A Robust Model of Bubbles with Multidimensional Uncertainty

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Abstract

One prominent theory of asset-price bubbles captures the notion of a greater fool’s bubble, where asymmetrically informed investors buy overvalued assets hoping to sell at a profit before the crash. However, existing models assume that prices are unresponsive to sales, that prices must satisfy certain parameter restrictions exactly, or that not all agents are rational. These assumptions have raised questions regarding the robustness of the results. To address these issues, I build a model with multidimensional uncertainty and market-clearing prices. In my model, noisy prices allow some agents to exit the market unnoticed, leading to bubble-riding behavior. In an extension, I also present a version of the model where all agents are rational, thereby demonstrating that the mechanism generating bubbles does not necessarily hinge on irrationality. This paper thus supports and advances previous theory by generating bubbles that are robust to the aforementioned critiques.

Keywords: Bubbles, Coordination, Noisy Prices
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1 Introduction

The existence of bubbles—defined as situations where the price of an asset differs from its fundamental value—has historically been difficult to reconcile with standard economic theory. As shown by Milgrom and Stokey (1982), Tirole (1982), Santos and Woodford (1997), and others, in a wide variety of environments, bubbles are inconsistent with equilibrium. For instance, in finite horizon models, bubbles are typically ruled out by backward induction, and in infinite horizon models, by transversality conditions. Furthermore, in reality, it is very difficult to identify bubbles even after the fact, because booms and busts can also be due to changes in fundamentals.1 Given these issues, it is not surprising that bubbles have traditionally been absent from most asset pricing theories, which are based on the efficient markets hypothesis (Fama (1965)), by which prices reflect all public information about fundamentals.

In the last few years, however, boom-bust episodes in asset markets—in the United States and other countries—have led to a great surge of interest in bubbles. Both the boom and bust in technology stocks of the late 1990s and the housing episode of the 2000s have been widely interpreted as bubbles in both media reports and policy discussions (see, for instance, Federal Reserve Chairman Bernanke’s (2010) speech on housing).2 At the same time, the academic literature on bubbles has also blossomed. On the empirical side, several papers have provided evidence supporting the view that bubbles are relevant. For example, Brunnermeier and Nagel (2004) document that, in the late 1990s, many hedge funds successfully timed the market, investing heavily in dotcom stocks during the boom, and selling before the crash. Moreover, there is an experimental literature (e.g., Smith et al. (1988), Lei et al. (2001)), which has found consistent and pervasive evidence of bubbles in the laboratory.

On the theoretical front, one strand of papers (including Caballero and Krishnamurthy (2006), Fahri and Tirole (2009), Ventura (2003), and others) has built on Tirole’s (1985) seminal work on bubbles and overlapping generations. In these models, due to a shortage of stores of value, bubbles fulfill a social need and improve efficiency, much like money improves efficiency in Samuelson (1958). These models often focus on steady states where bubbles last forever, either literally, or—in the case of stochastic bursting—in expected value.

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1 As Garber (2001) shows, even in episodes considered ‘textbook’ examples of bubbles, like the Dutch Tulipmania of 1634-37 or the South Sea Bubble in 1720, fundamentals-based interpretations cannot be ruled out.
2 While opinions vary, the boom-bust episode in the oil market in 2008 and the current boom in gold prices have also been interpreted as bubbles by many observers.
Another strand of literature (including Abreu and Brunnermeier (2003) (AB henceforth), Allen, et al. (1993), Conlon (2004), Harrison and Kreps (1978), Scheinkman and Xiong (2003), and others) focuses on asymmetric information and short sales constraints. Unlike overlapping generations models, where the pattern of trade is determined by agents’ ages, the information-driven trades in these models have a more speculative flavor. To avoid impossibility results à la Tirole (1982) and Milgrom and Stokey (1982), these models introduce a variety of features, such as heterogenous priors/heterogenous state-contingent marginal utilities, lack of common knowledge, behavioral traders and overconfidence. Within this second strand of literature, several models capture the idea of a ‘greater fool’s bubble’, where rational agents buy assets that they know are overvalued, attempting to time the market and sell at a profit to a ‘greater fool’ before the impending crash. In particular, the AB, Allen et al. (1993) and Conlon (2004) models fit this description remarkably well. In AB, for instance, rational agents hold a rapidly appreciating asset, and at some point observe a private signal revealing that it is overvalued. However, they do not know when others observe the signal. In equilibrium, some sell before the crash and make profits, and others suffer losses. Even so, if the probability of being in the ‘lucky’ group and the growth rate of the bubble are high enough, agents willingly invest in the bubble for some time, knowing that they are holding overvalued assets. While details differ in Allen et al. (1993) and Conlon (2004), the core ideas are similar, and involve agents playing a market timing game against each other.

The notion of a fool’s bubble is intriguing, because it captures, with a simple story, the idea of a bubble as a dramatic boom-bust cycle driven by speculation. However, existing models of fool’s bubbles are subject to criticism due to the assumptions they make to prevent prices from revealing private information. In Allen et al. (1993) and Conlon (2004), parameters, such as probabilities of different states of the world and dividends, must satisfy exact proportions. A small change in one parameter, holding others constant, makes bubbles collapse. In AB, bubbles

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3 Further approaches to modeling bubbles, which have also been influential, have focused on issues such as agency problems (Allen and Gorton (1993), Allen and Gale (2000), Barlevy (2008)), noise agent risk (Delong, et al (1990)), solvency constraints (Kocherlakota (2008)), and others.

4 As AB note, models of fool’s bubbles directly contradict the efficient markets hypothesis, since rational investors buy into the bubble, instead of attacking it. In practice, the Wall Street adage ‘The market can stay irrational longer than I can stay solvent’ warns about the danger of attacking bubbles. A dramatic example of this was the meltdown of the hedge fund managed by Michael Berger. According to a Bloomberg report, Mr. Berger bet against technology stocks in the mid and late 1990s. Amid rising prices and horrific losses, he began falsifying performance reports, hoping for the bubble to burst soon. The fraud was discovered in January of 2000, shortly before the crash.
are robust to small changes in parameters, but prices must be independent of selling pressure in the following sense. During the interval of time where ‘lucky’ agents (i.e., those who exit before the crash) are selling, the price cannot reflect these sales. Instead, it must continue to grow as if nobody was selling. The price reacts, by abruptly collapsing, only when sales reach a certain threshold. As AB note, this “invisibility-of-sales” assumption is indispensable to generate bubbles. Without it, prices would reveal private information as soon as sales began, effectively reducing the measure of agents who can sell before the crash to zero. Consequently, agents would sell as soon as they observed the ‘overvaluation’ signal and there would be no bubble.

AB, Allen et al. (1993) and Conlon (2004) also coincide in conjecturing that, in a model with multidimensional uncertainty, it could be possible to generate bubbles without these assumptions. Following this hint, I construct a discrete-time model based on AB, in which market-clearing prices reflect supply and demand monotonically, but demand has a noisy and unobservable component that makes the inference from prices imperfect. In the model’s basic version, I follow AB in assuming that bubble growth is fueled by demand from behavioral agents, who invest increasing amounts of resources into the bubble as their enthusiasm grows. However, I also assume that the precise amount by which behavioral demand shifts upward in a given period is unpredictable, due to some random variability which can be interpreted as preference or liquidity shocks hitting behavioral agents. In this environment, sales from rational agents tend to depress prices, but other rational agents who have not sold cannot distinguish whether depressed prices are due to noise or the fact that sales have begun.

Analyzing the model, I first consider the case without noise, where as soon as one type sells (a type includes those who observed the overvaluation signal in a given period), all uncertainty is revealed, triggering a crash in the next period. I derive a parameter restriction such that, without noise, agents sell immediately upon observing the signal. Maintaining this restriction, which is essentially an upper bound on the growth rate of the bubble, I increase the amount of noise so that it can conceal sales of one type, but not more. Prices (relative to trend) can then fall into one of three categories. High prices reveal with certainty that nobody has sold, medium prices reveal that sales may or may not have begun, and low prices reveal with certainty that sales have begun, thereby triggering the crash. This specification of noise, coupled with Markov strategies, is simple and analytically tractable. The strategies I consider are Markovian in the sense that agents’ sell-or-wait choices depend only on how much time has elapsed since
observing the signal and on whether the last price observed was high, medium, or low. Restricting attention to this class of strategies, I show that there is always a nonempty region in the parameter space where bubbles of arbitrary length arise in equilibrium.

In bubbly equilibria, as intuition suggests, agents wait the longest before selling if prices are high. But they also wait for a substantial number of periods if prices are medium. Given that different prices elicit different selling behavior and that prices reflect a mix of noise and sales, agents update their beliefs every period. Thus, as in Kai and Conlon (2008), private information slowly “leaks” through prices. To generate bubbles, the key challenge is ensuring that, at all times and for any prices, agents who are supposed to wait (according to equilibrium strategies) find it optimal to do so, despite the obvious crash risk. Sufficient conditions to ensure this can roughly be stated as follows. First, there must be enough noise to make it likely that, when the first type sells, the price does not reveal these sales. Second, the opportunity cost of selling preemptively after a medium price must be large. In turn, the opportunity cost is high when the bubble grows quickly and when noise-driven price slowdowns are relatively infrequent.

I also extend the model in two directions. First, I relax the assumption—made in the basic analysis for simplicity—that agents cannot reenter the market after selling, and show that, although some of the basic-analysis equilibria vanish, the overall picture remains unchanged, and bubbles with Markovian strategies still arise. In a second extension, I depart from AB by assuming that there are no behavioral agents in the model, and instead assume that all agents are rational. In that version/reinterpretation of the model, rational agents obtain an endowment every period, which they invest in the bubble as long as they believe that it will continue to grow. The shares bought with the new endowments are supplied by agents who are forced to sell by a preference shock, which captures lifecycle events or liquidity needs. If the mass of agents hit by the shock is random and unobservable, a price slowdown due to early types exiting the market cannot be distinguished from a slowdown due to a greater-than-usual mass of agents being hit by the shock. The informational role of prices and the mechanism generating bubbles are thus identical to the model with behavioral agents. This suggests that the AB model does not crucially hinge on irrationality. Once rational agents receive an inflow of resources to fuel bubble growth, and there is some exogenous reason for trading, behavioral agents are no longer necessary.

In sum, this paper extends AB by introducing noise and allowing prices to be market clearing. I show that speculative bubbles arise in a model that is robust to small changes in
parameters, and where prices always reflect selling pressure. This overcomes the critiques that affected previous models, and in this way, advances the theory of “greater fool’s bubbles”. Furthermore, by presenting a version of the model where all agents are rational, I show that the mechanism generating bubbles in AB need not hinge on the presence of behavioral agents.

The paper is organized as follows. In sections 2 and 3, respectively, I describe the model and define equilibrium. In section 4, I illustrate how bubbles arise in the basic analysis. In section 5, I present extensions and, in section 6, I conclude.

2 The Model
2.1 The Environment
To facilitate comparison with the literature, I follow AB closely. Besides discrete time, the only new features that I introduce are market-clearing prices and noise.

Time is discrete and infinite with periods labeled \( t = \ldots, -1, 0, 1, \ldots \). There are two assets, a risk-free asset with exogenous gross return \( R > 1 \), and a risky asset. The supply of the risky asset is fixed at \( 1 + t \), with \( t > 0 \), and its price at time \( t \) is denoted by \( p_t \). While \( t \leq 0 \), the risky asset’s fundamental value \( f_t \) and price \( p_t \) both equal \( R^t \). After time 0, a boom begins, as fundamental shocks cause \( f_t \) to grow at the average (gross) rate \( G > R \). Both \( f_t \) and \( p_t \) grow on average at this rate until time \( t_0 - 1 \). For \( t \geq t_0 \), \( f_t / f_{t-1} \) falls back to \( R \), and if \( p_t \) continues to grow faster than \( R \), a bubble arises. The bubble inflates until period \( T \geq t_0 \) and bursts in period \( T + 1 \), at which point equality between price and fundamental value is restored.\(^5\) The first period of overvaluation \( t_0 \) is geometrically distributed with probability function \( \phi \) given by

\[
\phi(t_0) = (e^\lambda - 1)e^{-\lambda t_0} \quad \text{for all } t_0 = 1, 2, \ldots,
\]

where \( \lambda > 0 \). As is well known, the expected value of \( t_0 \) is given by \( 1/(1 - e^{-\lambda}) \), and the greater \( \lambda \), the greater the probability of low values of \( t_0 \) relative to high values.

\(^5\) Kindleberger and Aliber (2005) argue that bubbles typically follow large fundamental shocks, or displacements, which cause dramatic shifts in prices. These price shifts are initially justified by fundamentals, but may turn into bubbles as markets overshoot. In keeping with this idea, AB mention episodes in stock markets after the arrival of new technologies (e.g., Internet in the 1990s, radio in the 1920s) as examples of bubbles. In these cases, prices rose dramatically, then crashed, and finally stabilized at a level higher than before the fundamental change but below the peak. In Doblas-Madrid (2008), I argue that the idea of a bubble as a temporary overreaction to fundamental events can also help to explain exchange rate overshooting in a series of currency crisis episodes.
There is a unit mass of rational, risk neutral agents with discount factor $1/R$. They do not observe $t_0$ exactly. Instead, every period from $t_0$ to $t_0 + N - 1$, a mass $1/N$ of rational agents observe a private signal revealing that the asset is overvalued, i.e., that $f_t$ is no longer growing at the rate $G$. This divides rational agents into $N$ types, indexed by $n = t_0, \ldots, t_0 + N - 1$. Conditional on $n$, the distribution of $t_0$ becomes

$$
\varphi(t_0 | n) = \begin{cases} 
\frac{e^{-\lambda t_0}}{e^{-\lambda (\max\{1,n-(N-1)\})} + \cdots + e^{-\lambda n}} & \text{if } \max\{1,n-(N-1)\} \leq t_0 \leq n \\
0 & \text{otherwise.}
\end{cases}
$$

Because agents observe $n$, but not $t_0$, they do not know whether they observe the signal early or late relative to others. As in AB, this uncertainty will play a key role in generating bubbles.

For all $n$ and $t$, short sales constraints limit $h_{nt}$, holdings of the risky asset by a type-$n$ agent at time $t$, between 0 and 1. Thus, aggregate rational holdings $H_t = (h_{0,t} + \cdots + h_{n+t,N-1})/N$ must also be in $[0,1]$ for all $t$. Initially, rational agents hold the maximum long position $h_{n,t} = 1$, $\forall t \leq 0$ and $n \in \{t_0, \ldots, t_0 + N - 1\}$. Agents will adjust their holdings over time as they observe prices and their signal. The timing of these sales will determine the duration of the bubble.

