A Finite Model of Riding Bubbles*

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Abstract

When asset prices boom over extended periods of time, many investors begin to worry about bubbles. However, even those who believe that assets are overpriced may stay in the market believing that they can rise even further before correcting. Abreu and Brunnermeier (2003; AB) model this idea in an environment with rational and behavioral agents, and more recently, Doblas-Madrid (2012; D-M) constructs a fully rational version of the AB model. These models conceptualize a bubble as a boom that is at first justified by fundamentals, but overshoots as asymmetrically informed agents ride the bubble hoping to sell to a greater fool. A critique of these papers is that, although bubbles are finite, they can only arise in equilibrium if prices can grow at extraordinary rates indefinitely. In this paper, I articulate this critique in a simplified D-M environment and show how it can be overturned by modifying investors' strategies. If the number of periods an investor plans to ride the bubble is conditional on her signal of fundamental value, one can sustain speculative bubbles in a finite model, where by construction it is impossible for prices to boom indefinitely.

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1 Introduction

In economics, a bubble is defined as a situation where price differs from fundamental value. This well-known definition is unfortunately not very meaningful, since fundamental value can be defined in many different ways in the context of very different environments. In some models, price diverges from fundamental value due to deviations from rational expectations on the part of some or all agents.[1] In fully rational models, on the other hand, agents buy an asset only if the benefits of owning the asset justify the price. Depending on the environment, those benefits may include dividends, liquidity, expectations of speculative gains, loosening of credit constraints, risk-shifting and others. Dividends are always considered fundamental, but whether other benefits are considered fundamental or bubbly varies widely. For instance, in the 'rational bubbles' literature à la Samuelson (1958) and Tirole (1985), bubbles provide liquidity benefits that enhance efficiency by overcoming a shortage of stores of value. Fiat money is a well-known example of this type of rational bubble. In Allen and Gale (2000), Allen and Gorton (1993) and Barlevy (2014), bubbles are based on risk-shifting. That is, agents use borrowed funds to buy assets and are willing to pay more than expected dividends because they shift losses to lenders in bad states of the world. Other models—some fully rational, others combining rational and irrational agents—have a speculative flavor, where agents are willing to pay above expected dividend for an asset if they have a chance to resell it at even higher prices to a so-called greater fool. In addition to partially rational models by Abreu and Brunnermeier (2003; AB henceforth), Delong et al. (1990), Harrison and Kreps (1978) and Scheinkman and Xiong (2003), some fully rational models in this vein include Allen et al. (1993), Conlon (2004) and Doblas-Madrid (2012; D-M henceforth).

The D-M model is a fully rational version of AB, which was the first paper to combine the speculative motive for riding bubbles with the notion of a bubble as an instance of overshooting to a fundamental shock.[2] In reality, asset price booms rarely last long without igniting a debate about bubbles, in which one side argues that prices are 'irrationally exuberant' and the other

[1] For instance, in Abreu and Brunnermeier (2003; AB henceforth) and Delong et al. (1990) there is a mix of rational and behavioral agents, while Lansing (2010) explores the bounded rationality case. In Harrison and Kreps (1978) and Scheinkman and Xiong (2003), expectation formation deviates from the standard rational expectations model due to heterogeneous beliefs/overconfidence.

[2] In a recent application, Asako and Ueda (2014) use the AB model examine the effect of public warnings on bubbles.

that prices are justified by fundamentals. The model captures this intuition by assuming that a price boom is in line with fundamentals up to period t_0 but becomes a bubble if it continues past that point. However, the crucial date t_0 is imperfectly observed. Different agents observe different private signals ranging from t_0 through $t_0 + N - 1$. That is, signals divide agents into N types $\{t_0, ..., t_0 + N - 1\}$. Given her signal, an agent knows that the true value of t_0 may be equal to her signal, or as many as $N - 1$ periods before her signal, or anywhere in between. In equilibrium, uncertainty about the bubble's starting date translates into uncertainty about the bursting date, hence diffusing the backward induction argument that typically rules out finite bubbles. Thus, agents play a market timing game against each other, trying to ride the bubble and make profits by selling to others before the crash. The bubbles generated by this model take the form of a finite boom-and-bust without desirable properties in terms of efficiency. Ex-post, the bubble is a zero-sum game that transfers resources from unlucky to lucky agents. Bubbles in the AB/D-M model are thus similar to pyramid schemes, which are viewed as undesirable, and in fact are forbidden by many countries' legal codes. This notion of bubbles contrasts with that of efficiency enhancing 'rational bubbles', which do not unravel because they last indefinitely, either literally or in expectation.

A critique of the AB/D-M model is that, while bubbles per se are finite, they can only arise if it is possible for asset prices to boom indefinitely. In this paper, I formally articulate this critique and show how to overturn it. To this end, I consider a version of the D-M model, simplified in some dimensions and generalized in others. In the model, every period a fraction of agents are hit by preference shocks which make them impatient, i.e., a shock that makes them liquidate assets in order to consume. The main simplification I introduce is that the fraction of agents hit by this 'impatience' shock is constant, instead of randomly variable. This assumption eliminates complications, making it easier to generalize other aspects of the analysis. Specifically, in this paper I carry out the analysis for general values of $N \geq 2$. That is, I relax the large-N assumption, which is maintained in D-M in order to abstract from the price impact of the sales by the early-signal agents who succeed in selling at the peak of the bubble.

In the model, prices boom as long as patient agents devote all of their growing endowments to acquire the shares sold by impatient agents. Specifically, agents' endowments grow at the rate G, which is higher than the risk-free rate R. If agents continue to fuel price growth at extraordinary rates after period t_0, the initially fundamental boom overshoots and becomes a bubble. For the benchmark analysis, I focus on the case where—as in AB and D-M—agents follow symmetric trigger strategies. That is, each agent plans to ride the bubble for τ^* periods after their signal. Restricting attention to this class of strategies, in Proposition 1 I characterize the values of bubble duration τ^* that can be sustained in equilibrium. All the equilibria described have the property that, regardless of how small τ^* is, prices grow at the rate G until period $t_0 + \tau^*$, where t_0 has infinite support. The infinite-boom critique states that, unless the volume of trade shrinks over time, it takes exponentially growing resources to buy a given number of shares at exponentially growing prices.[3] Assuming a vanishing trade volume, however, is directly at odds with the evidence—emphasized by Scheinkman and Xiong (2003)—that trading volume during bubble episodes tends to increase, not decrease over time. Even more problematic than this empirical failure is the fact that, with vanishing trading volume, the sales of the lucky early-signal agents would flood the market in the last period of the boom. For any finite N, it is straightforward to show that, if booming prices are driven by shrinking supply instead of booming demand, the market cannot absorb the sales of 'winners' of the market-timing game without prices collapsing. Being unable to realize their speculative profit, agents would have no incentive to ride the bubble.

Having established the necessity of booming endowments, a simple corollary establishes that the equilibria characterized by Proposition 1 hinge on exponential endowment growth being maintained in perpetuity. Under symmetric strategies, if endowment growth is expected to slow down at any point in the future—no matter how remote—bubble duration is determined by the slower future rate instead of the current high rate. This is a direct consequence of the sequential rationality requirement of the Perfect Bayesian Equilibrium concept. Agents ride the bubble if they believe that there is a good chance that other agents observe signals after they do and those agents are able and willing to pay higher prices later. Iterating this argument, any slowdown in the growth rate of endowments unravels any equilibrium based on an initial high

[3] Doblas-Madrid and Lansing (2014) consider a model where the bubble is fuelled by a self-reinforcing feedback loop between collateral prices and credit, instead of by exogenously growing endowments. This critique, however, also applies to their setup, since rapid bubble growth is possible only to the extent that lenders can make ever-larger loans at a fixed risk-free global interest rate. This small-open-economy assumption captures the idea that the funds lent into the model economy are negligible relative to the global capital market. However, said loans could not grow at a rate faster than the risky-free rate indefinitely without eventually exerting upward pressure on global interest rates.

growth rate.

