2 Limits And Continuity

2.1 The Limit Process

♣ The Idea of Limit

Now let $L$ be some real number. We say that the limit of $f(x)$ as $x$ tends to $c$ is $L$ and write

$$\lim_{x \to c} f(x) = L$$

provided that (roughly speaking)

as $x$ approaches $c$, $f(x)$ approaches $L$

or (somewhat more precisely) provided that

$f(x)$ is closed to $L$ for all $x \neq c$, which are close to $c$.

Do we always have to have a limit? No.

♣ How to find Limit?

1. Direct Substitution
2. Cancelation
3. By Graph

♣ One-Sided Limits:

1. Left-hand limit: $\lim_{x \to c^-} f(x) = L$ : as $x$ approaches $c$ from the left, $f(x)$ approaches $L$.
2. Right-hand limit: $\lim_{x \to c^+} f(x) = L$ : as $x$ approaches $c$ from the right, $f(x)$ approaches $L$.

For a full limit to exist, both one-sided limits have to exist and they have to be equal, i.e.

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$
Example 1

\[ \lim_{x \to 2} (4x + 5) \]

Example 2

\[ \lim_{x \to 3} \frac{x^3 - 2x + 4}{x^2 + 1} \]

Example 3

\[ \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \]

Example 4 If \( f(x) = \begin{cases} 3x - 4, & x \neq 0, \\ 10, & x = 0 \end{cases} \), find \( \lim_{x \to 0} f(x) \).
Example 5 If \( f(x) = \begin{cases} 
  x + 10, & x \leq -1, \\
  x - 2, & x > -1.
\end{cases} \), find \( \lim_{x \to -1^-} f(x) \), \( \lim_{x \to -1^+} f(x) \) and \( \lim_{x \to -1} f(x) \).

Example 6 For the function indicated below

\[
\begin{array}{c}
\hspace{1cm} y \\
\hline \\
\hspace{1cm} x
\end{array}
\]

\( (1) \) Find \( \lim_{x \to -2^-} f(x) \), \( \lim_{x \to -2^+} f(x) \) and \( \lim_{x \to -2} f(x) \).

\( (2) \) Find \( \lim_{x \to 4^-} f(x) \), \( \lim_{x \to 4^+} f(x) \) and \( \lim_{x \to 4} f(x) \).

\( (3) \) Find \( \lim_{x \to -5^-} f(x) \), \( \lim_{x \to -5^+} f(x) \) and \( \lim_{x \to -5} f(x) \).
2.3 Some Limit Theorems

**Theorem 1 (THE UNIQUENESS OF LIMIT)**
If \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} f(x) = M \), then \( L = M \).

**Theorem 2 (Operation Of Limit)**
If \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \), then
(i) \( \lim_{x \to c} [f(x) \pm g(x)] = L \pm M \),
(ii) \( \lim_{x \to c} \alpha f(x) = \alpha L \), where \( \alpha \) is a real number,
(iii) \( \lim_{x \to c} [f(x) \cdot g(x)] = L \cdot M \),
(iv) If \( M \neq 0 \), then \( \lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M} \),
(v) If \( M \neq 0 \), then \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \),
(vi) If \( L \neq 0 \) and \( M = 0 \), then \( \lim_{x \to c} \frac{f(x)}{g(x)} \) does not exist.

**Theorem 3 (Limit of Polynomial and Rational Function)**
For each polynomial \( P(x) = a_n x^n + \ldots a_1 x + a_0 \) and each real number \( c \)
\[ \lim_{x \to c} P(x) = P(c). \]
For another polynomial \( Q(x) \),
\[ \lim_{x \to c} R(x) = \lim_{x \to c} \frac{P(x)}{Q(x)} = R(c), \quad \text{provided } Q(c) \neq 0. \]
Example 1 (1) \( \lim_{x \to 0} (14x^5 - 7x^2 + 2x + 8) \), \hspace{1em} (2) \( \lim_{x \to -1} (2x^3 + x^2 - 2x - 3) \)

Example 2 (1) \( \lim_{x \to 2} \frac{3x - 5}{x^2 + 1} \), \hspace{1em} (2) \( \lim_{x \to 3} \frac{x^3 - 3x^2}{1 - x^2} \)

Example 3 (1) \( \lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} \), \hspace{1em} (2) \( \lim_{x \to 4} \frac{(x^2 - 3x - 4)^2}{x - 4} \)
Example 4  

(1) \[ \lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} \],  

(2) \[ \lim_{x \to 3} \frac{|x - 3|}{x - 3} \]

Example 5  

Evaluate the limit below for the function \( f(x) = 3x^2 \) at \( x = 3 \). 

\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
2.5 The Pinching Theorem; Trigonometric Limits

**Theorem 1 (THE PINCHING THEOREM)**
Let $p > 0$. Suppose that, for all $x$ such that $0 < |x - c| < p$,

$$h(x) \leq f(x) \leq g(x).$$

If

$$\lim_{x \to c} h(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$ 

♣ Trigonometric Limits

$$\lim_{x \to 0} \sin x = \sin 0 \quad \text{and} \quad \lim_{x \to 0} \cos x = \cos 1.$$ 

General case:

$$\lim_{x \to c} \sin x = \sin c \quad \text{and} \quad \lim_{x \to c} \cos x = \cos c.$$ 

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$ 

General case: for each number $a \neq 0$

$$\lim_{x \to 0} \frac{\sin ax}{ax} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos ax}{ax} = 0.$$
Example 1  If \(10x - 26 \leq f(x) \leq x^2 + 6x - 22\) for \(x \neq 0\), then find \(\lim_{x \to 2} f(x)\).

