by

\[
x = \frac{\sigma_1 + \sigma_2}{2(1 - \sigma_1 - \sigma_2)} \left( 1 + \sqrt{1 + \frac{4\sigma_1(1 - \sigma_1 - \sigma_2)}{(\sigma_1 + \sigma_2)^2}} \right).
\hspace{1cm} (44)

Substituting $\sigma_1 = (1 + e^{-\lambda})^{-1}$ and $\sigma_2 = 0$ back into (44) yields $\Gamma = e^2 / 2 \cdot (1 + \sqrt{1 + 4e^{-\lambda}})$.

Regarding $\Gamma^{**}$, note that (35) for $\tau^{**} = 1$ is (43) with $\sigma_1 = \pi / (1 + e^{-\lambda})$ and $\sigma_2 = 0$.

Substituting these values into (44) yields $\Gamma^{**} = \pi / [2(1 + e^{-\lambda} - \pi)] \cdot \left[ 1 + \sqrt{4(1 + e^{-\lambda}) / \pi - 3} \right]$.

For $\Gamma^*$, note that (37) at $\tau^* = 0$ is (43) with $\sigma_1 = (1 + e^{-\lambda})^{-1}$ and $\sigma_2 = e^{-\lambda} \pi / (1 + e^{-\lambda})$.

Substituting into (44), we obtain $\Gamma^* = (e^2 + \pi) / [2(1 - \pi)] \cdot \left[ 1 + \sqrt{1 + 4e^{-\lambda}(1 - \pi) / (1 + e^{-\lambda} \pi)^2} \right]$.

**APPENDIX B:** Proofs of Lemmas 2 and 4

**Proof of Lemma 2:** Consider an arbitrary type $n \geq 1$ at $t = n + \tau^{**} - j$, for any $\tau^{**} \geq 0$ and $j \geq 1$. Let $p_{t-1}$ be high. A type-$n$ agent may be inclined to sell preemptively at $t$ for fear that the bubble may burst at $t + 1$. In fact, if $t_0 = n - j$, type-$t_0$ agents will sell at $t = n + \tau^{**} - j$, causing a crash at $t + 1$ with probability $\pi$. In, and only in, the following cases (i)-(iv), type-$n$ agents are not tempted to sell at $t$ because $t_0 = n - j$ is either impossible (i-iii), or very unlikely (iv):

(i) If at least one of the prices $p_{t-(d+1)}, \ldots, p_{t-2}$ is medium, $t_0$ cannot be $n - j$. (Note that, since $t - (d + 1) = n - j + \tau^{**} - 1$, if $p_{t-s}$ was medium for some $s \in \{2, \ldots, d + 1\}$ and $t_0$ was $n - j$, type-$t_0$ agents would have sold at time $t - s + 1$, and $p_{t-1}$ would not be high.)

(ii) If $j \geq n$, $t_0$ cannot be $n - j$, since $t_0$ cannot be less than 1.

(iii) If $j \geq N$, $t_0$ cannot be $n - j$, since $t_0$ cannot be less than $n - (N - 1)$.
(iv) If \( j > \tau^{**} \), type-\( n \) agents have yet to observe the signal as of time \( t \). If \( \tau^{**} \geq N-1 \), sales cannot begin before all signals arrive. If \( \tau^{**} < N-1 \), \( \text{supp}_{n,j}(t_0) = \{ \tau_0 \mid \tau_0 \geq n-j \} \), and 
\[
\mu_{n,j}(n-j) = 1 - e^{-\lambda}.
\]
If \( e^{\lambda} < G/R \), type-\( n \) agents prefer not to sell at \( t \).

Having ruled out preemptive sales if one or more of (i)-(iv) hold, it remains to discuss situations where none of these conditions apply, i.e., cases where \( \min\{\tau^{**}, n-1, N-1\} \) and \( p_{t-s} \) is high \( \forall s = 1, \ldots, d+1 \). To rule out preemptive sales in these cases, it suffices to focus on the case where \( j = 1 \). To see why, note that at \( t = n + \tau^{**} - j \), \( \text{supp}_{n,j}(t_0) = \{ n-j, \ldots, n \} \), with the probability that \( t_0 = n-j \) given by \( \mu_{n,j}(n-j) = 1/(1 + e^{-\lambda} + \cdots + e^{-j\lambda}) \), which is greatest for \( j = 1 \). Thus, if type-\( n \) agents do not sell preemptively if \( j = 1 \), they will not do so either for \( j > 1 \). Let us then consider the situation faced by type-\( n \) agents at \( t = n + \tau^{**} - 1 \). High prices \( p_{t-(d+1)}, \ldots, p_{t-1} \) reveal to type-\( n \) agents that they were either first or second to observe the signal, i.e., that \( t_0 \) must be \( n-1 \) or \( n \). Probabilities \( \mu_{n,j}(n-1) \) and \( \mu_{n,j}(n) \), are respectively given by \( 1/(1 + e^{-\lambda}) \) and \( e^{-\lambda}/(1 + e^{-\lambda}) \). If \( t_0 = n-1 \), the first type will sell at \( t \). With probability \( \pi \), \( p_t \) will be low, causing a crash at \( t+1 \), and with probability \( 1-\pi \), \( p_t \) will be medium, and \( d+1 \) types will sell at \( t+1 \) at the expected price \( \alpha G^{r+1} \). If \( t_0 = n \), nobody will sell at \( t \), and type-\( n \) agents will sell at \( t+1 \) at a price \( \alpha G^{r+1} \). In sum, waiting is best if (35) holds and, in turn, (35) implies parts (a)-(c) of the Lemma. Q.E.D.