**Figure 1 — Timeline of events.**

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6 The case with $t_0 < N$ is exceptional, since types with $n < N$ know that they were not last to observe the signal. In fact, if $t_0$ happens to be 1, agents of type $n = 1$ know that $t_0 = n = 1$. 


In sum, as Figure 1 shows, there is a price boom that consists of a fundamental part for \( t \in \{1, \ldots, t_0 - 1\} \), and a bubble for \( t \in \{t_0, \ldots, T\} \). This last bubble period \( T \), as we will see, will be determined mainly by the timing of agents’ sales, but also partly by noise.

Rational agents coexist with behavioral agents, whose fuel the boom. Behavioral agents’ holdings of the risky asset at \( t \) are denoted by \( H_t^b \). While \( t \leq 0 \), \( H_t^b \) is constant at \( t > 0 \). I assume that, for all \( t \), \( H_t^b \) must be between a minimum \( 0 \) and a maximum \( t + \kappa \), where \( \kappa > 0 \) and \( t + \kappa < 1 \). As in AB, the assumption that \( \kappa < 1 \) captures the idea that behavioral agents cannot absorb all the shares that rational agents may sell. However, unlike in AB, in this model the bubble will burst before behavioral absorption capacity is exhausted. Specifically, prices will reveal information a mass \( \kappa \) of shares is sold to behavioral agents.

Initially, behavioral agents view both assets as perfect substitutes. Thus, letting \( f_t = R^t \) for all \( t \leq 0 \), behavioral demand for the risky asset while \( t \leq 0 \) is given by

\[
H_t^b = \begin{cases} 
0 & \text{if } p_t > f_t, \\
[0, t + \kappa] & \text{if } p_t = f_t, \\
t + \kappa & \text{if } p_t < f_t.
\end{cases}
\]

At \( t = 1 \), behavioral enthusiasm starts to rise. While \( 1 \leq t \leq t_0 - 1 \), growing optimism is justified by fundamentals, but at time \( t_0 \) it turns into ‘irrational exuberance’ which will last until time \( T \). In sum, while \( 1 \leq t \leq T \), behavioral demand for the risky asset is given by

\[
H_t^b = \max \left\{ t + \frac{G^t - p_t}{\alpha} + \epsilon_t, t + \kappa \right\}.
\]

In words, this demand is the sum of \( t \), a downward sloping term \((G^t - p_t) / \alpha \) (where \( \alpha > 0 \)), and a random term \( \epsilon_t \). As behavioral agents’ optimism grows, so do the resources they use to bid up the risky asset. Thus, demand shifts upward over time as \( G^t \) grows. But the size of each shift cannot be exactly predicted because of the noisy term \( \epsilon_t \). This term is uniform over \([-\bar{\epsilon}, \bar{\epsilon}]\) and represents preference/liquidity shocks affecting consumption needs or available resources at time \( t \). Since price changes due to rational sales are indistinguishable from price changes due to noise, \( \epsilon_t \) plays a key role in the model, by allowing some rational agents to sell without being noticed. Finally, as Figure 2 below shows, behavioral demand can never exceed a maximum \( t + \kappa \).
Figure 2 — During the boom, behavioral demand for the risky asset shifts upward. But the exact position of the demand curve at a given time $t$ depends on the realization of $\varepsilon_t$. Given $\bar{\varepsilon}$, the shaded areas denote the possible locations for the demand curves at times $t$ and $t+1$.

When the bubble bursts, I assume that the realization of $t_o$ is revealed to rational and behavioral agents. This implies that, for $t \geq T+1$, fundamental value is also known and given by

$$f_t = \left( \frac{G}{R} \right)^{t_o-1} R^t.$$  \hspace{1cm} (5)

When $t \geq T+1$, just like when $t \leq 0$, behavioral agents view both assets as perfect substitutes. Hence, for $t \geq T+1$, behavioral demand $H^B_t$ becomes (3) with fundamental value given by (5).

I next turn to the within-period timing of shocks and actions, which I summarize below in Figure 3. Type-$n$ agents start period $t$ with $h_{n,t-1}$ shares of the risky asset. The period proceeds in two steps. In Step 1, if $t \in \{t_o, \ldots, t_o + N-1\}$, type-$t$ agents observe signals. Type-$n$ agents ($\forall n$) choose $h_{n,t} \in [0,1]$ knowing past prices $p^{t-1} = \{p_{t-2}, \ldots, p_{t+1}\}$ and, if $t \geq n$, $n$. Also in Step 1, $\varepsilon_t$ is realized. In Step 2, orders are combined and trades executed at price $p_t$, determined by

$$H_t + H^B_t = 1 + t_s \quad \text{for all } t.$$  \hspace{1cm} (6)

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7 In the equilibria we will study later, prices $p_t, \ldots, p_T$ will often reveal the value of $t_o$ exactly. However, in some instances, this will hold only approximately, and a few values of $t_o$ will be consistent with prices. Nevertheless, to avoid burdening the reader with inessential complications, I will assume that $t_o$ is exactly revealed. Generalizing (5) to take the latter cases into account adds much complication in exchange for little or no insight.
As in Kyle (1985), this timing is Cournot-like in that agents submit orders to buy or sell before knowing others’ orders or the price, which is observed after all orders are combined. Naturally, given this supposition, a period in the model shall be interpreted as being short.\footnote{Rational agents are thus placing market orders, which they know will be executed, but they do not know at what price. (See Brunnermeier (2000) for a description of market orders, limit orders, etc.) In actual markets, orders are typically executed at prices that are close to the most recent prices observed by agents at the time submission. Only in situations which practitioners call fast markets, which include market crashes, the price may soar or plummet during the time interval between order submission and execution.}

Substituting $B_t$ and $t_H$ into (6), and assuming that $0 < B_t < t + \kappa$, we can solve for $p_i$.

Shifts in $H_i$ demarcate three phases, an initial phase for $t \leq 0$, a boom phase for $1 \leq t \leq T$, and a post-crash phase for $t \geq T + 1$. In the initial phase, behavioral agents hold $H_i = t$ only if $p_i = R'$. At these prices, even if they do not expect the boom, rational agents are willing to hold $H_i = 1$.

During boom periods, the market clearing price is given by

$$p_i = G' + \alpha (e_i - (1 - H_i)).$$ (7)

Before sales begin, i.e., while $H_i = 1$, it must be that $p_i \in [G' - \alpha \bar{e}, G' + \alpha \bar{e}]$. When $z > 0$ types sell, $H_i = 1 - z / N$, and $p_i \in [G' + \alpha (-\bar{e} - z / N), G' + \alpha (\bar{e} - z / N)]$. If $2\bar{e} > z / N$, the two ranges overlap, as $G' - \alpha \bar{e} < G' + \alpha (\bar{e} - z / N)$. Therefore, if $z / N < 2\bar{e}$, even with $z > 0$ types selling, there is positive probability that $p_i$ is consistent with $H_i = 1$. It follows that noise can hide sales by at most $\bar{e}$ types, with $\bar{e}$ given by

$$\bar{e} = \max \{z \mid z \in \mathbb{N}_0 \text{ and } z < 2\bar{e} N \},$$ (8)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{within-period-timing.png}
\caption{Within-period timing.}
\end{figure}
where \( \mathbb{N}_0 \) is the set of natural numbers, including zero.

Depending on its implications about how many types could be out of the market by time \( t \), the price \( p_t \) falls into one of \( \bar{z} + 2 \) categories \( c(p_t) \in \{0, \ldots, \bar{z} + 1\} \), defined as follows:

\[
c(p_t) = \begin{cases} 
  z & \text{if } G' + \alpha(\bar{c}-(z+1)/N) < p_t \leq G' + \alpha(\bar{c} - z / N) \text{ for } 0 \leq z \leq \bar{z} - 1 \\
  \bar{z} & \text{if } G' - \alpha \bar{c} \leq p_t \leq G' + \alpha(\bar{c} - \bar{z} / N) \\
  \bar{z} + 1 & \text{if } p_t < G' - \alpha \bar{c}.
\end{cases}
\]  

(9)

If \( c(p_t) = z \leq \bar{z} \), \( p_t \) is consistent with anywhere from 0 to \( z \) types having sold by time \( t \).

If, for instance, \( c(p_t) = 0 \), \( p_t > G' + \alpha(\bar{c} - 1/N) \) reveals that nobody has sold. Whenever \( c(p_t) = \bar{z} \), \( p_t \in [G' - \alpha \bar{c}, G' + \alpha(\bar{c} - \bar{z} / N)] \) is consistent with sales by 0, 1, \ldots, \( \bar{z} - 1 \), or \( \bar{z} \) types.

Finally, if \( c(p_t) = \bar{z} + 1 \), \( H_t = 1 \) is no longer possible, and \( p_t \) reveals that sales have begun.

The last boom period \( T \) is the first period with \( c(p_t) = \bar{z} + 1 \). The low price reveals to behavioral agents that sales have begun, and thus, \( H_t^B \) for \( t \geq T + 1 \) is (3) with \( f_t \) given by (5).\(^9\)

Post-crash, all agents are indifferent between any admissible holdings if \( p_t = f_t \), \( \forall t \geq T + 1 \).

### 2.2 Remarks

#### 2.2.1 Trading Volume

While the most striking feature of a bubble is a dramatic boom and bust, other notable features include price volatility and large and variable trading volume. In the model, I focus on price volatility as necessary for bubbles to arise and, for simplicity, exclude volume. But the model presented above can be thought of as a reduced version of a more complete model where volume is observed, but it is so volatile that it is very unlikely to reveal rational sales before the price. This could be formalized, for example, by assuming that volume \( \sigma_t \) has two components:

\[
\sigma_t = \sigma_t^B + \sigma_t^R,
\]

where \( \sigma_t^B \) is behavioral volume, i.e., shares behavioral agents sell to each other, and \( \sigma_t^R \) is rational volume, i.e., shares traded by rational agents. Since total volume \( \sigma_t \) is observed (at the same time as the price) but its components are not, the informational role of volume is similar to

\(^9\) Alternatively, I could assume that behavioral demand does not collapse after a low \( p_t \). Then, at \( T + 1 \), rational agents would try to sell more shares than behavioral agents could buy, leading to rationing. Some shares would sell at the pre-crash price, others at the post-crash price. This would complicate algebraic expressions and affect results quantitatively, but not qualitatively.
that of the price. Just like $p_t$ would reveal rational sales in the absence of $e_t$, $\sigma_t$ would reveal sales if $\sigma_t^b$ was not random. Naturally, in this more complete model, agents would draw their inference about rational sales, and hence about $t_0$, taking both price and volume into account. The model presented in this paper could thus be seen as based on a model where volume is so noisy that agents simply disregard it and focus on the price.

In sum, in this model there is, implicitly, a relationship between bubbles and volume, although the latter does not emerge endogenously. Instead, large and variable trading volume is a necessary condition for bubbles to arise.

2.2.2 Behavioral Resources and Maximum Bubble Duration

The assumption that behavioral demand during the boom is given by (4) relies on behavioral agents using rapidly growing resources to bid up the risky asset. In reality, an accelerated inflow of resources can plausibly be sustained for an extended period of time. As Kindleberger and Aliber (2005) remark, in a typical bubble, this inflow is often magnified by the arrival of novice investors, as well as by an expansion of credit. In the long run, however, this accelerated inflow must eventually slow down. This same issue arises in AB, where behavioral agents are assumed able to buy a certain mass of shares no matter how high the price becomes. However, AB impose an exogenous bound on bubble duration $\bar{T}$. Given the finding that, in experiments, bubbles often burst precisely when agents run out of money to bid prices up, AB’s exogenous cap may be seen as related to the eventual exhaustion of behavioral resources. While I am mindful of these issues, I focus on endogenous bursting due to rational sales, implicitly assuming that $T < t_0 + \bar{T}$, i.e., that the endogenous crash arrives before the limit to bubble duration (perhaps due to exhaustion of behavioral resources) is reached.

3 Equilibrium

I will define equilibrium and carry out the basic analysis under the restriction—which I will relax in section 5—that agents do not reenter the market after selling. Specifically, I assume:

**Restriction I - No Reentry** For all $n$ and $t$, $h_{n,t} \leq h_{n,t-1}$.

The equilibrium concept is Perfect Bayesian Equilibrium (PBE), consisting of strategies and beliefs $(n, t) \in [0, h_{n,t-1}]$ for all $t$, strategy $h_{n,t}$ specifies holdings $h_{n,t} \in [0, h_{n,t-1}]$ given past prices $p_{t-1}$ and, if $t \geq n$, $n$. A belief $\mu_{n,t}(t_0)$ is a probability distribution over values of $t_0$, also
conditional on \( p^{t-1} \) and, if \( t \geq n \), \( n \). In equilibrium, for all \( n \), \( h_{n,t} \) is optimal given \( \mu_{n,t}(t_0) \), and \( \mu_{n,t}(t_0) \) is consistent with the equilibrium strategies.