Example 2  \(\lim_{x \to 0} \frac{\sin 4x}{3x}\),

Example 3  \(\lim_{x \to 0} \frac{1 - \cos 2x}{5x}\),

Example 4  \(\lim_{x \to 0} \frac{\sin^2 2x}{19x^2}\)
Example 5 \[ \lim_{x \to 0} \frac{\tan x}{3x} \]

Example 6 \[ \lim_{x \to 0} x \cot 3x, \]

Example 7 \[ \lim_{x \to 0} \frac{\sin(\sin 20x)}{\sin 12x}, \]
2.2 Definition of Limit

**Definition 1 (THE LIMIT OF A FUNCTION)** Let $f$ be a function defined at least on an open interval $(c-p, c+p)$ except possibly at $c$ itself. We say that

$$\lim_{x \to c} f(x) = L$$

if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

*Remark:* Definition of One-sided limits are similar!

*Basic Limits*

$$\lim_{x \to c} |x| = |c|,$$

$$\lim_{x \to c} \sqrt{x} = \sqrt{c}, \text{ for any } c > 0$$
Example 1 For
\[ f(x) = x + 3, \quad L = 5, \quad c = 2, \quad \epsilon = 0.01, \]
find the largest value of \( \delta > 0 \) such that
\[
if \quad 0 < |x - c| < \delta, \quad then \quad |f(x) - L| < \epsilon.
\]

Example 2 Show that
\[
\lim_{x \to -1} (2 - 3x) = 5.
\]

Example 3 Show that
\[
\lim_{x \to 2} (x^2 - 4x + 15) = 11.
\]
2.4 Continuity

* Continuity at a Point

**Definition 2** Let $f$ be a function defined at least on an open interval $(c - p, c + p)$. We say that $f$ is continuous at $c$ if

$$\lim_{x \to c} f(x) = f(c).$$

**Remark 1 (Types of Discontinuities)** Let $c$ be a discontinuity of $f$.

- If $\lim_{x \to c^-} f(x)$ exists but not equal to $f(c)$, then $c$ is called a *removable* discontinuity.

- If $\lim_{x \to c^-} f(x)$ does not exist, then $c$ is called an *essential* discontinuity. In this case,
  - if $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist but not equal, then $c$ is called a *jump* discontinuity.
  - if $\lim_{x \to c^-} f(x) = \pm \infty$ or $\lim_{x \to c^+} f(x) = \pm \infty$, then $c$ is called an *infinity* discontinuity.
Theorem 1 If \( f \) and \( g \) are continuous at \( c \),

- \( f \pm g, \alpha f, f \cdot g \) are continuous at \( c \).
- \( f/g \) are continuous at \( c \) provided \( g(c) \neq 0 \).

If \( g \) is continuous at \( c \) and \( f \) is continuous at \( g(c) \), then the composition \( f \circ g \) is continuous at \( c \).

Definition 3 (ONE-SIDED CONTINUITY) A function \( f \) is called

continuous from the left at \( c \) if \( \lim_{x \to c^-} f(x) = f(c) \).

\( f \) is called

continuous from the right at \( c \) if \( \lim_{x \to c^+} f(x) = f(c) \).

Remark: \( f \) is continuous at \( c \) iff \( f(c), \lim_{x \to c^-} f(x) \) and \( \lim_{x \to c^+} f(x) \) exist and are equal.

* Continuity on Intervals

Definition 4 A function \( f \) is said to be continuous on an interval if it is continuous at each interior point of the interval and one-sidedly continuous at whatever endpoints the interval may contain.
Example 1 Determine whether or not the function is continuous at the indicated point. If not, determine if it is removable or essential discontinuity. If the latter, state if it is a jump discontinuity, an infinite discontinuity or neither.

\[ f(x) = \begin{cases} 
  x^2 + 4, & x < 2 \\
  5, & x = 2 \\
  x^3, & x > 2 
\end{cases} \]

Example 2 Find \( c \) such that \( f \) is continuous on \((-\infty, \infty)\), where

\[ f(x) = \begin{cases} 
  1 + cx, & x < 2 \\
  c - x, & x \geq 2 
\end{cases} \]
Example 3  The graph of \( f \) is given in the figure.

\[(a) \text{ At which points is } f \text{ discontinuous?}
\]

\[(b) \text{ For each discontinuity found in (a), determine if it is a removable or essential discontinuity. If the latter, state if it is a jump discontinuity, an infinite discontinuity or neither.}\]

Example 4  Use interval notation to indicate where \( f \) is continuous.

\[(1) \quad f(x) = 2x^3 - 5x + 1 \quad (2) \quad f(x) = \frac{x^2 - 3}{x^2 - 5x + 6} \quad (3) \quad f(x) = \sqrt{1 - x^2}\]
2.6 Two Basic Theorems

**Theorem 1 (THE INTERMEDIATE-VALUE THEOREM)**
If $f$ is continuous on $[a, b]$ and $K$ is any number between $f(a)$ and $f(b)$, then there is at least one number $c$ in the interval $(a, b)$ such that $f(c) = K$.

*Remark:* Let $f$ be a continuous function on $[a, b]$. If $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then there exists at least one number $c \in (a, b)$ such that $f(c) = 0$.

**Theorem 2 (THE EXTREME-VALUE THEOREM)**
If $f$ is continuous on a bounded closed interval $[a, b]$, then on that interval $f$ takes on both a maximum value $M$ and a minimum value $m$. 
Example 1  Use the intermediate-value theorem to show that there is a solution of the given equation in the indicated interval.
(a) \( 2x^3 - 4x^2 + 5x - 4 = 0; \quad [1, 2] \)

(b) \( x^3 = \sqrt{x + 2}; \quad [3, 5] \)

Example 2  Let \( f(x) = \frac{1}{x - 1} + \frac{1}{x - 4} \). Show that there is a number \( c \in (1, 4) \) such that \( f(c) = 0 \).