After presenting the critique, I proceed to show how it can be overturned. The solution I propose is to consider asymmetric, tapered strategies, by which the bubble riding time is shorter for higher-signal agents. To illustrate how such tapered equilibria can be sustained, I build a finite version of the model, where the support of t_0 is bounded above by a finite maximum $\overline{t_0}$, and this maximum is common knowledge. The tapered strategies I propose can resemble the symmetric trigger strategies for agents whose signals are much earlier than $\overline{t_0}$, but the bubble riding time τ falls as the horizon nears. I show that, as long as the speed of tapering is not too rapid, and bubble riding time is zero for agents whose signal is $\overline{t_0} - 1$ or higher, one can construct Perfect Bayesian Equilibria that resemble the symmetric ones for low realizations of t_0, but do not unravel despite the presence of a known terminal date. The resulting equilibria feature bubble riding, but do not rely on any notion of infinity. This tapered-strategies solution does not apply to the AB environment, where a symmetric trigger strategy equilibrium is proven to be unique.[4] Thus, although equilibrium multiplicity is generally viewed as a disadvantage, in this context the multiplicity in D-M has the advantage of allowing for asymmetric strategies in order to overcome the infinite-boom critique.

As far as the informational structure is concerned, the finite model presented in this paper is quite similar to the models developed by Allen et al. (1993), and especially Conlon (2004), where bubbles are not common knowledge, but are robust to n-order knowledge. In other important dimensions, however, the finite model presented here is quite different from the aforementioned papers, since here the asset price boom is made possible by binding—but gradually loosening—constraints, an element which is absent from Allen et al. (1993) and Conlon (2004).

The rest of the paper is organized as follows. In Section 2, I describe the infinite model and define equilibrium. In Section 3, I characterize bubble duration with symmetric strategies and present the infinite-resources critique. In Section 4, I present the finite model and present equilibria with tapered strategies. In Section 5, I conclude.

[4]Uniqueness in AB is proven under the belief restriction that whenever an agent exits the market, she believes that other agents who observed the signal before her are also leaving, or have already left the market. This belief restriction, however, is satisfied in the proposed tapered equilibrium.

2 The Infinite Model

2.1 Environment

The infinite model is based on D-M. Time is discrete with periods labeled $t = 0, 1, \ldots, t_{pay}$, where the terminal period t_{pay} is stochastically drawn from a distribution with infinite support. There is a continuum of rational agents indexed by $i \in [0, 1]$ who trade a safe asset and a risky asset. The safe asset yields an exogenous gross return R and can be consumed. The risky asset is a claim to dividends $\{d_t\}$. There is unit mass of risky shares, which in period t trade at price p_t.

At the beginning of period t, agents receive an endowment of $e_t = G^t$ units of the safe asset, where the growth rate G exceeds the risk-free rate R. It is not possible to borrow against the value of future endowments.[5] Agents are risk-neutral and may be hit by preference shocks à la Diamond-Dybvig (1983), which at random times force them to liquidate assets and consume. Specifically, every period a randomly chosen fraction $\theta \in (0, 1)$ of agents is hit by a shock that makes them impatient, setting their discount factor $\delta_{i,t}$ to 0. The remaining mass of agents $1 - \theta$ are patient, and have discount factor $\delta_{i,t} = 1/R$. Agents do not die after being hit by shocks. In fact, since shocks are i.i.d., the likelihood of being impatient in a given period is always θ regardless of previous shock realizations. It is important to note that, while preference shocks generate trades that are independent of price expectations, they never force agents to get caught in the crash. On the contrary, some agents avoid losses because of the shock.

Agent i starts period with $b_{i,t} \geq 0$ and $h_{i,t} \geq 0$ units of the safe and risky assets, respectively. The aggregates corresponding to $b_{i,t}$ and $h_{i,t}$ are given by B_t and H_t, where $H_t = 1$ at all times. At time 0, I assume that for all i, $b_{i,0} = 0$ and $h_{i,0} = 1$. After receiving endowments and learning the realization of their preference shocks, agents visit an asset market, modeled as a Shapley-Shubik post where trading proceeds in two steps. In Step 1, agents make bids and offers as follows. In one bin, agent i enters a bid of $m_{i,t} \in [0, b_{i,t} + e_t]$ units of the safe asset, which she wishes to spend buying risky shares. In another bin, she enters $s_{i,t} \in [0, h_{i,t}]$ risky shares she intends to sell. In

[5] Doblas-Madrid and Lansing (2014) endogenize this growth in investable funds in a credit market where collateral values and loans form a self-reinforcing feedback loop.

Step 2, bids and offers are combined and the price is given by the ratio

$$p_t = \frac{M_t}{S_t}, \tag{1}$$

where M_t and S_t denote the aggregate bid and the aggregate number of shares for sale, respectively. It is important to note that agents submit their buy and sell orders in Step 1, before observing the price, and cannot change their choices in Step 2. Agent i leaves the asset market with

$$h_{i,t+1} = h_{i,t} + \frac{m_{i,t}}{p_t} - s_{i,t} \tag{2}$$

risky shares and safe balances given by $b_{i,t} - m_{i,t} + p_t s_{i,t}$. After the asset market closes, agent i collects a dividend $d_t h_{i,t+1}$, and consumes a fraction $\chi_{i,t} \in [0,1]$ of her final safe balances. Hence, consumption $c_{i,t}$ is given by

$$c_{i,t} = \chi_{i,t} \left[b_{i,t} - m_{i,t} + p_t s_{i,t} + d_t h_{i,t+1} \right], \tag{3}$$

and safe savings equal

$$\frac{b_{i,t+1}}{R} = \left(1 - \chi_{i,t}\right) \left[b_{i,t} - m_{i,t} + p_t s_{i,t} + d_t h_{i,t+1} \right]. \tag{4}$$

To recapitulate within-period timing, agent i begins period t with portfolio $(b_{i,t}, h_{i,t})$, collects her endowment e_t and learns the realization of her preference shock. She then chooses $a_{i,t} = (m_{i,t}, s_{i,t}, \chi_{i,t})$, where $m_{i,t}$ and $s_{i,t}$ capture her asset trades and $\chi_{i,t}$ her consumption choice. After the asset market closes, she collects dividends $d_t h_{i,t+1}$, consumes $c_{i,t}$ and starts the next period with portfolio $(b_{i,t+1}, h_{i,t+1})$.