Proof of Lemma 4: The proof proceeds in two parts. In the first, I derive conditions under which type-\( n \) agents choose not to sell at \( t = n + \tau^{*} - 1 \) with medium \( p_{t-1} \) if \( n \geq d+3 \), \( \tau^{*} \geq 1 \), and \( p_{t-s} \) is high \( \forall s = 2, \ldots, d+2 \). In the second part, I show that, of all possible situations cases with \( t < n + \tau^{*} \) and medium \( p_{t-1} \), type-\( n \) agents are most tempted to sell in the case considered in the first part. Consequently, the conditions ruling out preemptive sales there also suffice to rule out preemptive sales by type-\( n \) agents in all other cases with \( t < n + \tau^{*} \) and medium \( p_{t-1} \).

Let us proceed to the first part by supposing that \( t = n + \tau^{**} - 1 \), \( p_{t-1} \) is medium, \( n \geq d+3 \), \( \tau^{*} \geq 1 \), and \( p_{t-s} \) is high \( \forall s = 2, \ldots, d+2 \). Then, type-\( n \) agents think that \( t_0 \) could be anywhere from \( n-(d+2) \) to \( n \), i.e., \( \text{supp}_{n,j}(t_0) = \{ n-(d+2), \ldots, n \} \). If \( n-(d+2) \leq t_0 \leq n-2 \), two or
more types will sell at \( t \), causing a crash. If \( t_0 = n-1 \), type \( n-1 \) will sell, causing a crash with probability \( \pi \). However, even if the payoff from waiting is set to zero for all \( t_0 < n \), type-\( n \) agents are still willing to wait at \( t \) if the expected (gross) return \( W_d \) earned in the event that \( t_0 = n \) is sufficiently large. Since \( e^{-\lambda (d+2)}/(1 + \cdots + e^{-\lambda (d+2)}) \) is the probability that \( t_0 = n \), (40) is a sufficient condition for type-\( n \) agents to be willing to wait at time \( t \).

To derive (39), note that \( W_d \) depends on how long a type-\( n \) agent can expect his shares to continue appreciating at the rate \( G \) after time \( t \), which may be up to \( d+1 \) periods. If \( t_0 = n \), every period from \( t+1 = n+\tau^* \) to \( t+d = n+\tau^{**}-1 \), the type-\( n \) agent will continue to ride the bubble if prices remain *high* and she is not hit by the shock, which will occur with probability \( \pi(1-\bar{\theta}) \) every period, and every period she will sell with probability \( 1 - \pi(1-\bar{\theta}) \), which is the likelihood that the last price is *medium* and/or she is hit by the shock. If all prices \( p_{t+1}, \ldots, p_{t+d-1} \) are *high* and she is not hit by the shock, she will sell the shares she currently owns at time \( t+d+1 = n+\tau^{**} \), regardless of whether \( p_{t+d} \) is *high* or *medium* and whether she is hit by the shock or not. Thus, \( W_d \) is given by

\[
W_d = (1 - \pi(1-\bar{\theta})) \left( G / R \right)^d + \pi(1-\bar{\theta}) \left( G / R \right) + \pi(1-\bar{\theta}) G / R + \pi(1-\bar{\theta}) G^2 / R + \pi(1-\bar{\theta}) G^3 / R + \cdots + \left[ \pi(1-\bar{\theta}) G^{d} / R \right].
\]