Example 3  Show that the equation \( x^3 - 4x + 2 = 0 \) has three distinct roots in \([-3, 3]\) and locate the roots between consecutive integers.
3 Limits And Continuity

3.1 The Limit Process
3 The Derivative; The Process of Differentiation

3.1 The Derivative

Definition 1 (THE DERIVATIVE OF A FUNCTION) A function $f$ is said to be differentiable at $x$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

If this limit exists, it is called the derivative of $f$ at $x$ and is denoted by $f'(x)$. 

(Tangent Lines) If \( f \) is differentiable at \( c \), the line that passes through the point \((c, f(c))\) with slope \( f'(c) \) is the tangent line at that point. As an equation for this line we can write

\[
y - f(c) = f'(c)(x - c) \quad \text{(point-slope form)}
\]

This is the line through \((c, f(c))\) that best approximates the graph of \( f \) near the point \((c, f(c))\).

**Theorem 1** If \( f \) is differentiable at \( x \), then \( f \) is continuous at \( x \).
Example 1  Using the definition of derivative, find $f'(x)$ where

$$f(x) = \frac{1}{x}$$

Example 2  

$$f(x) = \sqrt{x} \quad x \geq 0.$$  

(1) Using the definition of derivative, find $f'(x)$.

(2) Evaluate $f'(4)$.

(3) Find the equation for the tangent line at $x = 4$. 
Example 3 Find $f'(1)$ given that

$$f(x) = \begin{cases} 
  x^2, & x \leq 1, \\
  2x - 1, & x > 1.
\end{cases}$$

Example 4 Find the derivative of $f(x) = |x|$ at $x = 0$. 
### 3.2 Some Differentiation Formulas

If \( f(x) = \alpha \), \( \alpha \) any constant, then \( f'(x) = 0 \)

If \( f(x) = x \), then \( f'(x) = 1 \)

If \( f(x) = x^n \), then \( f'(x) = nx^{n-1} \)

\[(\alpha f)'(x) = \alpha f'(x), \text{ where } \alpha \text{ is a real number.}\]

\[(f + g)'(x) = f'(x) + g'(x)\]

\[(f - g)'(x) = f'(x) - g'(x)\]

\[(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x) \quad \text{(The Product Rule)}\]

\[
\left( \frac{f}{g} \right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \text{ where } g(x) \neq 0. \quad \text{(The Quotient Rule)}
\]

A special case: \( \left( \frac{1}{g} \right)'(x) = -\frac{g'(x)}{[g(x)]^2}, \text{ where } g(x) \neq 0. \)

- The derivative of a linear combination is the linear combination of the derivatives.

\[
(\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_1 f_1)'(x) = \alpha_1 f_1'(x) + \alpha_2 f_2'(x) + \ldots + \alpha_1 f_1'(x)
\]

- How to differentiate polynomials?

If \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 \),

then \( P'(x) = na_nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \ldots + 2a_2 + a_1 \).
Example 1  \( f(x) = 2 + \frac{4}{x} + \frac{6}{x^2} \). Find \( f'(x) \) and \( f'(4) \).

Example 2  Differentiate \( F(x) = 5(x^3 - 2x + 3)(4x^2 + 1) \) and find \( F'(-1) \).

Example 3  Let

\[ f(x) = \frac{2}{x - 2} + \frac{1}{2} \]

Find the equation of the tangent line to the curve at the point \( x = 6 \).
3.3 The $d/dx$ Notation; Derivatives of Higher Order

2$^{nd}$ derivative of $f$ is denoted by $f''$; $f'' = (f')'$ if $f'$ can be differentiated;

3$^{rd}$ derivative of $f$ is denoted by $f'''$; $f''' = (f''')'$ if $f''$ can be differentiated;

4$^{th}$ derivative of $f$ is denoted by $f^{(4)}$; $f^{(4)} = (f^{(3)})'$ if $f'''$ can be differentiated;

$\vdots$

$n^{th}$ derivative of $f$ is denoted by $f^{(n)}$; $f^{(n)} = (f^{(n-1)})'$ if $f^{(n-1)}$ can be differentiated;

♣ Leibniz notation ♣

1$^{st}$ derivative: $\frac{d}{dx}f(x)$,

2$^{nd}$ derivative: $\frac{d^2}{dx^2}f(x) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$,

3$^{rd}$ derivative: $\frac{d^3}{dx^3}f(x) = \frac{d}{dx}\left(\frac{d^2}{dx^2}f(x)\right)$,

$\vdots$

If $y = f(x)$, we can also write:

$$\frac{dy}{dx} \quad \text{(same as } y'(x)\text{)}, \quad \frac{dy}{dx}\bigg|_{x=a} \quad \text{(same as } y'(a)\text{)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right),$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right),$$

$\vdots$

Example 1 Find $\frac{dy}{dx}$ for $y = \frac{3x - 1}{5x + 2}$
Example 2  Find $\frac{du}{dx}$ for $u = (x)(x + 1)(x + 2)$

Example 3  Find $\frac{d^2}{dx^2} \left[ (x^2 - 3x) \frac{d}{dx} (x + x^{-1}) \right]$
3.4 The Derivative as a Rate of Change

For a differentiable function \( y = f(x) \),

\[ f'(x) = \text{the rate of change of } y \text{ respect to } x. \]

Example 1 Set \( y = (x - 2)x^{-2} \).

(1) Find the rate of change of \( y \) with respect to \( x \) at \( x = 2 \).

(2) Find the values(s) of \( x \) at which the rate of change of \( y \) with respect to \( x \) is 0.

Example 2 The total surface area of a right circular cylinder is given by the formula \( A = 2\pi r(r+h) \) where \( r \) is the radius and \( h \) is the height.

(1) Find the rate of change of \( A \) with respect to \( h \) if \( r \) remains constant.