For simplicity, I assume that risky shares pay a single dividend d_t at date t_{pay}. That is,

$$d_t = \begin{cases} \overline{F} R^{t_{pay}} & \text{if } t = t_{pay} \\ 0 & \text{if } t \neq t_{pay}. \end{cases} \tag{5}$$

As of time 0, the maturity t_{pay} and the magnitude of $\overline{F}$ are both unknown. I assume that $\overline{F}$ is given by

$$\overline{F} = \frac{1-\theta}{\theta}\left(\frac{G}{R}\right)^{t_0}, \tag{6}$$

where t_0 is a geometrically distributed random variable with support $\{0, 1, ...\}$ and (unconditional) pdf

$$\psi(t_0) = (1-\lambda)\,\lambda^{t_0}, \tag{7}$$

where $\lambda \in (0,1)$. This specification of dividends is designed to match the AB/D-M setup, where dividends are not explicitly specified, but it is assumed that an asset price boom is justified by fundamentals up to time t_0 and turns into a bubble if it continues past that point. During the boom, patient investors are fully invested in the risky asset. Since initial safe balances are zero, this results in an aggregate bid $M_t = (1-\theta)G^t$. Impatient investors are the only sellers, and thus the number of shares for sale is $S_t = \theta$ and the booming price is given by

$$p_t = \frac{1-\theta}{\theta}G^t. \tag{8}$$

The specification of $\overline{F}$ given by (6) is derived from the fact that the price at time t_0 equals $\overline{F}R^{t_0}$. A crucial assumption is that t_0 is not perfectly observed. Instead, at time 0, different investors observe different signals about the value of t_0. Signals range from t_0 to $t_0 + N - 1$, with an equal mass $1/N$ of agents observing each signal.[6] Private signals divide agents into N different types $\nu(i) \in \{t_0, ..., t_0 + N - 1\}$. Agent i knows $\nu(i)$, but not t_0. However, since agents understand that there are only N signals, they infer that t_0 cannot be earlier than $\nu(i) - (N-1)$ or later than $\nu(i)$. Combining this with the fact that t_0 cannot be negative, the support of t_0 conditional on $\nu(i)$ is given by

$$\text{supp}(t_0|\nu(i)) = \{\max\{0, \nu(i) - (N-1)\}, ..., \nu(i)\}, \tag{9}$$

while the conditional distribution of t_0 becomes

$$\psi(t_0|\nu(i)) = \begin{cases} \dfrac{\psi(t_0)}{\sum\limits_{\tau_0 \in \text{supp}(t_0|\nu(i))} \psi(\tau_0)} & \text{if } t_0 \in \text{supp}(t_0|\nu(i)) \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

[6] In AB and D-M, agents observe signals starting when the overvaluation begins. Here, I assume instead that signals are observed at time 0, which simplifies the exposition without affecting the main results.

Figure 1: [Insert here]

Signals order agents along a line, without them knowing their relative position in it. Since bubbles bring profits to agents early in the line and losses to those late in the line, it is crucial that all agents—including those late in the line—believe that there is a nontrivial probability that they could be early. This probability is higher the closer λ is to one.

To fix ideas, it is helpful to preview the analysis that will follow, where I focus on symmetric trigger-strategy equilibria. That is, type-$\nu(i)$ agents will plan to—while patient—ride the bubble for τ^* periods after observing their signal and sell at time $\nu(i) + \tau^*$. If all agents follow these strategies, the booming price is given by (8) for all $t \in \{0, ..., t_0 + \tau^* - 1\}$ as the only sales are the θ shares sold by impatient agents, and all patient agents invest fully in risky shares. When type-t_0 agents sell at the crash period $t_c = t_0 + \tau^*$, the number of shares for sale rises to $\theta + (1-\theta)/N$ as type-t_0 agents join impatient sellers. The mass of buyers correspondingly falls to $(1-\theta)(1-1/N)$ and the price is given by

$$p_{t_c} = \frac{(1-\theta)(1-1/N)}{\theta + (1-\theta)/N} G^{t_c}. \tag{11}$$

Once this price is observed, the value of $t_0 = t_c - \tau^*$ becomes known and the bubble bursts. Agents of type t_0 sell at the peak, while others—unless saved by an impatience shock—get caught in the crash. From time $t_c + 1$ onward, patient agents invest the fraction of their endowments in the risky asset needed to equate price p_t with discounted dividend $\overline{F}R^t$, and the rest in the safe asset. Finally, I assume that the payoff date t_{pay} is later than t_c, so that—as in AB and D-M—the bubble inflates and bursts before the risky asset pays dividends. The fundamental boom, bubble and crash arising under symmetric trigger strategies are graphically depicted in Figure 1.

2.2 Equilibrium

Since the definition and characterization of equilibrium are similar to D-M, here I will abbreviate the exposition as much as possible. A Perfect Bayesian Equilibrium (PBE) is strategies $a_{i,t} = (m_{i,t}, s_{i,t}, \chi_{i,t})$ and beliefs $\mu_{i,t}$, for all i, where strategies are best response to others' strategies, and beliefs are consistent with strategies. Given information $I_{i,t}$, which includes discount factor

$\delta_{i,t}$, signal $\nu(i)$, and price history up to the previous period $p^{t-1} = \{p_0, ..., p_{t-1}\}$, agent i's choices $a_{i,t}$ solve

$$\begin{aligned} V_{i,t}(b_{i,t}, h_{i,t}) &= \max_{(m_{i,t}, s_{i,t}, \chi_{i,t}) \geq 0} E\left[c_{i,t} + \delta_{i,t} V_{i,t+1}(b_{i,t+1}, h_{i,t+1}) | I_{i,t}\right] \\ \text{subject to} \quad m_{i,t} &\leq b_{i,t} + e_t,\, s_{i,t} \leq h_{i,t},\ \chi_{i,t} \leq 1,\ \text{(2), (3) and (4).} \end{aligned} \tag{12}$$

Equilibrium beliefs $\mu_{i,t}$ are probability distributions over values of t_0, consistent with $I_{i,t}$ and equilibrium strategies. At the start of time t, the set of values of t_0 consistent with signal $\nu(i)$, past prices p^{t-1}, and strategies is given by $\text{supp}_{i,t}(t_0)$. The equilibrium belief $\mu_{i,t}$ assigns probabilities to the different values in $\text{supp}_{i,t}(t_0)$ according to (10), adding over $\text{supp}_{i,t}(t_0)$ instead of $\text{supp}(t_0|\nu(i))$ in the denominator. Finally, equilibrium also requires that the market clearing condition (1) is satisfied.

3 Symmetric Equilibria

As in AB and D-M, I find equilibria using a guess-and-verify procedure. The strategies I propose are as follows. Impatient agents sell and consume everything. Patient agents of type $\nu(i)$ plan to ride the bubble for τ^* periods, unless they become impatient or the bubble is pricked before they sell. After the crash, they bid $G^{t_0} R^{t-t_0}$, which is less than their endowment G^t. They only consume in the final period t_{pay}. Letting $t_c = t_0 + \tau^*$ denote the crash period, strategies $a_{i,t}$ are specified as follows:

1. If $\delta_{i,t} = 0$:

$$(m_{i,t}, s_{i,t}, \chi_{i,t}) = (0, h_{i,t}, 1). \tag{13a}$$

2. If $\delta_{i,t} = 1/R$:

$$(m_{i,t}, s_{i,t}) = \begin{cases} (b_{i,t} + e_t, 0) & \text{if } t < \min\{\nu(i) + \tau^*, t_c + 1\} \\ (0, h_{i,t}) & \text{if } t = \nu(i) + \tau^* \text{ and } t < t_c + 1. \\ \left(G^{t_0} R^{t-t_0}, 0\right) & \text{if } t \geq t_c + 1. \end{cases} \tag{13b}$$

$$\chi_{i,t} = \begin{cases} 0 & \text{if } t < t_{pay} \\ 1 & \text{if } t = t_{pay}. \end{cases} \tag{13c}$$