To be consistent with a strategy profile, a belief \( \mu_{n,t}(t_0) \) must assign positive probability only to values of \( t_0 \) not ruled out by that strategy profile, given \( p^{t-1} \) and, if \( t \geq n \), \( n \). The set of values of \( t_0 \) that are not discarded is the support of \( t_0 \), denoted by \( \text{supp}_{n,t}(t_0) \). The signal \( n \) implies that \( \text{supp}_{n,t}(t_0) \subseteq \{ \max \{1, n-(N-1)\}, \ldots, n \} \). And prices and strategies rule out values of \( t_0 \) as follows. For each \( \tau \in \{1, \ldots, t-1\} \), given \( p_\tau \) and (7), agents know that \( \varepsilon_\tau \sim (1-H_\tau) \). Given this, and given strategies \( h_{n,t} \), each value of \( t_0 \) implies specific values of \( H_\tau \) and \( \varepsilon_\tau \) for all \( \tau \). A value of \( t_0 \) is discarded if it implies \( |\varepsilon_\tau| > \overline{\varepsilon} \) for some \( \tau \). Having determined \( \text{supp}_{n,t}(t_0) \), and since \( \varepsilon_\tau \) is uniform, Bayes’ rule implies that the likelihood of each value of \( t_0 \) in the support is

\[
\mu_{n,t}(t_0) = \frac{\phi(t_0)}{\sum_{t_0 \in \text{supp}_{n,t}(t_0)} \phi(t_0)} \quad \text{for all } t_0 \in \text{supp}_{n,t}(t_0). \tag{10}
\]

The equilibrium strategy \( h_{n,t} \) maximizes expected discounted utility \( E_{n,t}[V(h_{n,t-1})] \), where \( E_{n,t} \) denotes expectation given the equilibrium belief \( \mu_{n,t}(t_0) \). With risk neutrality, \( E_{n,t}[V(h_{n,t-1})] \) is the product of \( h_{n,t-1} \) times a per-share expected utility value \( E_{n,t}[v_{n,t}] \). To find \( E_{n,t}[v_{n,t}] \), note that, at each time \( t \), a type-\( n \) agent chooses to sell or wait. Under no-reentry, the expected utility from selling a share of the risky asset is \( E_{n,t} p_t \). On the other hand, the value (in expected utility terms) of that share if the agent waits until \( t+1 \) will be \( E_{n,t}[v_{n,t+1}] \). (This latter value can be computed via backward induction, since the utility value of post-crash holdings is the post-crash price, and—as discussed in Remark 2.2.2—in the cases of interest, the bubble will always burst within finite time.) In sum, for all \( n \) and \( t \), the equilibrium \( h_{n,t} \) solves

\[
E_{n,t}[v_{n,t}, h_{n,t-1}] = \max_{h_{n,t} \in [0, h_{n,t-1}]} E_{n,t} \left[ p_t h_{n,t-1} - p_t h_{n,t} + \frac{v_{n,t+1}}{R} h_{n,t} \right]. \tag{11}
\]

Due to risk neutrality, this program’s solution is simply to sell everything, setting \( h_{n,t} = 0 \), if \( E_{n,t}[p_t - v_{n,t+1}/R] \geq 0 \), and to wait, keeping \( h_{n,t} = h_{n,t-1} \), otherwise.
4 Equilibria with Bubbles: Basic Analysis

Since prices fall as agents sell, and, in turn, agents sell if they anticipate price declines, there is much scope for multiple equilibria. A given price realization may serve as a coordination device, and either trigger sales or be ignored, with both responses being individually optimal if and only if others respond to that price in the same way. In order to reduce the set of equilibria, in addition to Restriction I, I constrain strategies further by assuming the following.

Restriction II - Markov For all $n$ and for all $t \in \{2, \ldots, T\}$, the price history $p^{t-1}$ affects $h_{n,t}$ only through its most recent price $p_{t-1}$.

Restriction III - Price Discretization For all $n$, for all $t \in \{2, \ldots, T\}$, and for any two prices $\hat{p}_{t-1}$, $\tilde{p}_{t-1}$ with $c(\hat{p}_{t-1}) = c(\tilde{p}_{t-1})$, $h_{n,t}$ is the same after $\hat{p}_{t-1}$ or $\tilde{p}_{t-1}$.

Restriction IV - Symmetry For any types $n, m$, and any $\tau \geq 0$ with $\min\{n, m\} + \tau \geq 2$, $h_{n+\tau} = h_{m+\tau}$ whenever $c(p_{n+\tau-1}) = c(p_{m+\tau-1})$.

By Restriction II, actions $h_{n,t}$ depend only on the most recent price (and the signal if $t \geq n$). However, previous prices are not ignored, as beliefs $\mu_{n,t}$ depend on the full price history. Restriction III mitigates the multiplicity problem by supposing that agents view all prices within a category as equivalent, i.e., that agents respond to two prices differently only if they have different implications about how many types may have sold. Finally, the Restriction IV states that, besides $p_{t-1}$, the sell-or-wait choice $h_{n,t}$ depends on time since observing the signal $t-n$, rather than on the signal $n$ itself.

With these restrictions in place, I begin the analysis in subsection 4.1 by considering the case without noise. This is necessary because, with discrete time, a mass $1/N$ of agents sell before the crash, even if $\bar{\epsilon} = 0$. For very high $G/R$, the possibility of being among these first sellers may entice agents to try to ride the bubble. Thus, to properly analyze the role of noise, I derive a parameter restriction such that, if $2\bar{\epsilon} < 1/N$, the only equilibrium is one where agents sell immediately upon observing the signal. This restriction is simply that $\epsilon^2 < G/R < \Gamma$, where the threshold $\Gamma$ is a function of $\lambda$. Maintaining this restriction, in subsection 4.2, I illustrate how
bubbles arise once \(1/N < 2\varepsilon < 2/N\), i.e., when noise can hide sales of one type, but not two. In this case, which is quite tractable, I show that it is possible to construct bubbles.

Finally, in what follows, I will maintain two assumptions. First, I will focus on situations where rational agents sell less than \(\kappa\) shares before the crash, and the bubble bursts due to information revelation before behavioral absorption capacity is exhausted. Second, I assume that \(\alpha\) is positive, but close to zero. This implies that, while \(t \in \{1, \ldots, T\}\), \(p_t \approx G'\). In words, pre-crash price fluctuations matter only because of their informational content, and not because of their revenue effects, which are negligible. This greatly simplifies the formulas that govern equilibrium behavior and makes them easier to interpret, without qualitatively affecting results.

### 4.1 The Case where Noise Can Hide No Sales

Assume that \(2\varepsilon < 1/N\), so that \(\varepsilon = 0\). Then, sales are always detected, as \(G' + \alpha(\varepsilon - 1/N)\), the highest possible price if one type sells, is below \(G' - \alpha \varepsilon\), the highest price if nobody has sold. Given \(I-IV\), and the fact that \(c(p_t)\) must be zero for \(t = 1, \ldots, T - 1\), strategies depend solely on \(t - n\), i.e., on time since observing the signal. Specifically, consider

**Strategy Profile 1** — For any \(n \geq 1\), while \(t \leq T\), the strategy of a type-\(n\) agent is:

\[
h_{n,t} = \begin{cases} 
1 & \text{if } t < n + \tau^* \\
0 & \text{if } t \geq n + \tau^*,
\end{cases}
\]  

(12)

with \(\tau^* \geq 0\). Once \(t \geq T + 1\), the strategy is to maintain \(h_{n,t} = h_{n,t-1}\).

In words, the strategy is as follows: Prior to the crash, agents’ plan is to hold \(h_{n,t} = 1\) until \(\tau^*\) periods have passed since observing the signal, and then sell out. Post-crash, agents maintain whatever holdings \(h_{n,T}\) they had at the end of time \(T\). In equilibrium, type-\(t_0\) agents sell at \(t_0 + \tau^*\) (assume \(1/N < \kappa\), so behavioral agents can buy these shares), \(p_{t_0 + \tau^*}\) reveals these sales, and the crash happens at time \(T + 1 = t_0 + \tau^* + 1\).

In Proposition 1, I show that, if \(e^3 < G/R < \Gamma\) (where \(\Gamma = e^3(1 + \sqrt{1 + 4e^{-3}})/2\)), agents must sell as soon as they observe the signal. That is, agents are willing to follow (12) if and only if \(\tau^* = 0\).\(^{10}\) Inequality \(e^3 < G/R\) ensures that agents do not sell before observing the signal,

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10 Technically, \(\tau^* = \infty\) is also an equilibrium, albeit an implausible one in light of Remark 2.2.2.
while inequality $G/R < \Gamma$ ensures that they sell as soon as they observe it. If $G/R < \Gamma$, waiting for $\tau^* \geq 1$ periods after observing the signal cannot be optimal, because the bubble is not growing fast enough to compensate for potential losses in the event of a crash.

**Proposition 1** If $e^\delta < G/R < \Gamma$, where $\Gamma = e^\delta (1 + \sqrt{1 + 4e^{-\delta}})/2$, *Strategy Profile 1* and its implied beliefs constitute an equilibrium if and only if $\tau^* = 0$.

**Proof** Pre-boom and post-crash, for any parameters and $\tau^* \geq 0$, rational agents are willing to follow (12). While $t \leq 0$, they are indifferent between any holdings of the risky asset, and thus, $h_{n,t} = 1$ is weakly optimal. Similarly, in post-crash periods $t \geq T + 1$, the risky and riskless assets are again perfect substitutes, and thus, it is weakly optimal for agents to maintain their holdings.

For boom periods $t \in \{1, \ldots, T\}$, the proof has two parts. First, I show that, if $e^\delta < G/R$, there is an equilibrium with strategies given by (12) and $\tau^* = 0$. Second, I show that, if $G/R < \Gamma$, there are no equilibria with strategies given by (12) and $\tau^* \geq 1$.

For the first part, note that, in equilibrium with $\tau^* = 0$, type-$n$ agents (i) find it optimal to sell in period $n$, and (ii) find it optimal not to sell before period $n$. To see why (i) holds, note that, at time $n (= n + \tau^*)$, a type-$n$ agent can infer that $t_0 = n$ from the fact that the bubble has not burst. She also knows that other type-$n$ agents are selling, and that $p_n$ will reveal these sales, causing a crash at $n+1$. Clearly, selling is optimal since the expected time-$n$ price $G^n$ exceeds the discounted post-crash price $G^{n+1}$ that she will get if she waits. Next, to see why $e^\delta < G/R$ implies (ii), consider a type-$n$ agent at $t < n$. If $t \leq 0$, she has no reason to sell, since $t_0$ cannot be $t$, and a crash at $t+1$ is impossible. If $t \geq 1$, $t_0$ could be $t$, and thus the bubble could burst at $t+1$. With $\text{supp}_{n,t}(t_0)$ given by $\{\tau | \tau \geq t\}$, the probability of a crash at $t+1$ is $\mu_{n,t}(t) = 1 - e^{-\delta}$. The agent can sell at a price $G^n$, or she can wait, in which case with probability $e^{-\delta}$ she will be able to sell at $t+1$ for a higher (discounted) price $G^{t+1}/R$ and with probability $1 - e^{-\delta}$ she will obtain the post-crash price. Even if the post-crash price is zero, if $1 < e^{-\delta}G/R$, waiting is optimal. Hence, $e^\delta < G/R$ suffices to rule out preemptive sales while $t < n$.

For the second part (i.e., showing that there are no equilibria with $\tau^* \geq 1$ if $G/R < \Gamma$), suppose, by means of contradiction, that there is such an equilibrium even though $G/R < \Gamma$. In
any equilibrium with \( \tau^* \geq 1 \), type-\( n \) agents must be willing to wait at all times \( t < n + \tau^* \), including at \( t = n + \tau^* - 1 \). If \( t = n + \tau^* - 1 \) and the bubble has not burst, type-\( n \) agents know that their type was either first or second to observe the signal, i.e., \( \text{supp}_{n,t} (t_0) = \{n-1, n\} \). In this situation, a type-\( n \) agent’s sell-or-wait trade-off is as follows. Selling preemptively at \( t \) yields \( G^{n+\tau^* - 1} \), while waiting yields the (discounted) post-crash price \( G^{n-2} R^{\tau^* + 1} \) if \( t_0 = n-1 \), and \( G^{n+\tau^*}/R \) if \( t_0 = n \). In sum, deviating from (12) by selling preemptively at \( t \) is optimal if

\[
1 > \frac{1}{1 + e^{-\lambda}} \left( \frac{G}{R} \right)^{-(\tau^*+1)} + \frac{e^{-\lambda} G}{1 + e^{-\lambda}} R. \tag{13}
\]

Since the right hand side is decreasing in \( \tau^* \), if (13) holds for \( \tau^* = 1 \), it also does for \( \tau^* > 1 \). In Appendix A, I show that \( 1 + e^{-\lambda} > (G/R)^2 + e^{-\lambda} G/R \) holds if \( 1 < G/R < \Gamma \), where \( \Gamma = (1 + \sqrt{1 + 4e^{-\lambda}}) e^{\lambda}/2 \). Thus, there are no equilibria with \( \tau^* \geq 1 \) and \( G/R < \Gamma \). Q.E.D.

### 4.2 The Case where Noise Can Hide One Sale

Maintaining the restriction that \( \varepsilon^* < G/R < \Gamma \), which precludes bubbles without noise, I now increase \( \overline{\varepsilon} \) and derive conditions under which bubbles arise. Despite restrictions I-IV, when \( \overline{\varepsilon} > 1/N \), multiple equilibria appear. Nevertheless, since all equilibria with long bubbles share certain features, the analysis points to a set of conditions that are necessary for bubbles. In bubbly equilibria, the higher prices are, the longer are agents willing to postpone their sales after observing the signal. Since different prices elicit different behavior, price fluctuations reveal information about the value of \( t_0 \), i.e., there is gradual informational leakage, as in Kai and Conlon (2008). For instance, a recovery after a price slowdown reveals that \( t_0 \) exceeds a certain threshold, since the slowdown would have triggered more sales if \( t_0 \) was lower. By contrast, a string of consecutive high prices is consistent with \( t_0 \) being quite low, in which case several types would be awaiting the next slowdown, ready to sell. Confidence in the bubble is thus strongest after a recovery and weakest when, after many high prices, there is a slowdown. \(^{11}\)

---

\(^{11}\) This is similar to the version of AB with exogenous price drops or sunspots, where recoveries after price drops also reveal information, and multiple equilibria are inevitable since price drops may trigger sales or be ignored. However, unlike in AB, here price drops can be due to noise or sales, and agents take this into account in their inference. Furthermore, because time is discrete in this model, if agents exit the market and reenter when they see that prices recover, they miss out on a nontrivial amount of profits.
support bubbles in equilibrium, agents must be willing to postpone sales for some time after observing their signal. During this time, it must be optimal for agents to wait even if they see a price slowdown. Roughly, sufficient conditions to rule out such preemptive sales can be stated as follows. First, there must be enough noise to imply a sizable probability that, when the first agents sell, the price does not reveal the sales. Second, agents who sell preemptively during a slowdown must forgo a large profit if the bubble continues to grow after they have sold. This forgone profit is large if price slowdowns are not too frequent and bubble growth is fast enough.

To make these ideas precise, consider the case where $1/N < 2\bar{\varepsilon} < 2/N$, so that noise can hide sales by one type, but not two.\(^\text{12}\) According to (9), since $\bar{\varepsilon} = 1$, a price $p_t$ can be in one of three categories $c(p_t) = 0$, 1 or 2, which I will refer to as high, medium and low, respectively. High prices exceed $G' + \alpha(\bar{\varepsilon} - 1/N)$, and thus reveal that nobody has sold. Low prices are under $G' - \alpha\bar{\varepsilon}$, and thus reveal that sales have begun. Medium prices are between these two thresholds, and are therefore consistent with nobody having sold and with one type having sold. Before sales begin ($H_t = 1$), $p_t$ is high with probability

$$\pi = \frac{1/N}{2\bar{\varepsilon}}$$

(14)

and medium with probability $1 - \pi$. With one type out of the market ($H_t = 1 - 1/N$), $p_t$ is low with probability $\pi$ and medium with probability $1 - \pi$. If at least two types sell, $p_t$ must be low. Also note that, since $1/N < 2\bar{\varepsilon} < 2/N$, $\pi \in (\frac{1}{2}, 1)$.