(2) Find the rate of change of \( A \) with respect to \( r \) if \( h \) remains constant.

(3) Find the rate of change of \( h \) with respect to \( r \) if \( A \) remains constant.
3.5 The Chain Rule

(Chain-Rule in Leibniz’s Notation)

- If $y$ is a differentiable function of $u$ and $u$ is a differentiable function of $x$, then
  
  \[
  \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.
  \]

- In particular, if $u$ is a differentiable function of $x$, then
  \[
  \frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}.
  \]

- If $y$ is a differentiable function of $u$, $u$ is a differentiable function of $x$ and $x$ is a differentiable function of $t$, then
  \[
  \frac{dy}{dt} = \frac{dy}{du} \frac{du}{dx} \frac{dx}{dt}.
  \]

(Chain-Rule in Prime Notation)

If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at $x$ and

\[
(f \circ g)'(x) = f'(g(x))g'(x).
\]

Example 1 Find $\frac{dy}{dx}$ by the chain rule given that

\[
y = \frac{u - 1}{u + 1}, \quad u = x^2.
\]

Example 2 Find $\frac{dy}{ds}$ given that $y = 3u + 1$, $u = x^{-2}$ and $x = 1 - s$. 
Example 3  Differentiate the following functions:  

(1) \( \left( \frac{4x + 3}{5x - 2} \right)^3 \)  

(2) \( 2x^3(x^2 - 3)^4 \)

Example 4  Find \( f'(x) \) and \( f''(x) \) for \( f(x) = (x^2 - 5x + 2)^{10} \).
3.6 Differentiating the Trigonometric Functions

(Derivative Formulas of Trigonometric Functions)

\[
\begin{align*}
\frac{d}{dx}(\sin x) &= \cos x, & \frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\tan x) &= \sec^2 x, & \frac{d}{dx}(\cot x) &= -\csc^2 x \\
\frac{d}{dx}(\sec x) &= \sec x \tan x, & \frac{d}{dx}(\csc x) &= -\csc x \cot x
\end{align*}
\]

(Chain Rule Applied to Trigonometric Functions)

If \( u \) is a differentiable function of \( x \), then

\[
\begin{align*}
\frac{d}{dx}(\sin u) &= \cos u \frac{du}{dx}, & \frac{d}{dx}(\cos u) &= -\sin u \frac{du}{dx} \\
\frac{d}{dx}(\tan u) &= \sec^2 u \frac{du}{dx}, & \frac{d}{dx}(\cot u) &= -\csc^2 u \frac{du}{dx} \\
\frac{d}{dx}(\sec u) &= \sec u \tan u \frac{du}{dx}, & \frac{d}{dx}(\csc u) &= -\csc u \cot u \frac{du}{dx}
\end{align*}
\]

Example 1 Differentiate the following functions:

(1) \( f(x) = \cos x \sin x \)

(2) \( g(x) = \frac{1 - \sec x}{\tan x} \).
(3) $g(t) = \sec(t^2 + 1)$

(4) $u(x) = 7 \tan(2 \sin(4x))$

**Example 2** Find an equation for the tangent line of $y = \cos x$ at $x = \pi/3$.

**Example 3** Find $\frac{d^4}{dx^4} \cos x$. 
3.7 Implicit Differentiation; Rational Powers

(The Idea of Implicit Differentiation)
If $y$ is a differentiable function of $x$ and satisfies a particular equation in $x$ and $y$, in order to find $dy/dx$ we can follow these steps:

- Step 1: Differentiate both sides of the equation with respect to $x$. (Keep in mind $y$ is a function of $x$!)
- Step 2: Solve for $dy/dx$ from the equation obtained in Step 1.

(Rational Powers)
If $p, q$ are integers and $u$ is a differentiable function of $x$, then

\[
\frac{d}{dx}(x^{p/q}) = \frac{p}{q}x^{(p/q)-1}
\]
\[
\frac{d}{dx}(u^{p/q}) = \frac{p}{q}u^{(p/q)-1}\frac{du}{dx}
\]

Example 1 Assume that $y$ is a differentiable function of $x$ which satisfies

\[
\cos(x - y) = (2x + 1)^3 y
\]

Use implicit differentiation to express $dy/dx$ in terms of $x$ and $y$. 

Lecture Notes - chapter 3
Example 2 Find the slope of the tangent line to the curve $2x^3 + 2y^3 = 9xy$ at the point $(1, 2)$.

Example 3 Express $d^2y/dx^2$ in terms of $x$ and $y$ given

$y^3 - x^2 = 4$.

Example 4 Differentiate the following functions.

(a) $x^{-7/9}$
(b) $(1 + x^2)^{1/5}$
(c) $(\frac{x}{1 + x^2})^{1/2}$
4 The Mean-Value Theorem; Applications of the First And Second Derivatives

4.10 Related Rates of Change Per Unit Time

Recall:
If \( Q \) is any quantity that varies with time, then the derivative \( dQ/dt \) gives the rate of change of that quantity with respect to time.

Example 1 A spherical balloon is expanding. Given that the radius is increasing at the rate of 2 inches per minute, at what rate is the volume increasing when the radius is 5 inches?

Example 2 A particle moves clockwise along the unit circle \( x^2 + y^2 = 1 \). As it passes through the point \( (1/2, \sqrt{3}/2) \), its \( y \)-coordinate decreases at the rate of 3 units per second. At what rate does the \( x \)-coordinate change at this point?

A general method for solving problems of this type:
Step 1. Draw a suitable diagram, and indicate the quantities that vary.
Step 2. Specify in mathematical form the rate of change you are looking for, and record all relevant information.
Step 3. Find an equation that relates the relevant variables.
Step 4. Differentiate with respect to time \( t \) the equation found in Step 3.
Step 5. State the final answer in coherent form, specifying the units that you are using.
Example 3 A 13-foot ladder leans against the side of a building, forming an angle $\theta$ with the ground. Given that the foot of the ladder is being pulled away from the building at a rate of 0.1 feet per second, what is the rate of change of $\theta$ when the top of the ladder is 12 feet above the ground?