Beliefs evolve as follows. As previously discussed, the signal allows type-$\nu(i)$ agents to narrow down the support of t_0 to (9). Moreover, as the boom continues, agents gradually drop values from this support. Specifically, if p_{t-1} was a boom price—as given by (8)—a type-$\nu(i)$ agent knows at the start of period t that $t_c > t-1$, i.e., that $t_0 \geq t - \tau^*$. When she observes the price p_t she will again update the support of t_0, either by learning that $t_0 = t - \tau^*$ in the event that p_t shows a deceleration, or by learning that $t_0 > t - \tau^*$ otherwise. In sum, at the start of time t, given $I_{i,t} = \{\nu(i), \delta_{i,t}, p^{t-1}\}$, the support of t_0 is is given by

$$\textit{supp}_{i,t}(t_0) = \begin{cases} t_0 \in \{\max\{0, \nu(i) - (N-1), t - \tau^*\}, ..., \nu(i)\} & \text{if } 0 \leq t \leq t_c \\ \{t_c - \tau^*\} & \text{if } t > t_c. \end{cases} \tag{14}$$

This profile of strategies and beliefs is a PBE if agents are willing to: (i) Sell when the strategy dictates that they should sell and (ii) Wait when the strategy dictates that they should wait.

Regarding (i), impatient investors sell since they do not value the future. For patient investors, if $G/R \geq 1 + [\theta(N-1)]^{-1}$, condition (i) holds for any $\tau^* > 0$.[7] To see why, note that whenever a type-$\nu(i)$ agent sells at $t_c = \nu(i) + \tau^*$, she knows that other type-$\nu(i)$ agents are selling at t_c, so that the price will reveal these sales and the bubble will burst between periods t_c and $t_c + 1$. They also know at this point that $\nu(i) = t_0$. Selling is optimal as long as the price p_{t_c} exceeds the (discounted) post-crash price $p_{t_0} R^{\tau^*}$, which is the case if sales of the first type do not have a larger price effect than τ^* periods of appreciation at the rate $G > R$. More precisely, type-$\nu(i)$ agents wish to sell at $\nu(i) + \tau^*$ if

$$\frac{(1-\theta)(1-1/N)}{\theta + (1-\theta)/N} G^{t_0 + \tau^*} \geq \frac{1-\theta}{\theta} G^{t_0} R^{\tau^*},$$

[7] A complication arising from the discreteness of the model is that $\tau^* = 0$ is not an equilibrium, since type $\nu(i)$ agents would not be willing to sell at time $\nu(i)$ at a price below below $\overline{F} R^{\nu(i)}$. The $\tau^* = 0$ equilibrium would reappear under a large-N assumption. It would also reappear if (i) as in D-M, signals started at $t_0 + 1$ and were observed at time $\nu(i)$ instead of time 0, or (ii) if the definition of $\overline{F}$ was modified by replacing $(1-\theta)/\theta$ with $(1-\theta)(1-1/N)/[\theta + (1-\theta)/N]$.

which is equivalent to

$$\left(\frac{G}{R}\right)^{\tau^*} \geq 1 + \frac{1}{\theta(N-1)}. \tag{15}$$

Thus, if $G/R \geq 1 + [\theta(N-1)]^{-1}$, selling is optimal for any $\tau^* \geq 1$.

Condition (ii) states that preemptive sales should be ruled out. To understand the tradeoff between preemptive selling and continued buying, consider a patient type-$\nu(i)$ agent at $t = \nu(i)+\tau^*-1$, just one period before she is supposed to sell. At this point—if $\nu(i) > 0$—her support of t_0 has just two elements $\{\nu(i)-1, \nu(i)\}$, with respective probabilities $1/(1+\lambda)$ and $\lambda/(1+\lambda)$. The agent may continue buying as dictated by the strategy, or deviate by selling preemptively. If she sells, she avoids the crash and sells at a price given by (11) if $t_0 = \nu(i)-1$ and by (8) if $t_0 = \nu(i)$. If the agent buys, she gets caught in the crash if $t_0 = \nu(i)-1$, and rides the bubble for one more period if $t_0 = \nu(i)$. In sum, buying is optimal if

$$\begin{aligned}\left[\tfrac{1}{1+\lambda}\tfrac{(1-\theta)(1-1/N)}{\theta+(1-\theta)/N} + \tfrac{\lambda}{1+\lambda}\tfrac{1-\theta}{\theta}\right] G^{\nu(i)+\tau^*-1} \leq \\ \tfrac{1}{1+\lambda}\tfrac{1-\theta}{\theta} G^{\nu(i)-1} R^{\tau^*} + \tfrac{\lambda}{1+\lambda}\tfrac{(1-\theta)(1-1/N)}{\theta+(1-\theta)/N}\tfrac{G^{\nu(i)+\tau^*}}{R}\end{aligned}$$

Dividing through by $G^{\nu(i)+\tau^*-1}$ and rearranging terms, we can rewrite the above inequality as

$$1 + \lambda\left(1 + \frac{1}{\theta(N-1)}\right) \leq \left(1 + \frac{1}{\theta(N-1)}\right)\left(\frac{G}{R}\right)^{-\tau^*} + \lambda\frac{G}{R} \tag{16}$$

Although one can already solve for τ^* from here, the tradeoff faced by agents is easiest to read in the large-N case. Specifically, if $1/[\theta(N-1)]$ is very small, the slowdown in price growth between periods t_c-1 and t_c still reveals information, but otherwise has negligible revenue effects. This allows for the approximation

$$1 \leq \frac{1}{1+\lambda}\left(\frac{G}{R}\right)^{-\tau^*} + \frac{\lambda}{1+\lambda}\frac{G}{R}, \tag{17}$$

which simply states that, relative to selling, buying yields crash losses with probability $1/(1+\lambda)$ and one more period of appreciation with probability $\lambda/(1+\lambda)$. Thus, the higher λ and G/R, the greater the bubble duration τ^* that can be supported in equilibrium. This is stated more precisely below in Proposition 1.

Proposition 1 *Assume that $\lambda G/R > 1$, that $G/R \geq 1 + [\theta(N-1)]^{-1}$, and that agents play trigger strategies as defined by (13a)-(13c). Then, the values of τ^* that can be supported in equilibrium depend on parameters as follows: a) If $G/R < \left[1 + \lambda\left(1 + \{\theta(N-1)\}^{-1}\right)\right]/\lambda$, equilibrium can be supported for any integer between 1 and an upper bound $-\ln(\{\theta(N-1)/[1+\theta(N-1)]\} + (1 - \lambda\{\theta(N-1)/[1+\theta(N-1)]\}G/R))/\ln(G/R)$. b) If $G/R \geq \left[1 + \lambda\left(1 + \{\theta(N-1)\}^{-1}\right)\right]/\lambda$, any integer $\tau^* \geq 1$ can be supported in equilibrium.*