Since \( \pi(1-\bar{\theta})G / R \neq 1 \), this can be rewritten as (39). Given the assumption that \( \pi(1-\bar{\theta})G / R > e^\lambda \), \( e^{-\lambda (d+2)} W_d \) grows exponentially with \( d \). In turn, this implies that (40) holds for large enough \( d \). To see this, substitute (39) into (40) and rearrange terms to obtain

\[
1 - e^{-\lambda (d+3)} < e^{-\lambda (d+2)} \left[ \left( 1 - \pi(1-\bar{\theta}) \right) \left( G / R \right) + \left( \pi(1-\bar{\theta}) G / R \right)^{d+1} + \cdots + \left( \pi(1-\bar{\theta}) G^{d+1} / R \right) \right].
\]

(45)

The left-hand-side of (45) is increasing in \( d \), but approaches \( 1/(1-e^{-\lambda}) \) as \( d \to \infty \). On the right, all terms are positive, and since \( \pi(1-\bar{\theta})G / R > e^\lambda \), some terms grow exponentially with \( d \). Thus, (45) holds for \( d \) above some threshold \( \bar{d} \). Finally, I derived (39) and (45) assuming that all is lost in the crash, a good approximation if \( \tau^* \) is large. But for smaller \( \tau^* \), type-\( n \) agents are even less inclined to sell at \( t \), because they will lose less in the event of a crash.
The second part shows that, of all possible cases with \( t < n + \tau^* \) and medium \( p_{t-1} \), type-\( n \) agents are most inclined to sell if \( t = n + \tau^* - 1 \), \( \tau^* \geq 1 \), and \( p_{s} \) is high \( \forall s \in \{2, \ldots, d + 2\} \). In this case, the support of \( t_0 \) contains \( d + 2 \) ‘bad’ values \( n - (d + 2), \ldots, n - 1 \), for which \( t_0 < n \), and one ‘good’ value, \( t_0 = n \), for which waiting at \( t \) yields a large return \( W_d \). In all other cases with \( t < n + \tau^* \) and medium \( p_{t-1} \) the support of \( t_0 \) contains fewer bad values and/or more good values, making a crash at \( t + 1 \) less likely, and a large return more likely. To see this, observe how \( \text{supp}_{n,t}(t_0) \) changes when \( p_{t-1} \) is medium, but it is no longer the case that \( t = n + \tau^* - 1 \), \( \tau^* \geq 1 \), \( n \geq d + 3 \), and \( p_{s} \) is high \( \forall s \in \{2, \ldots, d + 2\} \). Every change in conditions makes waiting more attractive, by reducing the ratio of ‘bad’ to ‘good’ values of \( t_0 \) in \( \text{supp}_{n,t}(t_0) \).

(i) Let \( t = n + \tau^* - j \), \( j \geq 2 \), and \( j \leq \tau^* \), (with \( n \geq d + 3 \) and high \( p_{s} \) \( \forall s \in \{2, \ldots, d + 2\} \)). If \( j \leq N - (d + 2) \), the set \( \text{supp}_{n,t}(t_0) = \{n - j - (d + 1), \ldots, n\} \) contains \( d + 2 \) bad values \( n - j - (d + 1), \ldots, n - j \), and \( j \) good values \( n - j + 1, \ldots, n \). Moreover, the expected return if \( t_0 > n - j + 1 \) exceeds \( W_d \). If \( j > N - (d + 2) \), there are fewer than \( d + 2 \) bad values values, because \( n - j - (d + 1) < n - (N - 1) \).

(ii) If \( t = n + \tau^* - j \) and \( j > \tau^* \), (with \( n \geq d + 3 \) and high \( p_{s} \) \( \forall s \in \{2, \ldots, d + 2\} \)), \( \text{supp}_{n,t}(t_0) \) equals \( \{\tau | \tau \geq n - j - (d + 1)\} \), i.e., type-\( n \) agents have yet to observe the signal as of time \( t \). There may be up to \( d + 2 \) bad values, but also infinitely many good values.

(iii) If \( n < d + 3 \), (with \( t = n + \tau^* - 1 \), \( \tau^* \geq 1 \), and high \( p_{s} \) \( \forall s \in \{2, \ldots, d + 2\} \)), type-\( n \) agents know, from their signal, that \( t_0 \) cannot be \( n - (d + 2) \), since \( n - (d + 2) \leq 0 \). Type-\( n \) agents can thus eliminate \( d + 3 - n \) bad values from \( \text{supp}_{n,t}(t_0) \).

(iv) If there are \( k < d + 1 \) consecutive high prices before \( p_{t-1} \), \( \text{supp}_{n,t}(t_0) = \{n - (k + 1), \ldots, n\} \), i.e., the number of bad values falls from \( d + 2 \) to \( k + 1 \). This makes the good value \( t_0 = n \) relatively more likely. Also note that \( p_{s} \) can be high \( \forall s \in \{2, \ldots, d + 2\} \) only if \( t \geq d + 3 \).