Example 4 A conical paper cup 8 inches across the top and 6 inches deep is full of water. The cup springs a leak at the bottom and loses water at the rate of 2 cubic inches per minute. How fast is the water level dropping when the water is exactly 3 inches deep?

4.1 The Mean-Value Theorem

Theorem 1 The MEAN-VALUE THEOREM
If \( f \) is differentiable on the open interval \((a, b)\) and continuous on the closed interval \([a, b]\), then there is at least one number \( c \) in \((a, b)\) for which

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Theorem 2 Rolle’s Theorem
Suppose that \( f \) is differentiable on the open interval \((a, b)\) and continuous on the closed interval \([a, b]\). If \( f(a) \) and \( f(b) \) are both 0, then there is at least one number \( c \) in \((a, b)\) for which

\[
f'(c) = 0
\]

Remark 1 Rolle’s theorem is sometimes formulated as follows:
Suppose that \( g \) is differentiable on the open interval \((a, b)\) and continuous on the closed interval \([a, b]\). If \( g(a) = g(b) \), then there is at least one number \( c \) in \((a, b)\) for which

\[
g'(c) = 0
\]
Example 1 Show that the function

\[ f(x) = \sqrt{1 - x}, \quad -1 \leq x \leq 1 \]

satisfies the conditions of the mean-value theorem (MVT) and find \( c \) from the conclusion of the MVT.

See WebWork ("Similar Example") for more examples.
4.2 Increasing And Decreasing Functions

**Definition 1** A function $f$ is said to
(i) **increase** on the interval $I$ if for every two numbers $x_1, x_2$ in $I$,

$$x_1 < x_2 \implies f(x_1) < f(x_2);$$

(ii) **decrease** on the interval $I$ if for every two numbers $x_1, x_2$ in $I$,

$$x_1 < x_2 \implies f(x_1) > f(x_2).$$

**Theorem 1** Suppose that $f$ is differentiable on the interior $I$ and continuous on all of $I$.
(i) If $f'(x) > 0$ for all $x$ in the interior of $I$, then $f$ increases on all of $I$.
(ii) If $f'(x) < 0$ for all $x$ in the interior of $I$, then $f$ decreases on all of $I$.
(iii) If $f'(x) = 0$ for all $x$ in the interior of $I$, then $f$ is constant on all of $I$.

**Theorem 2**
(i) Let $I$ be an open interval. If $f'(x) = g'(x)$ for all $x$ in $I$, then $f$ and $g$ differ by a constant on $I$.
(ii) Let $I$ be an arbitrary interval. If $f'(x) = g'(x)$ for all $x$ in $I$, and $f$ and $g$ are continuous on $I$, then $f$ and $g$ differ by a constant on $I$. 
Example 1 Let $f(x) = \sqrt{1-x^2}$. Find the (largest) intervals on which $f$ increase and the intervals on which $f$ decreases.

Example 2 Let $f(x) = x - 2\sin x, 0 \leq x \leq 2\pi$. Find the (largest) intervals on which $f$ increase and the intervals on which $f$ decreases.

Example 3 Find the largest region over which the function is increasing or decreasing, for

$$f(x) = \frac{1}{|x-9|}$$
Example 4 Let \( g(x) = \frac{4}{5}x^5 - 3x^4 - 4x^3 + 22x^2 - 24x + 6 \). Find the (largest) intervals on which \( g \) increases and the intervals on which \( g \) decreases.

Example 5 Find \( f \) given that \( f'(x) = 6x^2 - 7x - 5 \) for all real \( x \) and \( f(2) = 1 \).
4.3 Local Extreme Values

**Definition 1 (Local Extreme values)**
Suppose that $f$ is a function and $c$ is an interior point of the domain. The function $f$ is said to have a **local maximum** at $c$ provided that

$$f(c) \geq f(x)$$
for all $x$ sufficiently close to $c$.

The function $f$ is said to have a **local minimum** at $c$ provided that

$$f(c) \leq f(x)$$
for all $x$ sufficiently close to $c$.

The local maxima and minima of $f$ comprise the local **extreme values** of $f$.

**Definition 2 (Critical Point)**
The interior points $c$ of the domain of $f$ for which

$$f'(c) = 0 \quad \text{or} \quad f'(x) \text{ does not exist.}$$

are called the **critical points** for $f$.

**Theorem 1** Suppose that $c$ is an interior point of the domain of $f$. If $f$ has a local maximum or local minimum at $c$, then

$$f'(c) = 0 \quad \text{or} \quad f'(x) \text{ does not exist.}$$
Theorem 2 (The First-Derivative Test) Suppose that $c$ is critical point for $f$ and $f$ is continuous at $c$. If there is a positive number $\delta$ such that:

(i) $f'(x) > 0$ for all $x$ in $(c - \delta, c)$ and $f'(x) < 0$ for all $x$ in $(c, c + \delta)$, then $f(c)$ is a local maximum.

(ii) $f'(x) < 0$ for all $x$ in $(c - \delta, c)$ and $f'(x) > 0$ for all $x$ in $(c, c + \delta)$, then $f(c)$ is a local minimum.

(iii) $f'(x) > 0$ keeps constant sign on $(c - \delta, c) \cup (c, c + \delta)$, then $f(c)$ is not a local extreme value.

Theorem 3 (The Second-Derivative Test) Suppose that $f'(c) = 0$ and $f''(c)$ exists.

(i) If $f''(c) > 0$, then $f(c)$ is a local minimum.