Proof. In a PBE, agents must find it optimal to sell and buy as dictated by (13a)-(13c). The optimality of selling when impatient follows directly from the fact that $\delta_{i,t} = 0$. For any $\tau^* \geq 1$, patient type-$\nu(i)$ agents are willing to sell at $\nu(i)+\tau^*$—provided that the bubble has not yet burst. The sales of other type-$\nu(i)$ agents will be revealed by $p_{\nu(i)+\tau^*}$, and given the fact that $G/R \geq 1+[\theta(N-1)]^{-1}$, the price will fall from period $\nu(i)+\tau^*$ to $\nu(i)+\tau^*+1$. After the bubble has burst, buyers are indifferent between safe and risky assets and are willing (and able) to bid the risky asset up to fundamental value. Buying behavior before the crash hinges on willingness of patient type-$\nu(i)$ agents to buy while $t < \nu(i) + \tau^*$. Agents of type $\nu(i) = 0$ are never tempted to sell preemptively, since they always know that $t_0 = \nu(i)$. For agents of type $\nu(i) > 0$, the optimality of buying is governed by (16), which captures the buy-vs-sell tradeoff at time $\nu(i) + \tau^* - 1$, when preemptive selling is most attractive. At time $\nu(i) + \tau^* - 1$, the only two values in $\text{supp}_{i,t}(t_0)$ are $\nu(i) - 1$ and $\nu(i)$. Therefore, the probability that $t_c = \nu(i) - 1 + \tau^*$ is the probability that $t_0 = \nu(i) - 1$, which is given by $1/(1+\lambda)$. At any earlier time $\nu(i) + \tau^* - s$, for $s > 2$, preemptive selling is less attractive because the probability that $\nu(i) + \tau^* - s = t_c$ is $1/(1 + \lambda + ... + \lambda^s)$, clearly under $1/(1+\lambda)$. The thresholds in (a)-(b) follow directly from (16), and the upper bound on bubble duration is found by solving (16) for τ^*. Finally, the restriction that $\lambda G/R \geq 1$ ensures that $\tau^* = 1$ is an equilibrium, also by (16). ■

Given our focus on finite bubbles, the parameters of interest are those described by (a), which put an upper bound on how large τ^* can be in equilibrium. That said, within this parameter region, as $\lambda G/R$ approaches $1 + \lambda(1 + [\theta(N-1)]^{-1})$, the upper bound on τ^* becomes arbitrarily large. Thus, we can always find parameters such that $\tau^* > N$. Bubbles lasting more than N periods are *strong bubbles*, in the sense of Allen et al. (1993), since all agents know that the risky asset is overvalued, but continue to buy it.

Finally, given a finite τ^*, one can choose $\tau_{pay} > \tau^*$, and let the payoff period t_{pay} be given by $t_0 + \tau_{pay}$.

3.1 The Infinite-Boom Critique

AB and D-M both focus on finite bubbles. In D-M, the focus is on the region of the parameter space where τ^* is bounded. In AB, there is an explicit restriction that bubble duration cannot exceed a maximum $\overline{\tau}$. By assumption, any bubble is assumed to burst for exogenous reasons once its duration, in terms of time, has reached $\overline{\tau}$. Thus, per se, the symmetric trigger-strategy bubbles characterized above are finite, and thus differ from the infinitely lived 'rational bubbles'.

However, even if bubble duration τ^* is finite, the equilibria presented above rely on the possibility—and in fact, the expectation—of an infinite asset price boom. To be precise, note that the condition $\lambda G/R \geq 1$, which only guarantees that agents are willing to wait for one period after their signal, already implies that the expected value of $\overline{F}$ is infinite. More precisely, the unconditional expectation of the discounted dividend $\overline{F}$ is given by

$$E[\overline{F}] = \frac{1-\theta}{\theta} E\left[G^{t_0}\right] = \frac{1-\theta}{\theta}(1-\lambda)\left(1 + \lambda\frac{G}{R} + \left(\lambda\frac{G}{R}\right)^2 + \cdots\right),$$

which is infinite if $\lambda G/R \geq 1$. Moreover, to generate long bubbles, $\lambda G/R$ must be not only greater than 1, but close to $1 + \lambda\{1 + 1/[\theta(N-1)]\}$.

Most importantly, in the equilibria characterized by Proposition 1, rapid endowment growth must be sustained indefinitely. Any deceleration in the endowment and price growth rate—for example if t_0 had finite support—would imply, by backward induction, that bubbles of duration τ^* cannot be sustained in equilibrium. This argument is presented more rigorously in Corollary 2.

Corollary 2 *Suppose that the rate of endowment growth e_{t+1}/e_t is given by G while $t < T$, but slows down to $\underline{G} < G$ for $t \geq T$. Then, regardless of T, in any Perfect Bayesian Equilibrium the upper bound on τ^* is given by the lower rate $\underline{G}$.*

Proof. The conclusion follows directly from the sequential rationality requirement of the PBE equilibrium concept. When a type-$\nu(i)$ agent decides to ride the bubble for τ^* periods, she counts on the fact that, if she is an early agent, later types $\nu(i), ..., \nu(i) + N - 1$ are also willing to ride the bubble for τ^* periods. In turn, those types would count on any possible later types to follow the same strategy. Iterating forward to and beyond period T, the maximum bubble duration in any PBE is found by substituting $\underline{G}$ into (16). ■

The infinite-resources critique is that the characterization of τ^* given by Proposition 1 is correct only if endowments can grow at the rate G indefinitely. If this assumption is essential, this critique severely limits the applicability of the results in D-M. In essence, the same critique also applies to the credit-market model developed by Doblas-Madrid and Lansing (2014), where the exponential growth in funds is generated by a self-reinforcing feedback loop between prices and credit. In that setup, prices can only grow exponentially if lenders can supply infinite funds without increasing interest rates.

Given expression (8) for the booming price, if the number of shares for sale θ shrank over time, this effect could in principle overcome the effect of slowing endowment growth. However, there are two reasons why shrinking θ is not a solution. First, the assumption of a vanishing θ would be in direct contradiction with the observation—emphasized by Scheinkman and Xiong (2003)—that trading volume increases during episodes of bubbles. Second, and even more importantly, the increase in shares for sale and decrease in the mass of buyers caused by the sales of type-t_0 agents at t_c would eventually overwhelm any decrease in θ. To see this, suppose that θ_t is falling over time. As long as $t < t_c$, the price growth rate is given by

$$\frac{p_t}{p_{t-1}} = \frac{\theta_{t-1}}{\theta_t} \frac{1 - \theta_t}{1 - \theta_{t-1}} \frac{e_t}{e_{t-1}},$$

which can be made arbitrarily large by lowering θ_t/θ_{t-1}. However, at time t_c,

$$\frac{p_{t_c}}{p_{t_c-1}} = \frac{\theta_{t_c-1}}{\theta_{t_c} + (1 - \theta_{t_c})/N} \frac{(1 - \theta_{t_c})(1 - 1/N)}{1 - \theta_{t_c-1}} \frac{e_{t_c}}{e_{t_c-1}},$$

which actually converges to zero as θ_{t_c-1} and θ_{t_c} approach zero.

This discussion is, of course, based on D-M. The AB model does not specify the supply and demand from behavioral agents, and thus is in principle open to other formulas. Nevertheless, in AB, the increase in the price is driven by demand by behavioral agents, which is supposed to support booming prices until a mass κ of shares has been sold to them. In expectation, the sum needed to purchase those shares is infinite.