Following restrictions I-IV, type-$n$ agents condition their sell-or-wait choice at time $t$ on time since observing the signal, i.e., on $t - n$, and on whether the last price $p_{t-1}$ was high ($c(p_{t-1}) = 0$) or medium ($c(p_{t-1}) = 1$). Specifically, agents’ plans are described by:

**Strategy Profile 2** — For any $n \geq 1$, the strategy of a type-$n$ agent is the following:

- For all $t \leq 1$, hold $h_{n,t} = 1$.
- For all $t \in \{2, \ldots, T\}$, let

\(^{12}\) Since, as previously mentioned, within-period timing assumptions are plausible only if periods are relatively brief, this is a restrictive assumption. However, analyzing this case is useful for expositional purposes since it is more tractable than the case where noise may hide sales by multiple types. (The latter is available upon request.)
\[ h_{n,t} = \begin{cases} 1 & \text{if } t < \min \{ t^*(n), t^{**}(n) \} \\ 0 & \text{if } t \geq \min \{ t^*(n), t^{**}(n) \}, \end{cases} \tag{15} \]

where \( t^*(n) = \min \{ t \mid t \geq n + \tau^* \land c(p_{t-1}) = 1 \} \), \( t^{**}(n) = \min \{ t \mid t \geq n + \tau^{**} \land c(p_{t-1}) = 0 \} \), and \( \tau^{**} \geq \tau^* \geq 0 \).

- For all \( t \geq T + 1 \), maintain \( h_{n,t} = h_{n,t-1} \).

In words, type-\( n \) agents hold the maximum long position before observing the signal (i.e., while \( t < n \)). After observing the signal, they wait for \( \tau^* \) periods until \( t = n + \tau^* \), then sell if \( p_{n+\tau^*-1} \) is medium and wait if it is high.\(^{13}\) They continue applying this sell-if-medium/wait-if-high rule for another \( d \) periods, where \( d = \tau^{**} - \tau^* \). In the event that prices remain high for all \( t \in \{ n + \tau^* - 1, \ldots, n + \tau^{**} - 1 \} \), they sell at \( n + \tau^{**} \), even though \( p_{n+\tau^{**}-1} \) is high. Finally, agents do not reenter the market after selling, and nobody does anything after time \( T \).

![Figure 4](image-url)

**Figure 4** — As soon \( t_0 \) is realized, \( p_t \) and \( f_t \) begin to diverge. Signals are observed from \( t_0 \) to \( t_0 + N - 1 \). (Bars above these periods, which decrease in height, denote conditional probabilities given by (2), for the signal \( n = t_0 + N - 1 \).) In this example, since \( \tau^{**} > N - 1 \), sales cannot begin until after all signals are observed. Also, \( d = \tau^{**} - \tau^* = 6 \). For the depicted realizations of \( \varepsilon_t \), the bubble bursts as late as possible. Since \( p_t \) is high \( \forall t \in \{ t_0 + \tau^* - 1, \ldots, t_0 + \tau^{**} - 1 \} \), sales begin at time \( t_0 + \tau^* \). Since \( p_{t_0+\tau^{**}} \) is medium, types \( t_0 + 1, \ldots, t_0 + 7 \) sell at \( T = t_0 + \tau^{**} + 1 \), causing a crash. If \( p_t \) had been medium for some \( \tau \in \{ t_0 + \tau^* - 1, \ldots, t_0 + \tau^{**} - 2 \} \) sales would have started at time \( \tau + 1 \), before \( t_0 + \tau^{**} \).

\(^{13}\) In the special case where \( n + \tau^* = 1 \), the sell-if-medium/wait-if-high rule does not apply, as \( c(p_0) \) is not defined. In this case, type-1 agents do not sell at \( t = 1 \); they begin to follow the sell-if-medium/wait-if-high at time 2 instead.
Depending on $\varepsilon_t$, sales can begin as soon as period $t_0 + \tau^*$, and as late as $t_0 + \tau^{**}$. The number of types that manage to sell before the crash ranges from 1 to $d + 2$ and also depends on the realizations of $\varepsilon_t$ in periods leading up to the crash. Specifically, if $p_{t_0 + \tau^* - 1}$ is medium, type $t_0$ sells at $t_0 + \tau^*$. If $p_{t_0 + \tau^*}$ is low, no one else sells, whereas if it is medium, type $t_0 + 1$ sells at $t_0 + \tau^* + 1$. If $p_{t_0 + \tau^* - 1}$ is high, the next $s$ prices (with $s \in \{0, 1, \ldots, d - 1\}$) are high and $p_{t_0 + \tau^* + s}$ is medium, $s + 2$ types sell at $t_0 + \tau^* + s$. And if for all $t \in \{n + \tau^* - 1, \ldots, n + \tau^{**} - 1\}$, prices remain high, type $t_0$ sells at $t_0 + \tau^{**}$. Next, if $p_{t_0 + \tau^{**}}$ is low, no more types sell, but if $p_{t_0 + \tau^{**}}$ is medium—as shown above in Figure 4—another $d + 1$ types also sell before the crash. As previously discussed, behavioral agents can only buy a mass $\kappa < 1$ of shares. Thus, $d + 2$ types can sell before the crash only if $d + 2 < \kappa N$. I assume that this holds.

To investigate the model’s ability to generate long bubbles, note that equilibrium bubble duration $T - t_0$ is at least $\tau^* + 1$ periods. Thus, the task at hand is to find conditions under which equilibria with large $\tau^*$ (and hence large $\tau^{**}$) can be supported. In any equilibrium, agents must be willing to sell if and only if (15) stipulates it. As discussed before, pre-boom ($t \leq 0$) and post-crash ($t \geq T + 1$), all types find it (weakly) optimal to follow equilibrium strategies. And at $t = 1$, they find it optimal not to sell, since nobody is selling and they can reap gains postponing their sales by at least one period. The analysis of agents’ choices while $t = 2, \ldots, T$ is more complex, and I therefore divide it into four parts: In Lemma 1, I state conditions under which type-$n$ agents choose to sell if $t = n + \tau^{**}$ and $p_{t - 1}$ is high. In Lemma 2, conditions such that they choose to wait if $t < n + \tau^{**}$ and $p_{t - 1}$ is high, and in Lemmas 3 and 4, respectively, conditions such that they are willing to sell if $t \geq n + \tau^*$ and $p_{t - 1}$ is medium, and wait if $t < n + \tau^*$ and $p_{t - 1}$ is medium. In Proposition 2, I combine the Lemmas into a set of conditions that are necessary to support equilibria with large $\tau^*$ and $\tau^{**}$. Finally, in Proposition 3, I show that the conditions in Lemma 2 are compatible with each other and with the no-bubbles-without-noise restriction $e^1 < G / R < \Gamma$.

Before plunging into the Lemmas, I will provide a brief preview of upcoming results. Making agents sell is easy. If a type-$n$ agent knows that other type-$n$ agents are selling, there is at least a probability $\pi$ that the bubble will burst next period. Given this, Lemmas 1 and 3 require
very little to make agents sell when they are supposed to. It is more difficult to make agents wait when they are supposed to wait, and therefore, Lemmas 2 and 4 need to impose some restrictions to rule out preemptive sales. In Lemma 2, it is only possible to support large \( \tau^{**} \) if \( G/R \) is above (or not far below) \( (1+e^{-\lambda})/(1+e^{-\lambda} - \pi) \). In Lemma 4, it is only possible to support large \( \tau^{*} \) if \( e^{\lambda}/\pi < G/R \). Under these conditions, agents are willing to wait because, if they sell early and the bubble continues to grow, they forgo large profits. To see exactly how these conditions affect agents’ choices, let us proceed to the Lemmas.

**Lemma 1** — Let \( n \geq 1 \) be an arbitrary type. If \( G/R < \Gamma, \pi \geq 1/(1+e^{-\lambda}), d + 2 < \kappa N, \tau^{**} \geq 1, t = n + \tau^{**} \) and \( p_{t-1} \) is high, type-\( n \) agents find it optimal to sell at time \( t \).

**Proof** — Consider a type-\( n \) agent at \( t = n + \tau^{**} \), with \( \tau^{**} \geq 1 \) and high \( p_{t-1} \). At this point, she knows that her type is first, i.e., that \( t_0 = n \). (If \( n = 1 \), she has known that \( t_0 = n \) since period 1; if \( n > 1 \), she learnt that \( t_0 = n \) from the fact that \( p_{n+t^{**}-1} \) was high.) Since other type-\( t_0 \) agents are selling at \( t \), \( p_t \) will be low with probability \( \pi \) and medium with probability \( 1 - \pi \). An individual type-\( t_0 \) agent can thus sell along with the other type-\( t_0 \) agents at an expected price \( G^{b_{t^{**}}^{t^{**}}} \), or wait. If she waits, with probability \( \pi \) she will earn the discounted post-crash price \( (G/R)^{b_{t^{**}}^{t^{**}}} R^{b_{t^{**}}^{t^{**}}} \) and with probability \( 1 - \pi \) she will sell at \( t + 1 \) with \( d + 1 \) other types at an expected price that—since \( d + 2 < \kappa N \)—equals \( G^{b_{t^{**}}^{t^{**}+1}} \). Thus, selling is optimal if

\[
1 \geq \pi \left( \frac{G}{R} \right)^{-\tau^{**+1}} + (1 - \pi) \frac{G}{R}.
\]  

(16)

If \( \pi = 1/(1+e^{-\lambda}) \), (16) is the same as (13). Thus, since \( \pi \geq 1/(1+e^{-\lambda}) \), \( G/R < \Gamma \) and \( \tau^{**} \geq 1 \), type-\( n \) agents are willing to sell. Q.E.D.

In Lemma 1, I have ignored the case where \( \tau^{**} = 0 \). Analyzing this case is not difficult, but it is tedious and, since our focus is on long bubbles, uninteresting. Similarly, assuming that \( \pi \geq 1/(1+e^{-\lambda}) \) simplifies the proof without actually imposing a binding restriction on parameters. This is because, as we will see in Proposition 3, to support long bubbles \( \pi \) must be above (or just a little bit below) a threshold \( (1+e^{-\lambda})/(1+e^{-\lambda} + e^{-2\lambda}) \). Since this threshold exceeds \( 1/(1+e^{-\lambda}) \) for any \( \lambda \), the parameter values of interest always satisfy \( \pi \geq 1/(1+e^{-\lambda}) \).
Lemma 1 establishes that, under general conditions, type-$n$ agents will sell at $n + \tau^{**}$ after a high $p_{n + \tau^{**} - 1}$. At this point, they know that they were the first to observe the signal and that they have successfully ridden the bubble. But how did they arrive here? In earlier periods $n + \tau^{**} - j$ (with $j \geq 1$), they did not know that they were first. Following (15) and waiting was risky, since $t_0$ could have been $n - j$, in which case type $n - j$ would have sold at $n + \tau^{**} - j$, causing a crash with probability $\pi$. Under what conditions was it optimal for them to take this risk? While I answer this fully in Lemma 2, the key condition ruling out preemptive sales can be understood by examining Figure 4 and focusing on period $t_0 + \tau^{**}$, which is preceded by $d + 1$ high prices $p_{b_0 + \tau^{**} - 1}, \ldots, p_{b_d + \tau^{**} - 1}$. As Figure 4 shows, in this situation, type-$t_0$ agents sell and others wait. To see why others wait, consider the sell-or-wait trade-off of type $t_0 + 1$, i.e., the second type to observe the signal. At $t = t_0 + \tau^{**}$, type-$t_0 + 1$ agents understand that, with probability $1/(1 + e^{-\lambda})$, they were second to observe the signal and the first type will sell at $t$, causing a crash with probability $\pi$. But they also assign a probability $e^{-\lambda}/(1 + e^{-\lambda})$ to the possibility that they were first, in which case nobody will sell at $t$ and a crash at $t+1$ is impossible. In sum, selling at $t$ yields $G'$, and waiting yields the post-cash price $(G/R)^{b-1}R'$ with probability $\pi/(1 + e^{-\lambda})$ and at $G'^{b-1}$ with probability $1 - \pi/(1 + e^{-\lambda})$. Waiting is optimal if

$$1 \leq \frac{1}{1 + e^{-\lambda}} \left( \frac{\pi}{R} \left( \frac{G}{R} \right)^{b-1} + (1 - \pi) \frac{G}{R} \right) + \frac{e^{-\lambda} G}{1 + e^{-\lambda} R}. \quad (17)$$

If $G/R \geq (1 + e^{-\lambda})/(1 + e^{-\lambda} - \pi)$, (17) holds for any $\tau^{**}$, no matter how large. In other words, for $G/R$ above this threshold, type-$t_0 + 1$ agents are willing to wait even if the post-crash price is zero. If $G/R$ is below $(1 + e^{-\lambda})/(1 + e^{-\lambda} - \pi)$, waiting is optimal only if the fraction of the price lost in the crash is not too large. Specifically, waiting is optimal only if

$$\tau^{**} \leq \frac{\ln \left( \frac{\pi}{(1 + e^{-\lambda}) - (1 + e^{-\lambda} - \pi)G/R} \right)}{\ln(G/R)} - 1. \quad (18)$$

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14 Note that the trade-off that type-$t_0$ agents face at $t_0 + \tau^{**} - 1$, with high prices for the last $d + 1$ periods, is identical to the one faced by type-$t_0 + 1$ agents at $t_0 + \tau^{**}$, also with high prices for the last $d + 1$ periods. In both cases, the support of $t_0$ has two values; agents know that they were either be first or second to observe the signal.
I have derived (17) for the specific situation of type-\(t_0+1\) agents at \(t_0+\tau^{**}\) after \(d+1\) high prices. However, in the proof of Lemma 2, I show that, of all possible situations with \(t < n + \tau^{**}\) and high \(p_{t-1}\), this is precisely the one where type-\(n\) agents are most tempted to sell.

**Lemma 2** — Suppose that \(e^\lambda < G/R\), \(d+2 < \kappa N\), \(t < n + \tau^{**}\) and let \(p_{t-1}\) be high. If \(G/R \geq (1+e^{-\lambda})/(1+e^{-\lambda} - \pi)\), type-\(n\) agents find it optimal not to sell at time \(t\), for any \(\tau^{**} \geq 0\). If \(G/R < (1+e^{-\lambda})/(1+e^{-\lambda} - \pi)\), type-\(n\) agents choose not to sell at time \(t\), only if (18) holds.