(ii) If $f''(c) < 0$, then $f(c)$ is a local maximum.
Example 1 List the critical point(s) and locate the local maxima/minima for

\[ f(x) = 2\sqrt{x} - \frac{x}{2} \]

Example 2 List the critical point(s) and locate the local maxima/minima for

\[ f(x) = \frac{x - \frac{5}{2}}{x^2 - 4} \]
Example 3 List the critical point(s) and locate the local maxima/minima for

\[ f(x) = |2x - 3| + 2 \]

Example 4 List the critical point(s) and locate the local maxima/minima for

\[ f(x) = 2x^3 - 3x^2 - 12x + 5 \]
4.4 Endpoint Extreme Values; Absolute Extreme Values

**Definition 3** (Endpoint Extreme values) If $c$ is an endpoint of the domain of $f$, then $f$ is said to have an endpoint maximum at $c$ provided that

$$f(c) \geq f(x) \quad \text{for all } x \text{ sufficiently close to } c.$$ 

It is said to have an endpoint minimum at $c$ provided that

$$f(c) \leq f(x) \quad \text{for all } x \text{ sufficiently close to } c.$$ 

**Definition 4** (Absolute Extreme values) The function $f$ is said to have an absolute maximum at $d$ provided that

$$f(d) \geq f(x) \quad \text{for all } x \text{ in the domain of } f.$$ 

$f$ is said to have an absolute minimum at $d$ provided that

$$f(d) \leq f(x) \quad \text{for all } x \text{ in the domain of } f.$$ 

A Summary for Finding All the Extreme Values (Local, Endpoint, and Absolute) of a Continuous Function $f$

- Step 1. Find the critical points—the interior points $c$ at which $f'(c) = 0$ or $f'(c)$ does not exist.
- Step 2. Test each endpoint of the domain by examining the sign of the derivative at nearby points.
- Step 3. Test each critical point $c$ by examining the sign of the first derivative on both sides of $c$ (the first-derivative test) or by checking the sign of the second derivative at $c$ itself (second-derivative test).
- Step 4. If the domain is unbounded on the right, determine the behavior $f(x)$ as $x \to \infty$; in unbounded on the left, check the behavior of $f(x)$ as $x \to -\infty$.
- Step 5. Determine whether any of the endpoint extremes and local extremes are absolute extremes.
Example 1 Find the critical points of \( f \) below. Then find and classify all the extreme values.

\[
x - y - 6 = 1 - 3 - 6
\]

Example 2 Find the critical points of the function

\[
f(x) = 1 + 4x^2 - \frac{1}{2}x^4, \quad x \in [-1, 3].
\]

Then find and classify all the extreme values.
Example 3  Find the critical points of the function

\[ f(x) = \begin{cases} 
  x^2 + 2x + 2, & -\frac{1}{2} \leq x < 0, \\
  x^2 - 2x + 2, & 0 \leq x \leq 2. 
\end{cases} \]

Example 4  Find the critical points of the function

\[ f(x) = 6\sqrt{x} - x\sqrt{x} \]

Then find and classify all the extreme values.

♣ Textbook, Page 179–Example 5 is a good example. Please read the example after class. ♣
4.5 Some Max-Min Problems

(Strategy of Solving Max-Min Problems)

- Step 1: Draw a representative figure and assign labels to the relevant quantities.
- Step 2: Identify the quantity to maximized or minimized and find a formula for it.
- Step 3: Express the quantity to be maximized or minimized in terms of a single variable; use the conditions given in the problem to eliminate the other variable(s).
- Step 4: Determine the domain of the function generated by Step 3.
- Step 5: Apply the techniques of the preceding sections to find the extreme value(s).

Example 1 A rectangular garden 200 square feet in area is to be fenced off against rabbits. Find the dimensions that will require the least amount of fencing given that one side of the garden is already protected by a barn.
Example 2 An isosceles triangle has a base of 6 units and a height of 12 units. Find the maximum possible area for a rectangle that is inscribed in the triangle and has one side resting on the base of the triangle. What are the dimensions of the rectangle(s) of maximum area?

Example 3 A window in the shape of rectangle capped by a semicircle is to have perimeter 36 feet. Find the radius of the semicircular part so that the window admits the most light.
4.6 Concavity and Points of Inflection

**Definition 5 (Concavity)** Let $f$ be a function differentiable on an open interval $I$. The graph of $f$ is said to be **concave up** on $I$ if $f'$ increases on $I$; it is said to be **concave down** on $I$ if $f'$ decreases on $I$.

**Remark:** Geometric Point of View on Concavity.

**Definition 6 (Concavity)** Let $f$ be a function continuous at $c$ and differentiable near $c$. The point $(c, f(c))$ is called a **point of inflection** if the graph of $f$ changes concavity at $c$.

**Theorem 1** Suppose that $f$ is twice differentiable on an open interval $I$.
- If $f''(x) > 0$ for all $x$ in $I$, then $f'$ increases on $I$, and the graph of $f$ is **concave up**.
- If $f''(x) < 0$ for all $x$ in $I$, then $f'$ decreases on $I$, and the graph of $f$ is **concave down**.

**Theorem 2** If the point $(c, f(c))$ is a **point of inflection**, then
$$f''(c) = 0 \text{ or } f''(c) \text{ does not exist.}$$
Example 1  The graph of $f$ is given in the below.

(a) Find the intervals on which $f$ increases and the intervals on which $f$ decreases.

(b) Find the intervals on which $f$ is concave up and the intervals on which $f$ is concave down.

(c) Give the $x$-coordinate of each point of inflection.