4 The Finite Model

In this Section, I consider a finite-horizon version of the model presented above. The timeline is given by $t = 0, 1, .., t_{pay}$, where t_{pay} is a known finite number. More precisely, I assume that the unconditional support of t_0 is given by

$$t_0 \in \{0, 1, ..., \overline{t_0}\}, \tag{18}$$

where $\overline{t_0} < \infty$. I assume a probability distribution that is simply a truncated version of (7). More precisely, the probability of each value of any given $t_0 \in \{0, 1, ..., \overline{t_0}\}$ equals

$$\varphi(t_0) = \frac{1-\lambda}{1-\lambda^{\overline{t_0}+1}}\lambda^{t_0}. \tag{19}$$

Moreover, I assume that agents observe signals $t_0, ..., t_0 + N - 1$, and that the terminal period t_{pay} is given by

$$t_{pay} = \overline{t_0} + N.$$

Conditional on the signal $\nu(i)$, the support of t_0 includes

$$\text{supp}(t_0|\nu(i)) = \{\min\{0, \nu(i) - (N-1)\}, ..., \max\{\nu(i), \overline{t_0}\}\}.$$

That is, the conditional distribution of t_0 is now truncated below by 0 and above by $\overline{t_0}$, so that all agents with signal $\nu(i) > \overline{t_0}$ know with certainty that they were not 'first in line'. Conditional probabilities are computed in a way similar to (10). The definiton of $\overline{F}$ is unchanged, although its unconditional expectation is finite and given by

$$E[\overline{F}] = \frac{1-\theta}{\theta}\frac{1-\lambda}{1-\lambda^{\overline{t_0}+1}}\frac{1-(\lambda G/R)^{\overline{t_0}+1}}{1-\lambda G/R}.$$

Figure 2: [Insert here]

Endowments, preferences, and all constraints faced by agents are the same as in the infinite model. The equilibrium concept is also PBE, and the definition of equilibrium is the same as previously, except for the fact that the bubble riding times are no longer symmetric across types.

4.1 Tapered Strategies

As before, I follow a guess-and-verify procedure to find equilibria. Given that there is a finite horizon, the conjectured equilibrium strategies gradually lower the bubble riding time τ as the signal $\nu(i)$ nears the horizon $\overline{t_0}$. To be precise, I consider strategies identical by (13a)-(13c), except for the fact that patient agents of type $\nu(i)$ plan to sell at $\nu(i) + \tau(\nu(i))$, where the bubble riding time $\tau(\nu(i))$ is tapered according to the schedule:

$$\tau(\nu(i)) = \begin{cases} \tau^* & \text{if } z_0 \leq \nu(i) < z_1 \\ \tau^* - 1 & \text{if } z_1 \leq \nu(i) < z_2 \\ \ldots & \\ \tau^* - k & \text{if } z_k \leq \nu(i) < z_{k+1} \\ & \\ 1 & \text{if } z_{\tau^*-1} \leq \nu(i) < z_{\tau^*} \\ 0 & \text{if } \nu(i) = z_{\tau^*}, \end{cases} \tag{20}$$

where $\{z_0, ..., z_{\tau^*}\}$ are integers with $z_0 = 0$ and $z_{\tau^*} = \overline{t_0} - 1$. I also assume that $z_{k+1} > z_k + 1$ for all $k \in \{1, ..., \tau^* - 1\}$, so that tapering is as gradual as possible, in the sense that—regardless of the realization of t_0—no more than two types sell at any given time. Also note that $\tau(\nu(i))$ is not defined for types $\nu(i) \in \{\overline{t_0}, ..., \overline{t_0} + N - 1\}$, because these types never sell. Unless they become impatient or the bubble bursts before $\overline{t_0}$, types $\nu(i) \geq \overline{t_0}$ are fully invested in risky shares while $t \leq \overline{t_0}$, and invest the fraction of their endowment needed to maintain the equality $p_t = \overline{F}R^t$ when $t > \overline{t_0}$.

If all agents follow these strategies, the crash period t_c is given by $t_0 + \tau(t_0)$ for $t_0 < \overline{t_0}$. If $t_0 = \overline{t_0}$, there is no 'crash period' since no patient agent ever sells, although period $\overline{t_0}$ acts as a crash period in the sense that, if the boom reaches $\overline{t_0}$, the value of t_0 is revealed and price growth rate falls to R after $\overline{t_0}$. Finally, note that if $t_0 = z_k - 1$ for any $k \in \{1, ..., \tau^*\}$, types $t_0 = z_k - 1$ and $t_0 + 1 = z_k$ sell at t_c. Otherwise, only one type, type t_0 sells at t_c The resulting dynamics, with bubble duration a decreasing function of t_0 are depicted in Figure 2.

As before, to verify that this strategy profile and its implied beliefs constitute a PBE, one must verify that agents of all types are willing to (i) Sell as dictated by their strategy and (ii) Buy as dictated by their strategy.

As was the case in the symmetric case, verifying willingness to sell (i) is obvious for impatient agents. However, for patient agents, it is more involved to verify (i) in the presence of asymmetric strategies. If $z_{k-1} < t_0 < z_k - 1$, for any $k \in \{1, ..., \tau^*\}$, patient type-t_0 agents are certain, when selling at time $\nu(t_0) + \tau(\nu(t_0))$ that they are the only type selling. Then, as in the symmetric stragegy case, for any positive $\tau(t_0)$, $G/R \geq 1 + [\theta(N-1)]^{-1}$ suffices to establish that selling is optimal. If $t_0 = z_k - 1$ for some $k \in \{1, ..., \tau^*\}$, agents of type t_0 know with certainty when they sell at $t_c = z_k - 1 + \tau(z_k - 1)$, that the next type $t_0 + 1 = z_k$ is also selling at t_c. With two types leaving the market at once, type-t_0 agents find it optimal to sell—for any positive $\tau(t_0)$—as long as $G/R \geq 1 + 2/[\theta(N-2)]$. (The derivation of this inequality is very similar to the case when only one type sells.) Agents of type $\nu(i) = z_k$ enter their orders to sell without knowing whether type-$z_k - 1$ agents exist. Given the information available to type-z_k agents at time $z_k + \tau(z_k)$, the probability that type $z_k - 1$ exists is the probability that $t_0 = z_k - 1$. Therefore, in their estimation there is a probability $1/(1+\lambda)$ that two types will leave the market at $z_k + \tau(z_k)$ and a probability $\lambda/(1+\lambda)$ that $t_0 = z_k$ and only one type will sell. In sum, type-z_k agents will sell at $t_c = z_k + \tau(z_k)$ if $p_{t_c} > p_{t_c+1}/R$, which in turn is the case if

$$\begin{aligned} & \frac{1}{1+\lambda} \frac{(1-\theta)\left(1-\frac{2}{N}\right)}{\theta + \frac{2(1-\theta)}{N}} G^{z_k+\tau(z_k)} + \frac{\lambda}{1+\lambda} \frac{(1-\theta)(1-\frac{1}{N})}{\theta + \frac{1-\theta}{N}} G^{z_k+\tau(z_k)} \\ \geq \quad & \frac{1}{1+\lambda} \frac{1-\theta}{\theta} G^{z_k-1} R^{\tau(z_k)+1} + \frac{\lambda}{1+\lambda} \frac{1-\theta}{\theta} G^{z_k} R^{\tau(z_k)} \end{aligned}$$

Dividing through by $G^{z_k+\tau(z_k)}(1-\theta)/[\theta\,(1+\lambda)]$ and rearranging terms yields

$$\frac{\theta\,(N-2)}{2+\theta(N-2)} + \lambda \frac{\theta\,(N-1)}{1+\theta(N-1)} \geq \left(\frac{G}{R}\right)^{-\tau(z_k)} \left(\frac{R}{G} + \lambda\right) \tag{21}$$

Of all types $(z_1, ..., z_{\tau^*})$, type z_{τ^*} is least likely to satisfy this selling condition, since $\tau(z_{\tau^*}) = 0$. If fact, if $t_0 = z_{\tau^*}$, selling at z_{τ^*} means selling at a price $G^{z_{\tau^*}}[(1-\theta)(1-1/N)]/[\theta + (1-\theta)/N]$, which is actually below the discounted "post-crash" price $G^{z_{\tau^*}}(1-\theta)/\theta$. Thus, if type-$t_{\tau^*}$ agents knew with certainty that $t_0 = z_{\tau^*}$ they would not sell. However, given their information at z_{τ^*}, t_0 could also be $z_{\tau^*} - 1$, in which case, if N is large enough, the price will in fact fall between periods z_{τ^*} and $z_{\tau^*} + 1$. More precisely, substituting $k = \tau^*$ in (21) and rearranging terms, we find that selling is optimal if

$$\frac{R}{G} \leq \frac{\theta(N-2)}{2+\theta(N-2)} - \frac{\lambda}{1+\theta(N-1)}. \tag{22}$$

Since (22) is more stringent than $G/R \geq 1 + 2/[\theta(N-2)]$, it is sufficient to establish condition (i). Note that, in the large-N case, this condition reduces to $G \geq R$.