If more than one of (i)-(iv) apply, several factors make waiting more desirable than the first part, and therefore, there are 1 or more good values and/or less than \( d + 2 \) bad values. Q.E.D.
APPENDIX C: Details of Section 5

Consider a type-$n$ agent at time $t = n + \tau** - j$, with $j \geq 1$ and high $p_{t-1}$. Instead of waiting, she can sell and reenter later if she so chooses. This will protect her agent against a crash if the bubble bursts, but she will forego capital gains between the time of sale and reentry, if the bubble does not burst. Recalling the proof of Lemma 2, we can see that, since in cases (i)-(iv), the crash probability is either zero or very small, selling and reentering cannot be optimal.

Thus, as in Lemma 2, the cases of interest are those with $j \leq \min \{\tau**, n-1, N-1\}$ and high $p_{t-s}$ $\forall s = 1, \ldots, d + 1$. To revisit these situations, let $\psi_j$ denote the (gross expected discounted) return earned by a type-$n$ agent if she follows the equilibrium strategy from $t = n + \tau** - j$ onward. Note that $\psi_1$ is the right-hand-side of (35), and that, if $2 \leq j \leq d + 1$,

$$\psi_j = \frac{1}{1 + \cdots + e^{-\lambda j}} [\pi \left(\frac{G}{R}\right)^{-(\tau**+1)} + (1 - \pi) \frac{G}{R}] + \left(1 - \frac{1}{1 + \cdots + e^{-\lambda j}}\right) \frac{G}{R} \left[(1 - \pi(1 - \bar{\theta})) + \pi(1 - \bar{\theta}) \psi_{j-1}\right].$$

That is, if $t_0 = n - j$, which is the case with probability $1/(1 + \cdots + e^{-\lambda j})$, the bubble will burst with probability $\pi$ and grow for one (and only one) more period with probability $1 - \pi$. But if $t_0 > n - j$, there is a probability $1 - \pi(1 - \bar{\theta})$ (i.e., the probability that $p_t$ is medium and/or the agent is hit by the shock) that the agent will sell next period, and otherwise she will continue to wait, earning an expected return of $G/R$ times $\psi_{j-1}$. Let us now compare this to the expected return from selling preemptively and reentering at $t + 1$ in the event that $p_t$ is high and the agent is not hit by the shock.35 This return is 1 if the agent does not reenter and $\psi_{j-1}$ if she does—which she will do with probability $\pi(1 - \bar{\theta})(1 - 1/ (1 + \cdots + e^{-\lambda j}))$. Hence, the return from selling and possibly reentering is

$$1 - \pi(1 - \bar{\theta}) \left(1 - \frac{1}{1 + \cdots + e^{-\lambda j}}\right) + \pi(1 - \bar{\theta}) \left(1 - \frac{1}{1 + \cdots + e^{-\lambda j}}\right) \psi_{j-1}.$$ 

Clearly, if (35) holds and $\psi_{j-1} = 1$, $\psi_j$ exceeds the value of this last expression. The difference only grows when taking into account the fact that, by Lemma 2, $\psi_{j-1} > 1$ for $2 \leq j \leq d + 1$. A similar argument (a bit more notation intensive, and available upon request) rules out preemptive selling with reentry option if $j$ exceeds $d + 1$, in which case agents will reenter the market after

35 Note that reentry is optimal if and only if, by Lemma 2, it is optimal for agents who are still in the market to wait.
safety even after a medium \( p_t \). It shall also be noted that, since staying in the market is preferable to selling preemptively (with reentry option) for all \( j \), agents have no incentive to sell preemptively and reenter after multiple periods. Staying out for more than one period serves only to compound the expected opportunity costs relative to staying in the market.

**APPENDIX D:** Formulas for general values of \( N \)

**THE CASE WITHOUT NOISE**

In the noiseless case, one type sells before the crash at an expected price given by \( \alpha_{0,1}G^t \), with

\[
\alpha_{0,1} = E \left[ \frac{1}{1/N + \theta_t (1-1/N)} \right] = \frac{1}{2\varepsilon} \int_{-\infty}^{\infty} \frac{1}{1/N + (\bar{\theta} + \varepsilon_i)(1-1/N)} -1 \, d\varepsilon_i = \frac{\ln \left( \frac{1}{N + (\bar{\theta} + \varepsilon)(1-1/N)} \right) - \ln \left( \frac{1}{N + (\bar{\theta} - \varepsilon)(1-1/N)} \right)}{2\varepsilon (1-1/N)} - 1. \tag{AD.1}
\]

The subscripts 0,1 denote the fact that zero types sell before time \( t \) and 1 type sells at time \( t \). In this notation, the \( \alpha \) in the text would be written as \( \alpha_{0,0} \). Clearly, if \( 1/N \approx 0 \), \( \alpha_{0,1} \approx \alpha \).