**Proof** — See Appendix B.

While I refer the reader to Appendix B for analysis of all possible cases with \(t < n + \tau^{**}\) and high \(p_{t-1}\), the reason why the situation captured by (17) is the one where preemptive selling (after a high price) is most tempting can be sketched as follows. Consider type-\(t_0+2\) agents at time \(t_0+\tau^{**}\) after \(d+1\) high prices. Given their information, they believe they could have been first, second, or third to observe the signal. Thus, for them, the crash probability is \(\pi/(1+e^{-\lambda} + e^{-2\lambda})\) (\(\pi\) times the probability of being third), clearly below \(\pi/(1+e^{-\lambda})\), the crash probability in (17). By the same logic, for types \(t_0+2\) and higher, the crash probability is even lower. Moreover, in many cases with \(t < n + \tau^{**}\) and high \(p_{t-1}\), type-\(n\) agents have no incentive to sell, since the crash probability is nil. For instance, if \(p_{t-s}\) is medium for some \(s \in \{2, \ldots, d+1\}\), all types know with certainty that nobody will sell at \(t\), and hence that a crash at \(t+1\) is impossible. This is due to the fact that, if \(t_0\) was \(t-\tau^{**}\), type-\(t_0\) agents would have sold at \(t-s+1\) after the medium \(p_{t-s}\). But then, \(p_{t-1}\) could not possibly be high.

From (18), we can analyze comparative statics for maximum bubble duration \(\tau^{**}\). Not surprisingly, \(\tau^{**}\) falls as \(\lambda\) increases, since the greater \(\lambda\), the greater the likelihood of lower values of \(t_0\) relative to higher values. The effect of \(\pi\) on \(\tau^{**}\) is also negative, since the greater \(\pi\), the more likely it is that, if one type sells at \(t\), the price reveals the sale.\(^{15}\) The effect of \(G/R\)

\(^{15}\) These comparative statics resemble those in AB (abstracting from AB’s exogenous cap on bubble duration). For some parameter values, there is no endogenous upper bound on bubble duration. For other values, there is a finite endogenous bubble duration, which is increasing in the rate of growth of the bubble and decreasing in \(\lambda\) (\(\lambda\) has the same meaning in AB and here). The fact that, in this model, \(\tau^{**}\) is decreasing in \(\pi\) corresponds loosely to the fact that, in AB, bubble duration is increasing in behavioral absorption capacity. Here, the higher \(\tau^{*}\), (and hence, the lower \(\pi\)) the more agents can sell unnoticed.
is not as straightforward, because increases in $G / R$ increase profits if the bubble does not burst, but also losses if it does burst. In general, the direction of the effect depends on parameter values. However, the parameter values of interest are those that allow long bubbles to arise, i.e., values of $G / R$ slightly below $(1 + e^{-\lambda}) / (1 + e^{-\lambda} - \pi)$. In this range, $\tau^{**}$ is increasing in $G / R$.

In sum, by Lemma 2, equilibria with large $\tau^{**}$ exist only if $G / R$ is above, or not far below, $(1 + e^{-\lambda}) / (1 + e^{-\lambda} - \pi)$, and if $d + 2 < \kappa N$. The latter condition, which is needed to ensure that behavioral agents can buy the $(d + 2) / N$ shares sold by rational agents, can only hold if $\tau^{*}$ is not too far below $\tau^{**}$. To investigate what values of $\tau^{*}$ can arise in equilibrium, in Lemmas 3 and 4, I study sell-or-wait tradeoffs after medium prices.

Lemma 3 is fairly straightforward. In Lemma 1, type-$n$ agents sold at $n + \tau^{**}$ with high $p_{n+\tau^{**}-1}$ knowing that they were first to observe the signal, and the only ones selling. Thus, the crash probability was $\pi$. By contrast, in Lemma 3, type-$n$ agents sell at $n + \tau^{*}$ after medium $p_{n+\tau^{*}-1}$, in general not knowing whether they were first to observe the signal or not. Besides their own type, others could also be selling, and this raises the crash probability above $\pi$. Agents are thus more inclined to sell after medium prices than after high prices, which implies that the conditions that sufficed to induce sales in Lemma 1 also suffice in Lemma 3.

**Lemma 3** — If $\pi \geq 1 / (1 + e^{-\lambda})$, $d + 2 < \kappa N$, $G / R < \Gamma$, $\tau^{*} \geq 0$ and $n + \tau^{*} \geq 2$, type-$n$ agents are willing to sell at time $t \geq n + \tau^{*}$ after a medium price $p_{t-1}$.

**Proof** — First consider the case with $n \geq 2$ and $t = n + \tau^{*}$. In Lemma 1, type-$n$ agents sell knowing that $t_0 = n$. Here, since $p_{t-1}$ is medium, they sell without knowing whether $t_0 = n$ or $t_0 < n$. If $t_0 = n$, the bubble will burst at $t + 1$ with probability $\pi$. And if $t_0 < n$, two or more types will sell at $t$, causing a crash at $t + 1$. The probability of a crash at $t + 1$ in this case is thus above $\pi$. This makes incentives to sell stronger at $n + \tau^{*}$ with medium $p_{n+\tau^{*}-1}$ than at $n + \tau^{**}$ with high $p_{n+\tau^{**}-1}$. Since $\pi \geq (1 + e^{-\lambda})^{-1}$ and $G / R < \Gamma$ suffice to make agents sell in the latter case, they also suffice in the former.

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16 If $n \geq 2$ and $p_{t-1}$ is medium, type-$n$ agents cannot rule out the possibility that $t_0 = n - 1$. This is true regardless of whether $p_{t-2}$ is high or medium. It also holds if $c(p_{t-1})$ is not defined, i.e., if $t = 2$ with $n = 2$ and $\tau^{*} = 0$. Thus, $\text{supp}_{p_{t-1}}(t_0)$ has at least two elements $\{n-1, n\}$. In may have even more, since, if $p_{t-1}$ is the first medium price after $k \geq 2$ consecutive high prices, $\text{supp}_{p_{t-1}}(t_0)$ is given by $\{n-k, \ldots, n\}$.
Continuing with $n \geq 2$, if $t > n + \tau^*$ and $p_{t-1}$ is medium, type-$n$ agents sell at $t$, since they know that at least two types ($n$ and $n+1$) will sell at $t$, which ensures a crash at $t+1$.\(^{17}\)

Finally, let $n = 1$. If $\tau^* > 0$ and $p_{(1+\tau^*)-1}$ is medium, type-1 agents sell at $t=1+\tau^*$, knowing that they are the only type selling. By Lemma 1, selling is optimal because $\pi \geq 1/(1+e^{-\lambda})$, $G/R < \Gamma$, and $\tau^* \geq 1$. If $\tau^* > 0$ and $p_{(1+\tau^*)-1}$ is high (or if $\tau^* = 0$), type-1 agents sell at $t > 1 + \tau^*$, as soon as $p_{t-1}$ is medium. In these cases, selling is optimal because two or more types are selling and a crash at $t+1$ is certain. **Q.E.D.**

The link between Lemmas 3 and 4 is akin to the one between Lemmas 1 and 2. In Lemma 3, after a medium price, some types sell, while others—who observed the signal later—wait. Lemma 4 examines under what conditions the latter find it optimal to wait despite a sizable risk of getting caught in the crash. Again, the Lemma’s proof is in Appendix B, but I will sketch the main argument here with the help of Figure 4. Consider type $n = t_0 + d + 2$ at $t = t_0 + \tau^{**} + 1$.

(Note that $t = t_0 + \tau^{**} + 1 = n + \tau^* - 1$, i.e., type $n$ is the lowest among the types who stay in the market at $t$.) After $d+1$ high prices, type $t_0$ sold at $t-1$, $p_{t-1}$ is medium, and now $d+1$ more types $t_0+1, \ldots, t_0+d+1$ will sell at $t$, while types $n$ and higher wait. Clearly, type-$n$ agents would not wait if they knew $t_0$. But given their information, they believe that there are $d+3$ possible values of $t_0$, $n-(d+2), \ldots, n$. They thus assign a probability $e^{-\lambda(d+2)}/(1 + \cdots + e^{-\lambda(d+2)})$ to the possibility of being ‘first in line’, i.e., to $t_0 = n$. Selling in this case would mean forgoing a sizable expected return $W_d$. If $t_0 = n$, for the next $d$ periods, with probability $1-\pi$, type-$n$ agents will sell and with probability $\pi$, the bubble will continue to grow. Overall, the return $W_d$ (see appendix B for full details) is given by

$$W_d = (1-\pi)\left(\frac{G}{R}\right)^{d+1} - 1 + \left(\frac{\pi}{R}\right)^{d+1}. \quad (19)$$

If $\pi G/R > e^\lambda$, as $d$ increases, $W_d$ grows faster than $e^{-\lambda(d+2)}/(1 + \cdots + e^{-\lambda(d+2)})$, falls, and thus

$$1 < \frac{e^{-\lambda(d+2)}}{1 + \cdots + e^{-\lambda(d+2)}} W_d \quad (20)$$

\(^{17}\) Type-$n$ agents find themselves in this situation if $p_{(1+\tau^*)-1}$ happens to be high.
holds for large $d$. In other words, if $\pi G / R > e^\delta$, there exist values of $d$ for which $W_d$ is so large that type-$n$ agents would be willing to wait at $t$, even if the price fell to zero for all $t_0 < n$.

In a second part of the proof, I show that, of all situations where the last price is medium and less than $\tau^*$ periods have passed since observing the signal, the one discussed above is the most critical one, in the sense that preemptive sales are most tempting. This is for two reasons. First, if there are less than $d + 1$ consecutive high prices leading up to the medium $p_{t-1}$, type-$n$ agents can rule out some of the earlier values of $t_0$, which makes $t_0 = n$ relatively more likely.

To see the second reason, consider type $n+1$ at time $t$. Just like for type $n$, for $n+1$, the support of $t_0$ at $t$ contains $d + 2$ ‘bad’ values with $t_0 < n$. But for type $n+1$, the support has two ‘good’ values, $t_0 = n$ and $t_0 = n+1$, with expected return $W_d$ or higher. Type $n+1$ is thus less tempted to sell than type $n$. By the same token, types $n+2$ and higher are less tempted than type $n+1$.

**Lemma 4** — Let $t < n + \tau^*$, and let $p_{t-1}$ be medium. If $\pi G / R > e^\delta$ and $d + 2 < \kappa N$, there is a threshold $\overline{d} > 0$ such that if $d \geq \overline{d}$, type-$n$ agents find it optimal not to sell at time $t$.

**Proof** — See Appendix B.

To recapitulate, in Proposition 2, I combine Lemmas 1-4 to obtain sufficient conditions to support bubbles in equilibrium.

**Proposition 2** — Suppose that $\pi \geq 1 / (1 + e^{-\delta})$ and that $e^\delta / \pi < G / R < \Gamma$. Then:

1. If $(1 + e^{-\delta}) / (1 + e^{-\delta} - \pi) \leq G / R$, there exists a threshold $\overline{d} \in \{1, 2, 3, \ldots\}$, such that any $(\tau^*, \tau^{**})$ with $\tau^* \geq 0$, $\tau^{**} \geq 1$, $d \geq \overline{d}$ and $d + 2 < \kappa N$ can be supported in equilibrium.
2. If $(1 + e^{-\delta}) / (1 + e^{-\delta} - \pi) > G / R$, there exists a threshold $\overline{d} \in \{1, 2, 3, \ldots\}$, such that any $(\tau^{**}, \tau^*)$ with $\tau^* \geq 0$, $1 \leq \tau^{**} \leq \ln \{\pi / [(1 + e^{-\delta}) - (1 + e^{-\delta} - \pi)G / R]\} / \ln(G / R) - 1$, $d \geq \overline{d}$ and $d + 2 < \kappa N$ can be supported in equilibrium.

**Proof** — Start with (2.1). Since $\pi \geq 1 / (1 + e^{-\delta})$, $d + 2 < \kappa N$, $G / R < \Gamma$ and $\tau^* \geq 1$, by Lemma 1, type-$n$ agents are willing to sell at $n + \tau^{**}$ with high $p_{n+\tau^{**}-1}$. And since $\pi \geq 1 / (1 + e^{-\delta})$, $G / R < \Gamma$, $d + 2 < \kappa N$, and $\tau^* \geq 0$, by Lemma 3, they are willing to sell at $t \geq n + \tau^* \geq 2$ with medium $p_{t-1}$. By Lemma 2, since $e^\delta / \pi < G / R$, $(1 + e^{-\delta}) / (1 + e^{-\delta} - \pi) \leq G / R$ and $d + 2 < \kappa N$, type-$n$ agents are willing to wait at $t < n + \tau^{**}$ with high $p_{t-1}$. By Lemma 4, since
$e^\delta / \pi < G / R$, and $d + 2 < \kappa N$, $\bar{d} > 0$ exists such that, if $d \geq \bar{d}$, type-$n$ agents are willing to wait at $t < n + \tau^*$ with medium $p_{t-1}$.

To prove (2.2), invoke Lemmas 1, 3 and 4 exactly as before. Then, by Lemma 2, since $G / R < (1 + e^{-\delta}) / (1 + e^{-\delta} - \pi)$, type-$n$ agents are willing not to sell at $t < n + \tau^{**}$ with high $p_{t-1}$ only for $\tau^{**} \leq \ln\{\pi / [(1 + e^{-\delta}) - (1 + e^{-\delta} - \pi)G / R] / \ln(G / R) - 1\}$. Q.E.D.

In sum, when noise cannot even hide sales by one type, Proposition 1 establishes that, if $e^\delta < G / R < \Gamma$, the only equilibrium (under strategic restrictions I-IV) has agents selling as soon as they observe the signal. And when there is enough noise to hide sales by one type, but not more, Proposition 2 establishes that arbitrarily long bubbles (with $d = \tau^{**} - \tau^*$, $d + 2 < \kappa N$, and $d \geq \bar{d}$) can be supported if $e^\delta / \pi < G / R < \Gamma$, $\pi \geq 1 / (1 + e^{-\delta})$ and $(1 + e^{-\delta}) / (1 + e^{-\delta} - \pi) < G / R$. In Proposition 3, I show that all these conditions are compatible.$^{18}$

**Proposition 3** — There exists a nonempty region of the parameter space where $e^\delta < G / R < \Gamma$, $\pi \geq 1 / (1 + e^{-\delta})$, $(1 + e^{-\delta}) / (1 + e^{-\delta} - \pi) < G / R$ and $e^\delta / \pi < G / R$.