Let us turn to (ii), the requirement that agents are willing to buy. After the crash, since the safe and risky assets are perfect substitutes, it is (weakly) optimal to buy the risky asset at fundamental value. Before the crash, one must rule out preemptive sales. As in the symmetric-strategies case, type-$\nu(i)$ agents find preemptive selling most tempting at time $t = \nu(i) + \tau(\nu(i)) - 1$, just one period before they are supposed to sell. With asymmetric strategies, preemptive sales become even more tempting when $\nu(i) = z_1 + 1$, and $z_2 = z_1 + 2$. If $\nu(i) = z_k + 1$, agent i's support of t_0 at time t is given by $\{\nu(i) - 2, \nu(i) - 1, \nu(i)\}$, and t will be equal to t_c if either $t_0 = \nu(i) - 2$ or $t_0 = \nu(i) - 1$. Moreover, if the bubble does burst at z_1, it will be a bigger bubble bursting, in the sense that the fraction of the price that will vanish will be bigger than at any other time z_k, for $k > 1$. Finally, if $z_2 = z_1 + 2$, two types will sell at $t + 1$ in the event that t_0 happens to be $\nu(i)$, reducing the reward from riding the bubble one more period. In sum, preemptive selling in this worst-case scenario is ruled out if

$$\left(\frac{1}{1+\lambda+\lambda^2} \frac{(1-\theta)\left(1-\frac{2}{N}\right)}{\theta+\frac{2(1-\theta)}{N}} + \frac{\lambda}{1+\lambda+\lambda^2} \frac{(1-\theta)\left(1-\frac{1}{N}\right)}{\theta+\frac{1-\theta}{N}} + \frac{\lambda^2}{1+\lambda+\lambda^2} \frac{1-\theta}{\theta} \right) G^{z_1+\tau^*-1} \leq$$
$$\frac{1}{1+\lambda+\lambda^2} \frac{1-\theta}{\theta} G^{z_1-1} R^{\tau^*} + \frac{\lambda}{1+\lambda+\lambda^2} \frac{1-\theta}{\theta} G^{z_1} R^{\tau^*-1} + \frac{\lambda^2}{1+\lambda+\lambda^2} \frac{(1-\theta)\left(1-\frac{2}{N}\right)}{\theta+\frac{2(1-\theta)}{N}} \frac{G^{z_1+\tau^*}}{R},$$

which can be rewritten as

$$\frac{\theta\,(N-2)}{2+\theta(N-2)} + \lambda \frac{\theta\,(N-1)}{1+\theta(N-1)} + \lambda^2 \leq \left(\frac{G}{R}\right)^{-\tau^*} + \lambda \left(\frac{G}{R}\right)^{-(\tau^*-1)} + \lambda^2 \frac{\theta\,(N-2)}{2+\theta(N-2)} \frac{G}{R}. \tag{23}$$

From here, note that even if τ^* is so large that $(G/R)^{-\tau^*}$ is negligible, preemptive sales can be ruled out if

$$\lambda^{-2} + \lambda^{-1}\left[1 + \frac{N}{N-2+\theta(N-1)(N-2)}\right] + \left[1 + \frac{2}{\theta\,(N-2)}\right] \leq \frac{G}{R}.$$

Our overall findings are summarized in the following Proposition

Proposition 3 *In the finite model, suppose that agents follow strategies given by (13a)-(13c), except that instead of of following symmetric strategies, bubble riding times given by (20). Then, if* $G/R \geq \lambda^{-2} + \lambda^{-1}[1 + \{1 - 2/N + \theta(N-1)(N-2)/N\}^{-1}] + [1 + 2/\theta\,(N-2)]$, *and* $R/G \leq \frac{\theta(N-2)}{2+\theta(N-2)} - \frac{\lambda}{1+\theta(N-1)}$, *bubbly equilibria with any bubble duration* $\tau^* < \overline{t_0}/2$ *can be supported.*

Proof. Under restriction $R/G \leq \frac{\theta(N-2)}{2+\theta(N-2)} - \frac{\lambda}{1+\theta(N-1)}$, condition (i) is met, and all agents wish to sell at $\nu(i) + \tau(\nu(i))$, even if $\tau(\nu(i)) = 0$. Restriction $G/R \geq \lambda^{-2} + \lambda^{-1}[1 + \{1 - 2/N + \theta(N-1)(N-2)/N\}^{-1}] + [1 + 2/\theta\,(N-2)]$ ensures that, for any τ^*, agents wish to buy before $\nu(i) + \tau(\nu(i))$. Finally, since no more than two types can sell at any given time, if τ^* was not below $\overline{t_0}/2$, it would not be possible to reduce $\tau(\nu(i))$ to zero unless $z_{k+1} = z_k + 1$ for some k. ■

Proposition 3 can allows us to reinterpret the infinite, symmetric equilibria from Proposition 1, which we can think of as the early part of a larger equilibrium with bounded distribution of t_0 and tapered strategies. While the parameter restrictions (22) and (23) are more complex than their symmetric-equilibrium counterparts, with a bounded distribution of t_0 and tapered strategies, one can assume that λ and G increase as needed at times $z_1, ..., z_{\tau^*}$ to satisfy (22) and (23).

Finally, these tapered equilibria developed in a version of D-M may be difficult to implement in an AB environment, where symmetric trigger strategy equilibria are proven to be unique. Uniqueness obtains under the belief restriction that when an agent sells she believes that all agents with lower signals have either sold, or are selling at the same instant as she is. However, this belief restriction is satisfied here. Thus, although the equilibrium multiplicity in D-M is in general a disadvantage, in this context it has the advantage of allowing us to consider tapered strategies in order to address the infinite-boom critique.