Without the large-\( N \) assumption, Strategy Profile 1 and the main message of Proposition 1 remain the same. But some details must be changed. Specifically, the proof of Proposition 1 derives conditions under which type-\( n \) agents are willing to (i) sell at \( n + \tau_0^* \) and (ii) not sell before then. If \( N \) is large, (i) holds automatically for any \( \tau_0^* \geq 0 \) since the expected price \( \alpha G^{n+\tau_0^*} \) exceeds the discounted post-crash price \( \alpha G^{n-1} R^{\tau_{n+1}} \). If the time-\( t \) price falls to \( \alpha_{0,1}G^{n+\tau_0^*} \) due to the first type’s sales, agents are only willing to sell under the mild condition that the price impact of the first type’s sales is smaller than the crash. That is, agents are willing to sell if

\[
(G/R)^{\tau_{n+1}} > \alpha / \alpha_{0,1}. \tag{AD.2}
\]

In part (ii), (24) must reflect the first type’s sales, which will happen at \( t \) and \( t+1 \) with respective probabilities \( \mu_{n_0}(n-1) = (1 + e^{-\lambda})^{-1} \) and \( \mu_{n_0}(n) = e^{-\lambda} / (1 + e^{-\lambda}) \). Thus, (24) becomes

\[
\left[ (\alpha_{0,1} + e^{-\lambda} \alpha)G^{n+\tau_{n+1}^*} \right] / (1 + e^{-\lambda}) \leq \left[ \alpha G^{n-2} R^{\tau_{n+1}} + e^{-\lambda} \alpha_{0,1}G^{n+\tau_0^*} / R \right] / (1 + e^{-\lambda}).
\]

Rearranging terms, we can rewrite the above inequality as

\[
1 \leq \frac{1}{\alpha_{0,1} / \alpha + e^{-\lambda} \left( \frac{G}{R} \right)^{(\tau_{n+1}^*)}} + \frac{e^{-\lambda} \alpha}{1 + e^{-\lambda} \alpha / \alpha_{0,1} R} \tag{AD.3}
\]
Naturally, (AD.3) converges to (24) as $\alpha_{0,1}/\alpha$ approaches 1. The threshold needed to support infinite bubbles increases to $e^x + \alpha/\alpha_{0,1}$. That is, if $G/R \geq e^x + \alpha/\alpha_{0,1}$, (AD.3) holds even if the post-crash price is zero. Unfortunately, there is no tractable formula for the generalized version of $\Gamma$, the threshold needed to support equilibria with $\tau^*_0 = 1$. However, we can see that since $\alpha > \alpha_{0,1}$, (AD.3) is in essence a version of (24) with more weight on the crash. Therefore, the threshold that $G/R$ has to clear for bubbly equilibria to exist exceeds $\Gamma$. As before, one can solve (AD.3) for $\tau^*_0$ to find the maximum bubble duration that can be supported in equilibrium. The resulting maximum $\tau^*_0$ is nonnegative if $\max\{e^x, \alpha/\alpha_{0,1}\} < G/R$ and well-defined (i.e., finite) if $G/R < e^x + \alpha/\alpha_{0,1}$.

The condition $e^x < G/R$ still suffices to preclude preemptive sales before signals arrive. To see why, recall that the condition follows from a type-$n$ agent’s sell-or-wait choice at $t < n$ with $t - \tau^* \geq \max\{1, t - N + 2\}$, and thus with $\text{supp}_{n,t}(t_0) = \{\tau_0 \in \mathbb{Z} | \tau_0 \geq t - \tau^*\}$. The expected $p_t$ is now $(e^{-\lambda} + (1-e^{-\lambda})\alpha_{0,1})G^{t+1}$ instead of $\alpha G^t$, because, if $t_0 = t$, which occurs with probability $\mu_{n,t}(t) = 1 - e^{-\lambda}$, one type will sell at $t$. However, the same applies to the expected $p_{t+1}$, which is the post-crash price with probability $1 - e^{-\lambda}$, and $(e^{-\lambda} + (1-e^{-\lambda})\alpha_{0,1})G^{t+1}$ with probability $e^{-\lambda}$.