Example 2  Determine the concavity and find the point(s) of inflection (if any) of the graph of

$$f(x) = x + \cos x, \quad x \in [0, 2\pi].$$
Example 3  Determine the concavity and find the point(s) of inflection (if any) of the graph of
\[ f(x) = 3x^{5/3} - 5x. \]

Example 4  Determine the concavity and find the point(s) of inflection (if any) of the graph of
\[ f(x) = \frac{12}{x - 6}. \]
4.7 Vertical and Horizontal Asymptotes; Vertical Tangents and Cusps

**Definition 7 (Vertical Asymptote)** $f$ is said to have a vertical asymptote $x = c$ if
as $x \to c$, $f(x) \to \infty$ or $-\infty$.

**Definition 8 (Horizontal Asymptote)** $f$ is said to have a horizontal asymptote $y = L$ if
as $x \to \infty$ or as $x \to -\infty$, $f(x) \to L$,
where $L$ is a finite real number.

**Remark**: The behavior of a rational function

$$f(x) = \frac{a_nx^n + \cdots + a_1x + a_0}{b_kx^k + \cdots + b_1x + b_0}, \quad (a_n \neq 0, b_k \neq 0)$$

as $x \to \infty$ and as $x \to -\infty$ can be easily seen through dividing by the highest power of $x$ present.
Definition 9 (Vertical Tangents) Suppose that $f$ is continuous at $c$. The graph of $f$ is said to have a vertical tangent at the point $(c, f(c))$ if
\[ as \; x \to c, \quad f'(x) \to \infty \text{ or } f'(x) \to -\infty. \]

Definition 10 (Vertical Cusp) Suppose that $f$ is continuous at $c$. The graph of $f$ is said to have a vertical cusp at the point $(c, f(c))$ if
\[ as \; x \text{ tends } c \text{ from one side}, \quad f'(x) \to \infty \]
and
\[ as \; x \text{ tends } c \text{ from the other side}, \quad f'(x) \to -\infty. \]

Example 1 Find vertical asymptote(s) of
\[ f(x) = \frac{3x + 6}{x^2 - 2x - 8}. \]
Example 2  Find horizontal asymptote(s) of

\[ g(x) = \frac{x + 1 - \sqrt{x}}{x^2 - 2x + 1} \]

Example 3  Determine if the graph of

\[ f(x) = (2 - x)^{1/5} \]

has a vertical tangent at \( x = 2 \).

Example 4  Determine if the graph of

\[ f(x) = 2 - (x - 1)^{2/5} \]

has a vertical cusp at \( x = 1 \).
4.8 Some Curve Sketching

(Strateg to Sketch the graph of a function \( f \))

1. **Domain**: List domain, endpoints, vertical asymptotes and horizontal asymptotes if any.

2. **Intercepts**: List \( x \)-intercepts and \( y \)-intercepts. (The \( y \)-intercept is the value of \( f(0) \); the \( x \)-intercepts are the solutions of \( f(x) = 0 \).)

3. **Symmetry/periodicity**: Determine if \( f \) is even, or odd, or periodic.

4. **Monotonicity**: Determine all critical points and determine the intervals on which \( f \) increases and the intervals on which \( f \) decreases; determine the vertical tangents and cusps.

5. **Concavity**: Determine the intervals on which \( f \) is concave up and the intervals on which \( f \) is concave down; determine the points of inflection.

6. **Points of interest and preliminary sketch**: Plot the points of interest in a preliminary sketch: intercept points, extreme points, points of inflection, vertical tangents and vertical cusps.

7. **The graph**: Sketch the graph of \( f \) by connecting the points in a preliminary sketch, making sure that the curve "rises", "falls" and "bends" in the proper way.
**Example 1** Sketch the graph of \( f(x) = \frac{1}{4}x^4 - 2x^2 + \frac{7}{4} \).
Example 2 Sketch the graph of

\[ f(x) = \frac{x^2 - 3}{x^3}. \]
4.9 Velocity and Acceleration; Speed

Suppose that an object moves along a straight line. We choose a point of reference as origin, a positive direction, a negative direction and a unit distance. Let \(x(t)\) be the position of the object at time \(t\), then

- The velocity \(v(t)\) of the object at time \(t\) is \(v(t) = x'(t)\).
- The acceleration \(a(t)\) of the object at time \(t\) is \(a(t) = v'(t) = x''(t)\).
- The speed \(\nu(t)\) of the object at time \(t\) is \(\nu(t) = |v(t)|\).

Remark:

- Positive velocity indicates motion in the positive direction. Negative velocity indicates motion in the negative direction.
- Positive acceleration indicates increasing velocity. Negative acceleration indicates decreasing velocity.
- If the velocity and acceleration have the same sign, the object is speeding up, but if the velocity and acceleration have opposite signs, the object is slowing down.

Free Fall Near the Surface of the Earth

Suppose that an object falls freely to the ground. Let \(y(t)\) be the height of the object at time \(t\) and let \(v_0\) and \(y_0\) be the velocity and height of the object at time \(t = 0\) respectively, then

\[
y(t) = -\frac{1}{2}gt^2 + v_0t + y_0,
\]

where \(g\) is the gravitational acceleration, which is given by

\[
g = 32 \text{ feet/sec}^2 \quad \text{or} \quad g = 9.8 \text{ meters/sec}^2.
\]
Example 1  An object moves along the $x$-axis; its position at each time $t$ is given by

$$x(t) = t^3 - 12t^2 + 36t - 27, \quad t \in [0, 9].$$

(a) Where does this object start and end up on the $x$-axis?

(b) When is the velocity of the object positive? Negative? Or 0?

(c) When is the acceleration of the object positive? Negative? Or 0?

(d) When does this object speed up? When does it slow down?
Example 2  A stone is dropped from a height of 98 meters. In how many seconds does it hit the ground? What is the speed at impact?

Example 3  An explosion causes some debris to rise vertically with an initial velocity of 72 feet per second.
(a) In how many seconds does this debris attain maximum height? What is this maximum height?