**Proof** — In Figure 5, I plot all restrictions in a diagram with $\pi$ on the horizontal and $G / R$ on the vertical axis. The pairs $(\pi, G / R)$ of interest lie in the interior of the rectangle defined by $1 / (1 + e^{-\delta}) \leq \pi \leq 1$ and $e^\delta \leq G / R \leq \Gamma$, and above the graphs of the functions $G / R = e^\delta / \pi$ and $G / R = (1 + e^{-\delta}) / (1 + e^{-\delta} - \pi)$. For $\pi \in [(1 + e^{-\delta})^{-1}, 1]$, both functions are continuous, $e^\delta / \pi$ is strictly decreasing and $(1 + e^{-\delta}) / (1 + e^{-\delta} - \pi)$ strictly increasing in $\pi$. They intersect at the point $(\pi_i, [G / R]_i)$, where $\pi_i = (1 + e^{-\delta}) / (1 + e^{-\delta} + e^{2\delta})$ and $[G / R]_i = e^\delta(1 + e^{-\delta} + e^{2\delta}) / (1 + e^{-\delta})$. For any $\delta > 0$, $(\pi_i, [G / R]_i)$ is in the interior of the rectangle, as $\pi_i$ is clearly between $1 / (1 + e^{-\delta})$ and $1$, $[G / R]_i$ is clearly above $e^\delta$, and, as I show in Appendix C, $[G / R]_i < \Gamma$. Since $(\pi_i, [G / R]_i)$ is in the interior of the rectangle, there exists a region in the $(\pi, G / R)$ plane (the shaded area in Figure 5), where all parameter restrictions hold. Q.E.D.

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$^{18}$ Here I focus on the first part of Proposition 2 and ignore the second, by which, if $G / R < (1 + e^{-\delta}) / (1 + e^{-\delta} - \pi)$, $\tau^{**}$ less than $\ln\{\pi / [(1 + e^{-\delta}) - (1 + e^{-\delta} - \pi)G / R] / \ln(G / R) - 1\}$ can be supported. Since the focus of the analysis is on long bubbles, and the upper bound on $\tau^{**}$ is small unless $(1 + e^{-\delta}) - (1 + e^{-\delta} - \pi)G / R \approx 0$, considering the case where $G / R < (1 + e^{-\delta}) / (1 + e^{-\delta} - \pi)$ in Proposition 3 would add more complication than insight.
For any $\lambda > 0$, the region where all parameter restrictions are satisfied—the shaded area in Figure 5—is delimited by one straight side and two curved sides.\(^{19}\)

Figure 5 — Compatibility of Parameter Restrictions from Propositions 1 and 2.

Having shown that there exist parameters for which no bubbles arise without noise, but long bubbles arise if noise can hide one sale, we can now construct bubbly equilibria as follows. Pick a pair $(\pi, G/R)$ inside the shaded area above.\(^{20}\) Pick some large $\tau^*$, and let $\tau^* = \tau^{**} - \tilde{d}$, where $\tilde{d}$ is the smallest integer for which (20) holds. Finally, let $N$ be an integer above $(\tilde{d} + 2)/\kappa$, and set $\varepsilon = 1/(2\pi N)$. More concretely, consider the following examples:

**Numerical Example 1** — Let $\lambda = 0.001$, implying $\Gamma \approx 1.619$. Set $\pi = 0.75$ and $G/R = 1.618$. Since $G/R > (1 + e^{-\lambda})/(1 + e^{-\lambda} - \pi) \approx 1.6005$, $\pi G/R > e^{\lambda}$, and $G/R < \Gamma$, $(\pi, G/R)$ lies inside the shaded area in Figure 5. Set (arbitrarily) $\tau^{**} = 56$. Since (20) holds for $d \geq 6$, let $\tau^* = 50$. Finally, if $\kappa = 1/2$ and $N = 40$, $(6 + 2) < \kappa N$. It is implied that $\varepsilon = 1/(2\pi N) = 1/60.$

\(^{19}\) A few algebra steps suffice to show that $e^{\lambda}/\Gamma > 1/(1 + e^{-\lambda})$ and that $(1 + e^{-\lambda})(\Gamma - 1)/\Gamma < 1$, where $e^{\lambda}/\Gamma$ and $(1 + e^{-\lambda})(\Gamma - 1)/\Gamma$ are, respectively, the values of $\pi$ for which the graphs of the functions $G/R = e^{\lambda}/\pi$ and $G/R = (1 + e^{-\lambda})/(1 + e^{-\lambda} - \pi)$ cross the $G/R = \Gamma$ line.

\(^{20}\) Technically, there are no restrictions on $\lambda$. But intuition suggests that it should be relatively small. Specifically, it would seem implausible to have a large value of $N$ (representing very disperse opinions about $t_n$) unless the expected $t_n$, given by $1/(1 - e^{-\lambda})$, was also relatively large.
To illustrate this example in more detail, in Table 1, I track agents’ beliefs and expected returns in the last few periods of a bubble with \((\tau^*, \tau^{**}) = (50, 56)\). To fix ideas, let \(t_o = 50\). As in Figure 4, I assume realizations of noise for which the bubble bursts as late as possible. Specifically, after a medium \(p_{95}\), prices remain high for ten periods \(96, …, 105\). Type 50 sells at time \(t_0 + 56 = 106\), \(p_{106}\) is medium, types 51, …, 57 sell in period 107, \(p_{107}\) is low, causing a crash at time 108.

**Numerical Example 2** — Maintain \(\lambda = 0.001\), \(\pi = 0.75\), \(N = 40\), \(\kappa = 1/2\) and \(\bar{\sigma} = 1/60\). But let \(G/R = (1 + e^{-0.001} - \nu)/(1 + e^{-0.001} - \pi)\) for some small \(\nu < 0\), so that \((\pi, G/R)\) lies just below Figure 4’s shaded area. Then, equilibria with \((\tau^*, \tau^{**})\) can be supported if \(\tau^* \geq 0\), \(\tau^{**} \leq \ln(0.75/\nu) / \ln(G/R) - 1\), and \(\tau^{**} - \tau^* \geq \bar{d} = 6\). Long bubbles can be supported only if \(\nu\) is very small. For instance, if \(\nu = 0.001 (\nu = 10^{-8})\), it must be that \(\tau^{**} \leq 13 (\tau^{**} \leq 37)\).

5 Extensions

In this section, I present two extensions of the model. In the first, I allow agents to reenter the market after selling. In the second, I remove behavioral agents, and sketch a fully rational version of the model.

5.1 Re-entering the Market after Selling

The no-reentry assumption does much to simplify the analysis, and is quite defensible in high-transaction-cost markets such as real estate. But in other asset markets, such as stock markets, transaction costs are low and frequent trading is typical. In these markets, the no-reentry assumption it is clearly unrealistic. Furthermore, it is not obvious that it is innocuous. It is thus worth asking whether bubbles continue to arise once agents are able to buy and sell whenever they want. In this subsection, I answer this question in the affirmative. Specifically, I show that there are bubbly equilibria where agents follow (15) even though they are allowed to reenter the market after selling. In these bubbles, agents could sell and reenter, but choose not to. To examine whether the reentry option makes a difference, I will revisit Lemmas 1-4.

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21 The equilibrium concept from Section 3 should be expanded to account for the fact that, with allowed reentry, the payoff from selling is no longer given by the price. In the expanded definion (not presented here to conserve space, but available upon request), there is an expected return associated with being in the market and one associated with being out of the market, which reflects the reentry option. Every period, agents choose between the two returns. As in Section 3, these returns are well defined, since it is possible to work backwards from the post-crash payoffs.
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Table 1 — Numerical Example 1: $\lambda = 0.001, \pi = 0.75, G / R = 1.618, \kappa = 1 / 2, N = 40, \bar{E} = 1 / 60$. Equilibrium $(\tau^*, \tau^{**}) = (50, 56)$. To fix ideas, let $t_0 = 50$. As in Figure 4, the realizations of noise are such that the bubble bursts as late as possible. If a cell has no shading, the type is still in the market. Light shading denotes that the type is currently selling, and darker shading that the type already sold in a previous period. Each cell contains, at the top, $\text{supp}_{n,t}(t_0)$. When agents see a high $p_{t_0}$, they rule out $t \leq t_0$. High prices in periods 97-102 reveal no new information. Only as agents see high $p_{103}, p_{104}$ and $p_{105}$, they can successively discard 47, 48 and 49 from $\text{supp}_{n,t}(t_0)$. From period $t = 103$ onward, each cell also reports, at the bottom, the expected return from waiting relative to selling. If this is above (below) one, agents wait (sell). Post-crash, agents are indifferent between buying, selling and doing nothing. The boldfaced 1.011 for type 50 at time 105 and 51 at 106 is the right hand side of (17) evaluated at these parameter values. As we move down/left from these cells, the payoffs associated with waiting increase, as the probability of a crash at $t + 1$ falls. The boldfaced 1.032 for type 58 at time 107 equals the right-hand-side of (20). As I show in the proofs of Lemmas 2 and 4, these cells represent the situations where types who are supposed to wait are most tempted to sell preemptively. In periods 97-102, agents know for sure that nobody is selling, and thus, have no incentive to sell. Details on how to calculate payoffs for all cells are available upon request. (As noted in footnote 8, the assumption that $\text{supp}_{n,108}(t_0) = \{50\}$ for all $n$ is literally true if $p_{107} < G^{107} - \alpha[\bar{E} - 7 / N]$, and approximately true otherwise. That is, if $p_{107} > G^{107} - \alpha[\bar{E} - 7 / N]$, type-50 agents have $\text{supp}_{50,108}(t_0) = \{50\}$, while others have $\text{supp}_{n,108}(t_0) = \{50, 51\}$.)

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In situations captured by Lemmas 1 and 3, it is easy to see why agents choose not to reenter the market after selling. Along the equilibrium path, whenever an agent sells, she knows that the crash will arrive within one or, at most, two periods. Reentry would thus have to take place either at time $T$, i.e., when the price is about to collapse, or at time $T+1$, i.e., after the crash. Clearly, agents will not choose to reenter.

The reentry option turns out not to make a difference in situations captured by Lemma 2, either. Roughly (see Appendix D for details), reentry does not matter for the following reason. The reentry option makes preemptive sales more desirable by reducing their potential opportunity cost. After selling, if the bubble does not burst, reentering agents forego just one, instead of many, periods of appreciation. However, as I show in the proof of Lemma 2, of all situations with $t < n + \tau^{**}$ and high $p_{r-1}$, the most critical situation for type-$n$ agents is the one captured by (17). In this situation, even if the bubble does not burst at $t$, type-$n$ agents will sell at time $t+1$. This means that, even with forbidden reentry, the relevant opportunity cost to rule out preemptive sales already consists of just one period of forgone profit. Since the reentry option cannot reduce this opportunity cost, it cannot tilt the balance in favor of preemptive selling.

Given the above reasoning, it is not surprising that the situations covered by Lemma 4 are the ones where the reentry option makes the greatest difference. In Lemma 4, if $\pi G / R > e^d$ and $d \geq \tilde{d}$, type-$n$ agents wait at $t = n + \tau^* - 1$ with medium $p_{r-1}$ (preceded by $d+1$ high prices) for fear of missing out on a large return $W_d$ if $t_0 = n$. This return is accumulated over a number of periods that may reach up to $d+1$ periods. Allowing reentry lowers this opportunity cost, since an agent who sells at $t$ and then sees a high $p_t$ can reenter at $t+1$, foregoing only part of $W_d$. The sell-or-wait choice is no longer governed by (20), since the return from selling (on the left) now includes a reentry return $W_{d-1}$ with probability $\pi e^{-\lambda(d+2)} / (1 + \cdots + e^{-\lambda(d+2)})$—the probability that $t_0 = n$ and $p_t$ is high. Hence, waiting is now optimal if

$$1 + \pi \frac{e^{-\lambda(d+2)}}{1 + \cdots + e^{-\lambda(d+2)}} (W_{d-1} - 1) < \frac{e^{-\lambda(d+2)}}{1 + \cdots + e^{-\lambda(d+2)}} W_d.$$

Since $W_{d-1} > 1$ for all $d \geq 1$, agents prefer selling and reentering (if $p_t$ is high) to selling without the option to reenter. Still, if $d$—and hence $W_d$—is large enough, not selling is better than selling and reentering. This is because agents who sell and reenter forego one period of
growth, and thus, part of $W_d$. For large enough $d$, this foregone part—which is related to the difference between $W_d$ and $W_{d-1}$—is important enough to deter preemptive sales. To see this more precisely, use (19) and rearrange terms to rewrite the above inequality as

$$\frac{1-e^{-\lambda(d+3)}}{1-e^{-\lambda(d+2)}} - e^{-\lambda(d+2)} \left[ \pi + \frac{G}{\pi G / R - 1} \right] < e^{-\lambda(d+2)} \left( \frac{G}{R} \right)^d \frac{\pi (G / R - 1)^2}{\pi G / R - 1}.$$  

(21)

Note that the left hand side increases with $d$, but approaches $1/(1-e^{-\lambda})$ as $d \to \infty$. The right hand side, since $\pi G / R > e^{\lambda d}$, grows exponentially with $d$. Thus, there is a positive $\bar{d} > d$ such that (21) holds if $d \geq \bar{d}$. However, once reentry is allowed, equilibria with $\bar{d} \leq d < \bar{d}$ vanish.$^{23}$

Other than the fact that the minimum $d$ is lengthened from $d$ to $\bar{d}$, there are no new requirements that equilibria with bubbles must satisfy once reentry is allowed. Hence, within the class of equilibria where agents follow (15), the possibility of reentry makes a quantitative, but not a qualitative, difference. The mechanisms protecting bubbles from preemptive sales remain the same, and long bubbles still arise. For parameter values as in Numerical Example 1, $\bar{d} = 13$, and thus, pairs $(\tau^*, \tau^{**}) = (k, k + d)$, with $k \geq 0$ and $13 \leq d < 18$ satisfy all inequalities. However, equilibria from Example 1 with $6 \leq d < 12$ vanish, as they are not “reentry proof”.

5.2 A Fully Rational Version of the Model

Thus far, I have followed AB and included both rational and behavioral agents in the model. The presence of behavioral agents, while unorthodox, is defensible for two reasons. First, proponents of the efficient markets hypothesis (e.g., Fama (1965)) do not claim that all investors are rational in reality, but rather that, as long as there are some rational investors, they will be quick to arbitrate away any mispricings. Second, a ‘stylized fact’ of bubbles is that they tend to attract novice investors, who may not realize that they are buying overvalued assets.$^{24}$

These arguments notwithstanding, the presence of behavioral agents raises the concern that the model’s ability to generate bubbles may hinge on irrationality. To address this concern, in this subsection, I develop a fully rational version of the model where bubbles continue to arise.