**The Case with Noise**

If noise may hide sales by one type, pre-crash sales may happen over two periods. Price effects of such sales are captured by (19), from which we derive an expected $p_t/G^{t}$ given by

$$\alpha_{z,t_{t+1}} = E\left[\left(1 - \frac{z_t}{N}\right)\left(1 - \frac{z_{t+1}}{N}\right)\frac{1 - \frac{z_t}{N}}{N + \theta_{t+1} (1 - (z_t + z_{t+1})/N)} - 1\right] =$$

$$\left(1 - \frac{z_t}{N}\right)\left(1 - \frac{z_{t+1}}{N}\right)\ln\left(\frac{z_{t+1}}{N} + (\bar{\theta} + \bar{\epsilon}) \left(1 - \frac{z_t + z_{t+1}}{N}\right)\right) - \ln\left(\frac{z_{t+1}}{N} + (\bar{\theta} - \bar{\epsilon}) \left(1 - \frac{z_t + z_{t+1}}{N}\right)\right) - 1. \quad \text{(AD.4)}$$

---

36 Since $G/R = 1$ is not a solution, the algebraic simplifications from Appendix A cannot be reproduced here. The cubic formula does yield solutions, but the expressions are unwieldy to be point of intractability.
Of course, \( \alpha_{0,1} \) and \( \alpha_{0,0} = \alpha \) are particular cases of \( \alpha_{z_i,z_{i+1}} \). Clearly, if \( (z_t + z_{t+1}) / N \) is small, \( \alpha_{z_i,z_{i+1}} (z_t + z_{t+1}) / N \) is close to \( \alpha \).

Without the large-\( N \) assumption, we must abandon the simplified expression for \( \bar{\varepsilon}_{t,z} \) given by (28) and instead use the more precise but also more complex expression (27). Similarly, we can no longer use the single approximate probability \( \pi \) as given by (31). Instead, we must take into account that when all types are still in the market, \( p_t \) is high with probability \( \pi_{H0} \), given by (29), and once the first type sells, the price is low with probability \( \pi_{L1} \), given by (30).

Without assuming that \( \pi_{H0} \approx \pi_{L1} \approx \pi \), and that \( (d+2) / N \) is small, so that \( \alpha_{z_i,z_{i+1}} \approx \alpha \), we can still specify Strategy Profile 2. However, the conditions needed to support a given equilibrium change slightly. Specifically, Lemmas 1-4 are derived from inequalities (34), (35), (37) and (39)-(40). These inequalities need to be modified as follows:

(a) In Lemma 1, inequality (34) must be altered to reflect the effect of one type’s sales on \( p_t \), the effect of another \( d+1 \) types’ sales on \( p_{t+1} \) if \( p_t \) is not low, and the fact that \( p_t \) will be low with probability \( \pi_{L1} \) instead of \( \pi \). Thus, (34) becomes

\[
\alpha_{0,1} > \pi_{L1} \alpha \left( \frac{G}{R} \right)^{(\tau^{**+1})} + (1-\pi_{L1}) \alpha_{1,d+1} \frac{G}{R},
\]

To find the analog of threshold \( \pi / (1-\pi) \), evaluate (AD.5) at \( \tau^{**} = 0 \), solve for \( G/R \), and choose the solution that approaches \( \pi / (1-\pi) \) for large \( N \). The analog of threshold \( 1/(1-\pi) \) is \( \alpha_{0,1} / ((1-\pi_{L1}) \alpha_{1,d+1}) \). For \( G/R \) between these two values, there is a well defined positive minimum equilibrium \( \tau^{**} \), which can be found solving (AD.5) for \( \tau^{**} \).

(b) Regarding Lemma 2, inequality (35) becomes

\[
\frac{\alpha_{0,1} + e^{-\lambda} \alpha}{1 + e^{-\lambda}} < \frac{\pi_{L1} \alpha \left( \frac{G}{R} \right)^{(\tau^{**+1})} + (1-\pi_{L1}) \alpha_{1,d+1} \frac{G}{R} + e^{-\lambda} \left[ \pi_{H0} \alpha_{0,1} + (1-\pi_{H0}) \alpha_{0,d+1} \right] \frac{G}{R}}{1 + e^{-\lambda}},
\]

where with probability \( 1/(1 + e^{-\lambda}) \) the situation is the same as in (AD.5). On the other hand, with probability \( e^{-\lambda} / (1+e^{-\lambda}) \), no types sell at \( t \) and thus, depending on whether \( p_t \) is high
or medium, one or \(d+1\) types sell at \(t+1\). The probability that \(p_t\) is high is \(\pi_{H/0}\), and 
\[
\left(\alpha_{0,1} + e^{-\lambda} \alpha \right) / \left(1 - \pi_{L/3}\right) \alpha_{1,d+1} + e^{-\lambda} (\pi_{H/0} \alpha_{0,1} + (1-\pi_{H/9}) \alpha_{0,d+1}) \right) \) is the analog of threshold 
\[
(1 + e^{-\lambda}) / (1 - \pi). \] 
As in the noiseless case, there is no tractable formula for the 
generalized version of \(\Gamma^*\), the threshold needed to support equilibria with \(\tau^* = 1\). \(^{37}\) It is 
possible, however, to solve (AD.6) for \(\tau^*\) in order to find the upper bound \(\bar{\tau}\).