5 Conclusion

Recent episodes of boom-and-bust, followed by deep recessions in countries such as the United States, Spain, and Ireland, have led to a great resurgence of interest in the topic of asset price bubbles. However, after decades of dominance of the efficient market hypothesis, several strands of the theoretical literature on bubbles still have much ground to gain, in terms of microfoundations, before they are compatible with standard macroeconomic models. This is particularly the case for models of finite, speculative bubbles, which are not as well established as, for instance, the overlapping generations models of infinitely lived bubbles. In this paper, I seek to take one step in bridging this gap, focusing on the model by Doblas-Madrid (2012; D-M), which in turn is a fully rational version of Abreu and Brunnermeier (2003; AB). In this model, a fundamental boom overshoots and turns into a bubble as asymmetrically informed investors bid prices up hoping to resell at a profit before the crash. A critique of the model is that, while bubbles are finite, they can only arise if the fundamental boom preceding it can last indefinitely. If the model's ability to generate bubbles hinges on the possibility of an infinite, exponentially-growing source of inputs, this raises important questions regarding the plausibility of the assumptions behind the theory. In this paper, I articulate the critique in a simplified version of D-M. I find that the critique does apply to the symmetric trigger-strategy equilibria proposed by AB and D-M. However, I also find that the critique can be avoided by considering tapered strategies, which gradually reduce bubble duration for longer fundamental booms. To illustrate how bubbles arise under asymmetric strategies, I construct a finite version of the D-M model, where bubbles are very similar to those in the infinite model, unless the duration of the fundamental boom is close to the model's terminal period. In sum, I construct a rational model of riding bubbles that does not hinge on any notion of infinity, but instead relies on lack of common knowledge, much like the models developed by Allen et al. (1993) and Conlon (2004).

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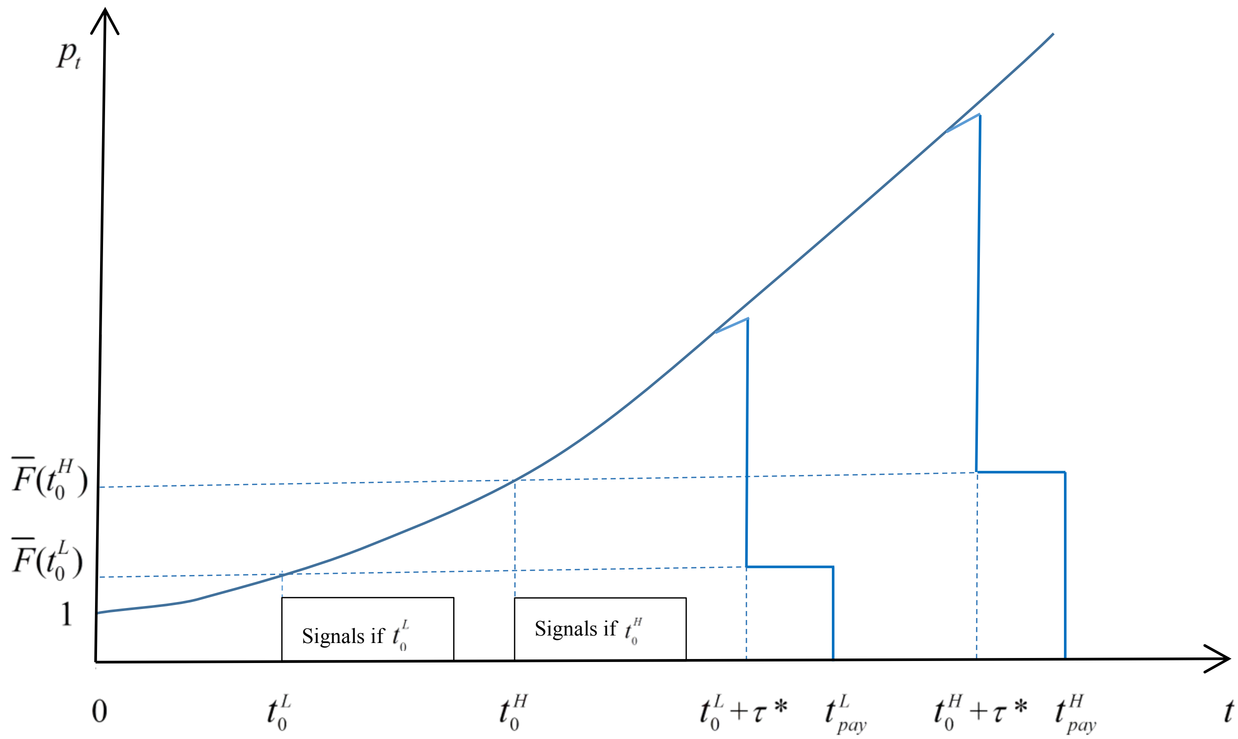

Figure 1 – Under symmetric strategies, the bubble riding time $\tau *$ is independent of t_0. Even if $\tau *$ is finite, t_0, $t_0 + \tau *$ and t_{pay} can be unboundedly large.

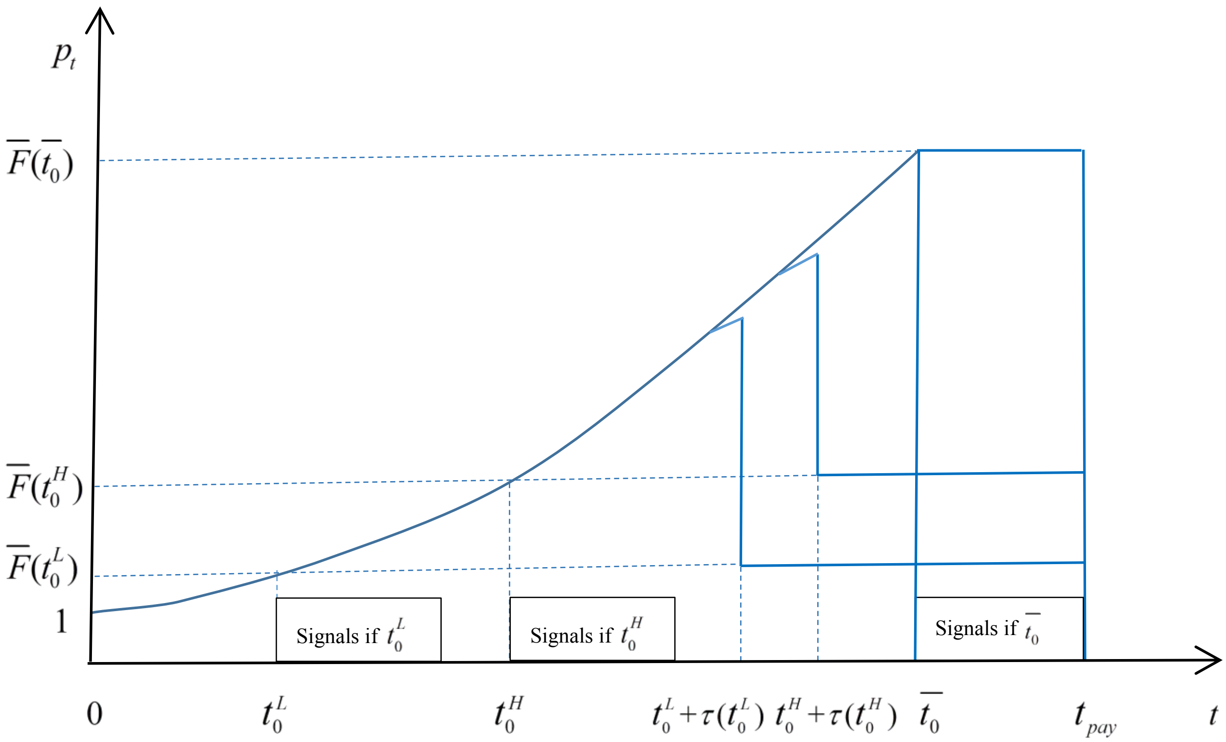

Figure 2 – In a tapered equilibrium, if $t_0^L < t_0^H$, $\tau\left(t_0^L\right) \geq \tau\left(t_0^H\right)$. That is, bubble riding time decreases as signals approach the upper bound $\overline{t}_0$.