$^{23}$ Inequality (21) dissuades type-$n$ agents from selling at $t = n + \tau^* - j$ with medium $p_{n,i}$, also if $j > 1$ or less than $d + 1$ high prices precede $p_{n,i}$. As in Lemma 4, both of these changes make preemptive selling less tempting.

$^{24}$ In the words of Kindleberger and Aliber (2005), in a typical bubble, “More and more firms and households that previously had been aloof from speculative ventures begin to participate in the scramble for high rates of return.”
Two new elements—an endowment and a preference shock—allow rational agents to perform functions previously performed by behavioral agents. The endowment provides an inflow of resources with which to fuel growth. The preference shock captures exogenous reasons to trade, such as life events, that compel agents to sell the risky asset. Moreover, if the mass of agents hit by the preference shock is random and unobservable, noisy price fluctuations can conceal the sales of the first types. This is precisely the nature of uncertainty when the model includes behavioral agents.

In the fully rational version of the model, there is a unit-measure continuum of rational, risk-neutral agents. There are no behavioral agents. Supply of the risky asset is fixed at 1. As before, the riskless asset has gross return $R$. The boom starts at $t = 1$, and is justified by fundamentals at first, but becomes a bubble at time $t_0$, distributed as in (1). Signals arrive at $t = t_0, \ldots, t_0 + N - 1$, the bubble lasts until time $T$, and equality between price and fundamental value is restored at $T + 1$.

Agents are endowed with $e_t > 0$ units of the risk-free asset every period. Endowments cannot be capitalized (i.e., agents cannot pledge $e_t$ in order to borrow at earlier dates $s < t$). Starting at $t = 1$, $e_t$ grows by a factor of $G$.\(^{25}\) Agents invest their growing endowments into the risky asset, fueling price growth. The other novelty in the model is a preference shock, capturing life events that force some agents to sell their assets and consume. (Assume that only the risk-free asset may be consumed.) To model this, I assume that at time $t$, a randomly chosen mass $\theta_t \in (0, 1)$ of agents are hit by a shock that makes their discount factor zero. This ensures that agents who have been hit by the shock value present consumption infinitely more than future consumption. This makes them sell their entire holdings of the risky asset and consume. Those

\(^{25}\)Two different mechanisms may generate this growing endowment. One could be increasing borrowing ability, which would arise in a model where lenders accepted the risky asset as collateral. The more the bubble grew, the more agents could borrow to fuel growth even further. The second could be gradual arrival of new agents into the market. Arrivals after time 1 could be rationalized by assuming gradual information diffusion, perhaps due to time constraints limiting the number of markets that an agent may follow at a given time. Before the boom, only a few agents would happen to be invested in the risky asset, and as the boom continued, the number of agents following (and investing in) the risky asset would grow exponentially. Once more, under either mechanism, remark 2.2.2 continues to apply, as the growing inflow of resources would eventually have to slow down in the long run. However, I again focus on endogenous crashes, i.e., I assume that sales lead to a crash before the growth of the bubble begins to collide against other limits in the environment.
who are not hit by the shock have a discount factor equal to $1/R$. Agents do not see how many agents are hit by the shock (i.e., $\theta_t$ is unobservable). Moreover, $\theta_t$ is not deterministic. Instead, 

$$\theta_t = \bar{\theta} + \sigma, \text{ where } 0 < \bar{\theta} < 1 \text{ and } \sigma_t \text{ is random i.i.d., uniform over } [-\bar{\sigma}, \bar{\sigma}], \text{ with } 0 < \bar{\sigma} < \min\{\bar{\theta}, 1-\bar{\theta}\}.$$

Since $\theta_t$ does not vary across types, it does not affect $h_{n,t}$, which denotes average holdings of the risky asset across type-$n$ agents.\footnote{Naturally, $h_{n,t}$ is the integral over all individual agents $i$ within type $n$, of their holdings $h_{n,i}(t)$, divided by the mass of type-$n$ agents $1/N$. For types that have not left the market, individual holdings differ across individuals, as agents recently hit by the shock hold fewer shares than agents who have not been hit for a while. For our purposes, however, it is not necessary to keep track of within-type distributions.} While all types are in the market, $h_{n,t} = 1$ for all $n$. During these periods, type-$n$ agents hit by the shock sell $\theta_t$ shares, and all others use $e_t$ to buy those shares. There is no exogenous cap on individual holdings, although short-sales constraints make it impossible to borrow infinite amounts of one asset to invest in the other. Once $z_t$ types sell, $h_{n,t}$ becomes zero for types who have sold, and $1/(1-z_t/N)$ for types who stay in the market. As before, a period proceeds in two steps. In Step 1, $\sigma_t$ is realized, type-$t$ agents observe their signal if $t \in \{t_0, \ldots, t_0 + (N-1)\}$, and agents who wish to trade place their orders. In Step 2, orders are combined and executed at the market clearing price $p_t$, and agents hit by the shock consume.

In periods $t > 0$ before sales begin, a mass $\theta_t$ of agents (i.e., those hit by the shock) demand zero shares of the risky asset, while the remaining mass $1-\theta_t$ demand $1 + e_t/p_t$ shares. With supply fixed at 1, the market clears if 

$$(1-\theta_t)(1 + e_t/p_t) = 1,$$

or, equivalently, if 

$$p_t = \left((\theta_t)^{-1} - 1\right)e_t.$$ 

This can also be derived by simply noting that $(1-\theta_t)e_t$ units of the riskless asset buy $\theta_t$ shares of the risky asset. If $e_t = \zeta G'$ and $\zeta = \bar{\theta} / (1-\bar{\theta})$, prices during the boom fluctuate around $G'$, and fall to $G'^{(t+1)}R^{(T+1)-(t+1)}$ in the crash, exactly as in the model with behavioral agents. For other values of $\zeta$, all prices for $t \geq 1$ are rescaled by a constant. This does not affect results.
When sales first begin, and $z_t > 0$ types exit the market, the mass of shares for sale rises to $\theta_i + (1-\theta_i)z_t / N$, and the resources used to buy them fall to $(1-\theta_i)(1-z_t / N)e$. Therefore,

$$p_t = \left( \frac{\theta_i + (1-\theta_i)z_t}{N} \right)^{-1} e_t.$$

With $\theta_i = \bar{\theta} + \sigma_i$, and assuming that $\sigma_i$ and $z_t / N$ are small enough that $\sigma_i z_t / N \approx 0$, we have

$$p_t \approx \left( \frac{\bar{\theta} + \sigma_i + (1-\bar{\theta})z_t}{N} \right)^{-1} e_t. \quad (22)$$

Thus, $\bar{\sigma} / (1-\bar{\theta})$ plays the same role that $\bar{e}$ played in the model with behavioral agents. As long as $\sigma_i + (1-\bar{\theta})z_t / N$ is below $\bar{\sigma}$, agents in the market cannot discern whether a low price—or, equivalently, a high volume—is due to noise or to early types selling their shares. As before, if $1 / N > 2\bar{\sigma} / (1-\bar{\theta})$, noise cannot hide sales by even one type. If $2 / N > 2\bar{\sigma} / (1-\bar{\theta}) > 1 / N$, noise may hide sales by one type, but not two. In that case, let $\pi \equiv [1 / N] / [2\bar{\sigma} / (1-\bar{\theta})]$.

From this point onward, we can specify Strategy Profiles 1 and 2 as given by (12) and (15), augmented to accommodate shock-induced sales. Then, the analysis of Section 4 applies, with two modifications. First, the assumption that $\alpha \approx 0$ in Section 4 ensures that, during the boom, price fluctuations relative to trend matter only due to their informational content. That is, the fluctuations are so small that $p_t \approx G^t$ during the boom is a good approximation as far as revenue is concerned. To make a similar simplification here, note that, given (22), pre-crash price fluctuations around trend are small only if $(1-\bar{\theta})z_t / N$ is small. As previously discussed, under Strategy Profile 2, the maximum number of types that could possibly sell before the crash is $d + 2$. Thus, only if $(1-\bar{\theta})(d+2) / N$ is small, the growth rate of $p_t$ during the boom is approximately equal to $G$.

The second modification concerns the term $W_d$ in Lemma 4. In Section 4, I compute $W_d$ under the premise that, for $d$ periods, type-$n$ agents sell with probability $1-\pi$ (i.e., if the price is medium) and continue to ride the bubble with probability $\pi$ (i.e., if the price is high). Now, agents continue to ride the bubble with probability $\pi(1-\bar{\theta})$ (i.e., if the price is high and the agent is not hit by the shock). Thus, in a modified Lemma 4, the parameter restriction to rule out early sales (for $d$ above some threshold) would be $\pi(1-\bar{\theta})G / R > e^3$ instead of $\pi G / R > e^3$. 

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Besides the restriction that \((1 - \overline{\theta})(d + 2)/N\) must be small, this is the only parameter restriction that changes. In Proposition 1 and Lemmas 1-3, inequalities (13), (16) and (17) are unaffected by the new assumptions. These inequalities capture situations where a type-\(n\) agent with discount factor \(1/R\) chooses to sell now or to sell next period, having no intention to ride the bubble for multiple periods under any circumstances. It is for this reason that the preference shock has no effect on these inequalities.

By examining Figure 5, which summarizes equilibrium conditions, we can see that the main argument in the proof of a modified Proposition 3 would be very similar to that of the original Proposition. In the modified Figure 5, the only line that would change would be \(G/R = e^\lambda/\pi\), which would be replaced by \(G/R = e^\lambda/[\pi(1-\overline{\theta})]\). Clearly, for low enough \(\overline{\theta}\), there would still be nonempty shaded area, and thus, long bubbles would arise.27

6 Conclusion
This paper extends existing models of greater fool’s bubbles (Allen, et al. (1993), Conlon (2004), and especially Abreu and Brunnermeier (2003)) by introducing noisy prices and price responsiveness to selling pressure. These features make it possible to circumvent the main critiques of these models, which assumed that exact parameter proportions held or that prices were to some extent independent of sales. By generating bubbles in a model with multidimensional uncertainty, I show that bubbles can arise even if prices are fully endogenous. Moreover, these bubbles are robust to small changes in parameters. In an extension, I address another critique of prior models, namely the presence of behavioral agents. I specifically present a fully rational version of the model, thereby demonstrating that the mechanism generating bubbles does not necessarily hinge on irrationality. In short, the mechanism generating bubbles in the aforementioned papers does not crucially depend on special assumptions.

Future work should attempt to further refine models of bubbles. For example, as Brunnermeier (2001) points out, bubbles burst abruptly in models, but often deflate gradually in reality. A version of the current model where the noisy component was not bounded might generate this sort of gradual decline, since, for any price growth rate, some agents could still think that prices may rebound, and after the peak, agents would become gradually convinced that

27 To construct equilibria with long bubbles, one would pick a \(\overline{\theta}\) low enough to have a nonempty shaded area in the modified Figure 5. Then, pick a \((G/R, \pi)\) pair in this area, compute a new \(\overline{\theta}\) (taking into account the modified way of computing \(W_j\)) and, finally, choose \(N\) large enough to make \((1 - \overline{\theta})(d + 2)/N\) small.
the growth is over. The analysis could also be extended by focusing on equilibria where agents sell and reenter the market frequently. In such an extension, high and variable trading volume would arise endogenously, instead of being assumed.
APPENDIX A — Derivation of $\Gamma$

Evaluate (13) at $\tau^* = 1$, set $x = G / R$ and find roots of $1 = x^{-2} / (1 + e^{-x}) + xe^{-x} / (1 + e^{-x})$:

\[
1 + e^{-x} = x^{-2} + e^{-x} x \iff (1 + e^{-x})x^2 = 1 + e^{-x} x^3 \iff x^2 - 1 = e^{-x} (x^3 - x^2) \iff (x + 1)(x - 1) = e^{-x} x^2 (x - 1).
\]

Clearly, $x = 1$ is a root. For $x \neq 1$, we have, $x + 1 = e^{-x} x^2 \iff 0 = x^2 - e^x x - e^x$. The quadratic formula yields roots $x = (1 \pm \sqrt{1 + 4e^{-x}}) / 2$. Let $\Gamma = e^x / (1 + \sqrt{1 + 4e^{-x}}) / 2$ be the positive root, which is always above one. ($\Gamma$ is increasing in $\lambda$ and approaches $(1 + 5) / 2 \approx 1.618$ as $\lambda \to 0$.)

Inequality (13) evaluated at $\tau^* = 1$ holds if $1 < G / R < \Gamma$ and fails if $G / R \geq \Gamma$.

APPENDIX B — Proofs of Lemmas 2 and 4

Proof of Lemma 2 — Consider an arbitrary type $n \geq 1$ at $t = n + \tau^{**} - j$, for any $\tau^{**} \geq 0$ and $j \geq 1$. Let $p_{t-1}$ be high. A type-$n$ agent may be inclined to sell preemptively at $t$ for fear that the bubble may burst at $t + 1$. In fact, if $t_0 = n - j$, type-$t_0$ agents will sell at $t = n + \tau^{**} - j$, causing a crash at $t + 1$ with probability $\pi$. In, and only in, the following cases (i)-(iv), type-$n$ agents are not tempted to sell at $t$ because $t_0 = n - j$ is either impossible (i-iii), or very unlikely (iv):

(i) If at least one of the prices $p_{t-(d+1)}, \ldots, p_{t-2}$ is medium, $t_0$ cannot be $n - j$. (Note that, since $t - (d + 1) = n - j + \tau^* - 1$, if $p_{t-s}$ was medium for some $s \in \{2, \ldots, d + 1\}$ and $t_0$ was $n - j$, type-$t_0$ agents would have sold at time $t - s + 1$, and $p_{t-1}$ would not be high.)

(ii) If $j \geq n$, $t_0$ cannot be $n - j$, since $t_0$ cannot be less than 1.

(iii) If $j \geq N$, $t_0$ cannot be $n - j$, since $t_0$ cannot be less than $n - (N - 1)$.

(iv) If $j > \tau^{**}$, type-$n$ agents have yet to observe the signal as of time $t$. If $\tau^{**} \geq N - 1$, sales cannot begin before all signals arrive. If $\tau^{**} < N - 1$, $\text{supp}_{n,t}(t_0) = \{\tau_0 | \tau_0 \geq n - j\}$, and $\mu_{n,t}(n - j)$ is $1 - e^{-\lambda}$. Since $\lambda < G / R$, type-$n$ agents prefer not to sell at $t$.