(c) In Lemma 3, the equivalent of (37) depends on whether \(p_{t-2}\) is high or medium. To see why, 
recall that (37) captures a type-\(n\) agent’s choice when \(\supp_{n,t} (t_0) = \{n-1, n\}\), i.e., in the 
case—of all possible cases with \(t \geq n + \tau^*\) and medium \(p_{t-1}\)—where selling is least tempting. 
If \(N\) is large, (37) applies both to cases where \(p_{t-2}\) and \(p_{t-1}\) are medium and to cases where 
\(p_{t-3}\), \(p_{t-2}\) and \(p_{t-1}\) are medium, high and medium, respectively. Here, the two situations lead 
to different inequalities:

- If \(p_{t-2}\) and \(p_{t-3}\) are both medium, the analog of (37) is 
\[
\alpha_{1,1} + e^{-\lambda} \left(1 - \frac{1}{N}\right) \alpha_{0,1} > \frac{\alpha \left(1 - \frac{1}{N}\right)}{1 + e^{-\lambda}} \left(1 - \frac{1}{N}\right) \left(1 + e^{-\lambda}\right) \left(1 - \frac{1}{N}\right) \left(1 + e^{-\lambda}\right) . \quad \text{(AD.7)}
\]

Note that the probability that \(t_0 = n-1\) is \(1 / (1 + e^{-\lambda} (1 - 1 / N))\), instead of \(1 / (1 + e^{-\lambda})\). This is 
because, given an observed medium \(p_{t-1}\), the likelihood of this price with one type selling is 
higher than the likelihood if no types have sold. \(^{38}\) Also, revenue effects of sales of the first 
and second types are captured by \(\alpha_{0,3}\) and \(\alpha_{1,1}\), and \(\pi_{L/3}\) replaces \(\pi\). The analog of threshold 
\(1 + e^{-\lambda}) / (1 - \pi)\) is \(e^{-\lambda} N / (N-1) + \alpha_{0,1} / \alpha_{1,1} / (1 - \pi_{L/3})\). If \(G / R\) is above this threshold there is 
no equilibrium, as agents are unwilling to sell for any \(\tau^*\). It is not possible to derive a

\(^{37}\) Once more, since \(G / R = 1\) is not a solution, the algebraic simplifications from Appendix A cannot be reproduced 
here, and the cubic formula yield solutions that are too unwieldy to be tractable.

\(^{38}\) An observed \(\tilde{p}_{t-1}\) implies \(\tilde{\alpha} = \tilde{p}_{t-1} / \tilde{G}^{-1}\). If no types have sold, \(\tilde{\alpha} = \alpha_{n,0}\), and the relevant cdf \(F_{n,0}(\tilde{\alpha})\) is 
\(\Pr[\alpha_{0,0} \leq \tilde{\alpha}] = \Pr[\tilde{\alpha} > (1 + \tilde{\alpha}^{-1} - \tilde{\alpha}) = (\tilde{\alpha} - (1 + \tilde{\alpha}^{-1} + \tilde{\alpha}) / (2 \tilde{\alpha})\), with \(f_{n,0}(\tilde{\alpha}) = (1 + \tilde{\alpha}^{-2}) / (2 \tilde{\alpha})\). If \(\tilde{\alpha} = \alpha_{n,1}\), we have 
\(F_{n,1}(\tilde{\alpha}) = (\tilde{\alpha} - [N / (N-1)]^{-1} + \tilde{\alpha}) / (2 \tilde{\alpha})\), and \(f_{n,1}(\tilde{\alpha}) = [N / (N-1)]^{-1} / (2 \tilde{\alpha})\). 
Given this, \(\mu_{n,0}(n-1) = [\phi(n-1) / f_{n,0}(\tilde{\alpha})] / [f(n-1) / f_{n,1}(\tilde{\alpha}) + \phi(n) / f_{n,0}(\tilde{\alpha})]\), and \(\mu_{n,0}(n) = 1 - \mu_{n,1}(n-1)\).
tractable expression for the equivalent of threshold $\Gamma^*$, but it is possible, however, to solve (AD.7) for $\tau^*$ in order to find the lower bound on bubble duration $\underline{\tau}^*$.

- If $p_{t-3}$, $p_{t-2}$, and $p_{t-1}$ are medium, high and medium, the analog of (37) is

$$\frac{\alpha_{0,2} + e^{-\lambda} \alpha_{0,1}}{1 + e^{-\lambda}} > \frac{\alpha \left( \frac{G}{R} \right)^{-(\tau^*+2)} + e^{-\lambda} \left[ \pi_{t,0} \alpha \left( \frac{G}{R} \right)^{-(\tau^*+1)} + (1 - \pi_{t,0}) \alpha_{1,1} \frac{G}{R} \right]}{1 + e^{-\lambda}},$$

(AD.8)

Here, the probability that $t_0 = n-1$ remains $1/(1+e^{-\lambda})$, as the agent knows that— even if $t_0 = n-1$—since $p_{t-2}$ is high, no types sold at $t-1$. That is, the medium $p_{t-1}$ must be due to a high $\varepsilon_{t-1}$. Thus, $\alpha_{0,2}$ captures the fact that, if $t_0 = n-1$, two types will sell at $t$. The analog of threshold $(1 + e^\lambda)/(1 - \pi)$ is $(e^\lambda \alpha_{0,2} + \alpha_{0,1}) / [(\alpha_{1,1} - \pi_{t,0})]$. For $G/R$ above this level there is no equilibrium, as agents are unwilling to sell for any $\tau^*$. As above, it is not possible to derive a tractable expression for the equivalent of threshold $\Gamma^*$, but it is possible, to solve (AD.8) for $\tau^*$ to derive the lower bound on bubble duration $\underline{\tau}^*$.

(d) By Lemma 4, if (39)-(40) hold, type-$n$ agents are willing to wait at $t = n + \tau^* - 1$ with medium $p_{t-1}$ and $d+1$ high prices before $t-1$. These prices imply that $\text{supp}_{n,t}(t_0)$ equals $\{n-(d+2),\ldots,n\}$. As before, a sufficient condition for type-$n$ agents to be willing to wait is that $W_d$, the expected return on the bubbly asset if $t_0 = n$ is large enough to make waiting optimal even if the price fell to zero for all other values of $t_0$. That is,

$$1 < \frac{e^{-\lambda(d+2)}(N-1)/N}{1 + (e^{-\lambda} + \ldots + e^{-\lambda(d+2)})(N-1)/N} W_d.$$  

(AD.9)

Compared to (39), the probability that $t_0 = n$ in (AD.9) is lower, since a medium price is more likely for $t_0 = n-(d+2)$ than for any other admissible value of $t_0$. This is due to the fact that, if $t_0 = n-(d+2)$, one type would have sold at $t-1$, whereas for other values no types would have sold. The expected return $W_d$ is now given by
\[ W_d = \left\{ \sum_{s=1}^{d} (\pi_{H|0}(1-\theta))^{s-1} (1-\pi_{H|0}) \alpha_{0,s} + \pi_{H|0} \tilde{\alpha} \left( \frac{G}{R} \right)^s \right\} + \\
+ \left[ \pi_{H|0}(1-\theta) \right]^d (1-\pi_{H|0}) \alpha_{0,d+1} + \pi_{H|0} \tilde{\alpha} \alpha_{0,1} + \pi_{H|0}(1-\theta) \alpha_{0,1} \left( \frac{G}{R} \right)^{d+1}, \quad (AD.10) \]

where \( \tilde{\alpha} \), which reflects the expected price impact of time-\( t \) sales, is given by

\[ \tilde{\alpha} = \alpha_{0,d+1} + \left[ \alpha_{0,d+1} e^{-\lambda} + \alpha_{0,d} e^{-2\lambda} + \cdots + \alpha_{0,1} e^{-\lambda(d+1)} + \alpha e^{-\lambda(d+2)} \right] \frac{(N-1)/N}{1 + (e^{-\lambda} + \cdots + e^{-\lambda(d+2)})(N-1)/N}. \]

Numerator divided by \( \tilde{\alpha} \) in (AD.10) capture price effects of sales at \( t+1, \ldots, t+(d+1) \).

Details notwithstanding, we can still glean from (AD.10) that \( W_d \) is greater than

\( (\alpha_{0,1}/\tilde{\alpha}) \pi_{H|0}(1-\theta) G/R \) \( \uparrow \). From here, by the same argument from the proof of Lemma 4, it follows that if \( \pi_{H|0}(1-\theta) G/R > e^\lambda \), (AD.9) must hold for sufficiently large \( d \). Finally, as in the noiseless case, \( e^\lambda < G/R \) suffices to preclude preemptive sales before signals arrive.

**References**


Moinas, S., and S. Pouget, *Bubbles and Irrational Bubbles: An Experiment*, 2009, Manuscript, University of Touluse, IAD and Toulouse School of Economics.