Having ruled out preemptive sales if one or more of (i)-(iv) hold, it remains to discuss situations where none of these conditions apply, i.e., cases where $j \leq \min\{\tau^{**}, n - 1, N - 1\}$ and $p_{t-s}$ is high $\forall s = 1, \ldots, d + 1$. To rule out preemptive sales in these cases, it suffices to focus on the case where $j = 1$. To see why, note that at $t = n + \tau^{**} - j$, $\text{supp}_{n,t}(t_0) = \{n - j, \ldots, n\}$, with the probability that $t_0 = n - j$ given by $\mu_{n,t}(n - j) = 1 / (1 + e^{-\lambda} + \cdots + e^{-\lambda j})$, which is greatest for $j = 1$. Thus, if type-$n$ agents do not to sell preemptively if $j = 1$, they will not do so either if
Let us then consider the situation faced by type-$n$ agents at \( t = n + \tau^* - 1 \). High prices \( p_{t-(d+1)}, \ldots, p_{t-1} \) reveal to type-$n$ agents that they were either first or second to observe the signal, i.e., that \( t_0 \) must be \( n-1 \) or \( n \). By (10), probabilities \( \mu_{n,n}(n-1) \) and \( \mu_{n,n}(n) \), are respectively given by \( 1/(1+e^{-\lambda}) \) and \( e^{-\lambda}/(1+e^{-\lambda}) \). If \( t_0 = n-1 \), the first type will sell at \( t \). With probability \( \pi \), \( p_t \) will be low, causing a crash at \( t+1 \), and with probability \( 1 - \pi \), \( p_t \) will be medium, and \( d+1 \) types will sell at \( t+1 \) at the expected price \( G^{t+1} \). (Since \( d+2 < \kappa N \), behavioral agents will be able to buy the shares). If \( t_0 = n \), nobody will sell at \( t \), and type-$n$ agents will sell at \( t+1 \) at a price \( G^{t+1} \). In sum, waiting is best if (17) holds. If \( G/R \geq (1+e^{-\lambda})/(1+e^{-\lambda} - \pi) \), (17) holds for any \( \tau^* \). Otherwise, it holds only for \( \tau^* \) under the threshold given by (18). Q.E.D.

**Proof of Lemma 4** — The proof proceeds in two steps. In \( \text{Step 1} \), I derive conditions under which type-$n$ agents choose not to sell at \( t = n + \tau^* - 1 \) with medium \( p_{t-1} \) if \( n \geq d+3, \ \tau^* \geq 1, \) and \( p_{t-s} \) is high \( \forall s = 2, \ldots, d+2 \). In \( \text{Step 2} \), I show that, of all possible situations cases with \( t < n + \tau^* \) and medium \( p_{t-1} \), type-$n$ agents are most tempted to sell in the situation considered in \( \text{Step 1} \). Consequently, the conditions ruling out preemptive sales in \( \text{Step 1} \) also suffice to rule out preemptive sales by type-$n$ agents in all other situations with \( t < n + \tau^* \) and medium \( p_{t-1} \).

\( \text{Step 1} \) — Suppose that \( t = n + \tau^* - 1, \) \( p_{t-1} \) is medium, \( n \geq d+3, \ \tau^* \geq 1, \) and \( p_{t-s} \) is high \( \forall s = 2, \ldots, d+2 \). Then, type-$n$ agents think that \( t_0 \) could be anywhere from \( n-(d+2) \) to \( n \), i.e., \( \text{supp}_{n,s}(t_0) = \{n-(d+2), \ldots, n\} \). If \( n-(d+2) \leq t_0 \leq n-2 \), two or more types will sell at \( t \), causing a crash. If \( t_0 = n-1 \), type \( n-1 \) will sell, causing a crash with probability \( \pi \). In sum, if \( t_0 < n \), waiting brings losses. To simplify, assume that, if \( t_0 < n \), the payoff from waiting is zero. Given this, type-$n$ agents will wait at \( t \) only if the expected (gross) return \( W_d \) if \( t_0 = n \) is sufficiently large. Since \( e^{-\lambda(d+2)}/(1 + \cdots + e^{-\lambda(d+2)}) \) is the probability that \( t_0 = n \), (20) is a sufficient condition for type-$n$ agents to be willing to wait at time \( t \).

To derive (19), note that \( W_d \) depends on how long type-$n$ agents ride the bubble after \( t \), which could amount to a maximum of \( d+1 \) periods. If \( t_0 = n \), every period from \( t+1 = n + \tau^* \) to \( t+d = n + \tau^{**} - 1 \), type-$n$ agents will sell if the last price is medium (which will occur with probability \( 1 - \pi \)) and wait if it is high (which will occur with probability \( \pi \)). If all prices
\( p_1, \ldots, p_{t+d-1} \) are high, they will sell at \( t + d + 1 = n + \tau^* \), regardless of whether \( p_{t+d} \) is high or medium. Thus, \( W_d \) is given by

\[
W_d = (1 - \pi) \frac{G}{R} + \left( \frac{G}{R} \right)^2 + \cdots + \left( \frac{G}{R} \right)^{d+1}.
\]

Since \( \pi G / R \neq 1 \), this can be rewritten as (19). Since \( \pi G / R > e^\lambda \), \( e^{-\lambda(d+2)} W_d \) grows exponentially with \( d \). In turn, this implies that (20) holds for high enough \( d \). To see this, substitute (19) into (20) and rearrange terms to obtain

\[
\frac{1 - e^{-\lambda(d+3)}}{1 - e^{-\lambda}} < e^{-\lambda(d+2)} (1 - \pi) \left( \frac{G}{R} \right)^{d+1} \left( \frac{\pi G}{R} \right) + \frac{G}{R}.
\]

(23)

The left-hand-side of (23) is increasing in \( d \), but it approaches \( 1/(1 - e^{-\lambda}) \) as \( d \to \infty \). On the right, all terms are positive, and since \( \pi G / R > e^\lambda \), some terms grow exponentially with \( d \). Thus, (23) holds for \( d \) above some threshold \( \bar{d} \). Finally, I derived (19) and (23) assuming that all is lost in the crash, a good approximation if \( \tau^* \) is large. But for small \( \tau^* \), type-\( n \) agents are even less inclined to sell at \( t \), because they will lose less in the event of a crash.

**Step 2** — Of all possible cases with \( t < n + \tau^* \) and medium \( p_{t-1} \), type-\( n \) agents are most inclined to sell if \( t = n + \tau^* - 1 \), \( \tau^* \geq 1 \), and \( p_{t-s} \) is high \( \forall s \in \{2, \ldots, d + 2\} \). In this case, the support of \( t_0 \) contains \( d + 2 \) ‘bad’ values \( n - (d + 1), \ldots, n - 1 \), for which \( t_0 < n \), and one ‘good’ value, \( t_0 = n \), for which waiting at \( t \) yields a large return \( W_d \). In all other cases with \( t < n + \tau^* \) and medium \( p_{t-1} \) the support of \( t_0 \) contains less bad values and/or more good values, making a crash at \( t + 1 \) less likely, and a large return more likely. To see this, observe how \( \text{supp}_{p_0}^n(t_0) \) changes when \( p_{t-1} \) is medium, but it is no longer the case that \( t = n + \tau^* - 1 \), \( \tau^* \geq 1 \), \( n \geq d + 3 \), and \( p_{t-s} \) is high \( \forall s \in \{2, \ldots, d + 2\} \). Every change in conditions makes waiting more attractive, by reducing the number of bad values or increasing the number of good values of \( t_0 \) in \( \text{supp}_{p_0}^n(t_0) \).

(i) Let \( t = n + \tau^* - j \), \( j \geq 2 \), and \( j \leq \tau^* \), (with \( n \geq d + 3 \) and high \( p_{t-s} \) \( \forall s \in \{2, \ldots, d + 2\} \)). If \( j \leq N - (d + 2) \), the set \( \text{supp}_{p_0}^n(t_0) = \{n - j - (d + 1), \ldots, n\} \) contains \( d + 2 \) bad values \( n - j - (d + 1), \ldots, n - j \), and \( j \) good values \( n - j + 1, \ldots, n \). Moreover, the expected return
if \( t_0 > n - j + 1 \) exceeds \( W_d \). If \( j > N - (d + 2) \), there are less than \( d + 2 \) bad values, because \( n - j - (d + 1) < n - (N - 1) \).

(ii) If \( t = n + \tau^* - j \) and \( j > \tau^* \), (with \( n \geq d + 3 \) and high \( p_{r-s} \\forall s \in \{2, \ldots, d + 2\} \)), \( \text{supp}_{n,t}(t_0) \) equals \( \{\tau \mid \tau \geq n - j - (d + 1)\} \), i.e., type-\( n \) agents have yet to observe the signal as of time \( t \). There may be up to \( d + 2 \) bad values, but there are infinitely many good values.

(iii) If \( n < d + 3 \), (with \( t = n + \tau^* - 1 \), \( \tau^* \geq 1 \), and high \( p_{r-s} \\forall s \in \{2, \ldots, d + 2\} \)), type-\( n \) agents know, from their signal, that \( t_0 \) cannot be \( n - (d + 2) \), since \( n - (d + 2) \leq 0 \). Type-\( n \) agents can thus eliminate \( d + 3 - n \) bad values from \( \text{supp}_{n,t}(t_0) \).

(iv) If there are \( k < d + 1 \) consecutive high prices before \( p_{r-s} \), \( \text{supp}_{n,t}(t_0) = \{n - (k + 1), \ldots, n\} \), i.e., the number of bad values falls from \( d + 2 \) to \( k + 1 \). This makes the good value \( t_0 = n \) relatively more likely. Also note that \( p_{r-s} \) can be high \( \forall s \in \{2, \ldots, d + 2\} \) only if \( t \geq d + 3 \).

If more than one of (i)-(iv) apply, several factors make waiting more desirable than in Step 1. The number of good values exceeds 1 and/or the number of bad values falls below \( d + 2 \). Q.E.D.

**APPENDIX C — Proof that \([G/R]_t < \Gamma \).**

\[
[G/R]_t < \Gamma \iff \frac{\mathcal{F}(1+e^{-2\lambda})}{1+e^{-\lambda}} < \frac{\mathcal{F}(1+\sqrt{1+4e^{-\lambda}})}{2} \iff 2\left(1+\frac{e^{-2\lambda}}{1+e^{-\lambda}}\right) < 1+\sqrt{1+4e^{-\lambda}} \\
\iff 1+\frac{2e^{-2\lambda}}{1+e^{-\lambda}} < \sqrt{1+4e^{-\lambda}} \iff \frac{4e^{-2\lambda}}{1+e^{-\lambda}} + \frac{4e^{-4\lambda}}{(1+e^{-\lambda})^2} < 4e^{-\lambda} \iff \frac{\mathcal{H}e^{-2\lambda}}{1+e^{-\lambda}} + \frac{\mathcal{H}e^{-4\lambda}}{(1+e^{-\lambda})^2} < \mathcal{H}e^{-\lambda} \\
\iff e^{-2\lambda}(1+e^{-\lambda}) + e^{-4\lambda} < e^{-2\lambda}(1+e^{-\lambda})^2 \iff e^{-\lambda} + e^{-2\lambda} + e^{-3\lambda} < 1+2e^{-\lambda} + e^{-2\lambda} \iff e^{-3\lambda} < 1+e^{-\lambda}.\text{Q.E.D.}
\]

**APPENDIX D — Details of Subsection 5.1**

Consider a type-\( n \) agent at time \( t = n + \tau^* - j \), with \( j \geq 1 \) and high \( p_{r-s} \). Instead of waiting, agents can sell with the option to reenter later. This has the benefit of protecting the agent against a crash if the bubble bursts, and the cost of foregoing capital gains (between the time of sale and reentry) if the bubble continues to grow. Clearly, since the benefit from preemptive selling is avoiding a crash, preemptive selling is suboptimal in cases (i)-(iv), as listed in the proof of Lemma 2. In all these cases, the crash probability is either zero or very small.

Thus, as in Lemma 2, the situations of interest are those with \( j \leq \min\{\tau^*, n - 1, N - 1\} \) and high \( p_{r-s} \ \forall s = 1, \ldots, d + 1 \). To revisit type-\( n \) agents’ sell-or-wait tradeoffs in these situations,
let us first invest a little in notation, letting $\psi_j$ denote the (gross expected discounted) return earned by a type-$n$ agent if she follows the equilibrium strategy from $t = n + \tau** - j$ onward. Note that $\psi_1$ equals the right-hand-side of (17), and that, if $2 \leq j \leq d + 1$,

$$\psi_j = \pi \frac{1}{1 + \cdots + e^{-\lambda j}} \left( \frac{G}{R} \right)^{(e^{**} + 1)} + \frac{G}{R} \left[ (1 - \pi) + \pi \left( 1 - \frac{1}{1 + \cdots + e^{-\lambda j}} \right) \psi_{j-1} \right].$$

That is, if the agent waits at $t$, the bubble will burst with probability $\pi / (1 + \cdots + e^{-\lambda j})$, the agent will sell at $t + 1$ if $p_t$ is medium—which will happen with probability $(1 - \pi)$—and with the remaining probability, $p_t$ will be high and the agent will earn $G / R$ times $\psi_{j-1}$. Now compare this with the expected return from selling preemptively and reentering at $t + 1$ if $p_t$ is high. This return is 1 if the agent does not reenter and $\psi_{j-1}$ if she reenters—which she will do with probability $(1 - \pi)$. Hence, the return from selling and possibly reentering is

$$1 - \pi \left( 1 - \frac{1}{1 + \cdots + e^{-\lambda j}} \right) + \pi \left( 1 - \frac{1}{1 + \cdots + e^{-\lambda j}} \right) \psi_{j-1}.$$

Clearly, if (17) holds and $\psi_{j-1} = 1$, $\psi_j$ exceeds the value of this last expression. The difference only grows when taking into account the fact that, by Lemma 2, $\psi_{j-1} > 1$ for $2 \leq j \leq d + 1$. A similar argument (a bit more cumbersome notationally, and available upon request) rules out preemptive selling with reentry option if $j$ exceeds $d + 1$, in which case agents will reenter the market after selling even after a medium $p_t$.

Finally, note that, since staying in the market is better than selling preemptively (with reentry option) for all $j$, agents have no incentive to sell preemptively and reenter after multiple periods. Staying out for more than one period serves only to compound the expected opportunity costs relative to staying in the market.
References