(b) What is the speed of the debris as it reaches a height of 32 feet (i) going up? (ii) coming back down?
4.11 Differentials

The change in $f$ from $x$ to $x + h$ can be approximated by the product $f'(x)h$, i.e.

$$f(x + h) - f(x) \approx f'(x)h.$$ 

**Definition 1** For $h \neq 0$ the difference $f(x + h) - f(x)$ is called the increment of $f$ from $x$ to $x + h$ and is denoted by $\triangle f$:

$$\triangle f = f(x + h) - f(x).$$

The product $f'(x)h$ is called the differential of $f$ at $x$ with increment $h$ and is denoted by $df$:

$$df = f'(x)h.$$ 

**Definition 2** Given a differentiable function $f$ and a point $c$ in the domain of $f$, the linear approximation $L(x)$ to $f(x)$ at the point $x = c$ is given by

$$L(x) = f(c) + f'(c)(x - c).$$
Example 1 Use a differential to estimate the change in \( f(x) = \frac{x^{2/5}}{5} \)
(a) as \( x \) increases from 32 to 34,  
(b) as \( x \) decreases from 1 to \( \frac{9}{10} \).

Example 2 Give an estimate of \( \cos 40^\circ \).
Example 3  Find the linear approximation $L(x)$ to the function $f(x) = 3x^2 + 8x$ at the point $x = 3$ and use it to estimate the value of $f(\frac{28}{3})$.

Example 4  The length of a side of a cube was measured to be 4 cm with a possible error of up to 1/10 cm. Use a differential to estimate the maximum error in the calculated volume of the cube.
4.12 Newton-Raphson Approximations

Newton-Raphson Method for locating a root of an equation \( f(x) = 0 \):

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

Example 1  Use the Newton-Raphson method to estimate the real solution of

\[ x^3 + 3x + 1 = 0 \]

starting from the initial estimate \( x_1 = 0 \). Then find the next two approximations \( x_2 \) and \( x_3 \).
12 Infinite Series

12.1 Sigma Notation

We use the sigma notation \( \sum \) to write

\[
\sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n.
\]

**Example 1** Evaluate \( \sum_{k=1}^{4} (3k - 1) \).

**Example 2** Find a function \( f(k) \) such that, term by term,

\[
4 - 8 + 12 - 16 + 20 = \sum_{k=1}^{5} f(k).
\]

**Example 3** Evaluate \( \sum_{k=1}^{89} (8k + 9) \).
5 Integration

5.1 An area Problem; A Speed-Distance Problem

5.2 The definite integral of a continuous function

Question: How to find the area of $\Omega$, which is bounded by $y = f(x), x = a, x = b$ and $x$–axis?
A particular case: A Speed-Distance Problem

\[ \text{distance} = \text{speed} \times \text{time}. \]

How to calculated the distance traveled from time \( a \) to time \( b \) if \( \nu \) does not remain constant?

**Definition 1**

Let \( f \) be a function defined on a closed interval \([a, b]\) with \( f(x) \geq 0 \) for all \( x \in [a, b] \). Then the region under the graph of \( f \) has area \( A \) if there is a number \( A \) such that

\[
\lim_{n \to \infty} \Delta x \sum_{k=1}^{n} f(x_k^*) = A,
\]

where \( \Delta x = (b-a)/n \) and \( A \) does not depend on the choice of \( x_k^* \in [a + (k-1)\Delta x, a + k\Delta x] \).

**Definition 2**

Let \( a, b \in \mathbb{R} \) and \( f \) be a function defined on \([a, b]\). Then \( f \) is Riemann integrable from \( a \) to \( b \) means there is a number, called the **definite integral** of \( f \) from \( a \) to \( b \) denoted by \( \int_{a}^{b} f(x)dx \), such that

\[
\int_{a}^{b} f(x)dx = \lim_{n \to \infty} \Delta x \sum_{k=1}^{n} f(x_k^*),
\]

where \( \Delta x = (b-a)/n \) and \( \int_{a}^{b} f(x)dx \) does not depend on the choice of \( x_k^* \in [a + (k-1)\Delta x, a + k\Delta x] \).

**Two basic formulas**

The integral of a constant function:

\[
\int_{a}^{b} k \, dx = k(b-a)
\]

The integral of the identity function:

\[
\int_{a}^{b} x \, dx = \frac{1}{2}(b^2 - a^2)
\]
Example 1 Let \( f(x) = -\frac{1}{2}x + 4 \) for \( x \in [1,4] \). Find the area of the region under the graph of \( y = f(x) \) between \( x = 1 \) and \( x = 4 \).

Example 2 Give both overestimate and underestimate of the area under the graph of

\[
f(x) = 27 + 6x - x^2
\]

from \( x = -3 \) to \( x = 9 \) using an "upper sum" and "lower sum" of areas of 4 rectangles of equal width.
Example 3 The following table gives the velocity $v(t)$ (in feet per sec) at different instances of time $t$ (in sec) of a particle moving along a horizontal axis.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$</td>
<td>50</td>
<td>49</td>
<td>46</td>
<td>40</td>
<td>28</td>
<td>0</td>
</tr>
</tbody>
</table>

Estimate the distance traveled by the particle between $t = 0$ sec and $t = 20$ sec using 5 time intervals of equal length with (1) left endpoints (2) right endpoints.

Example 4 Find the area under the curve $f(x) = -\frac{1}{2}x + 4$ from $x = 1$ to $x = 4$, using $n$ equal-width rectangles and left endpoints, and taking the limit as $n \to \infty$. 
Example 5  Find the area under the curve \( f(x) = 100 - 3x^2 \) from \( x = 1 \) to \( x = 5 \), using \( n \) equal-width rectangles and right endpoints, and taking the limit as \( n \to \infty \).

Example 6  Evaluate the following integral below by interpreting it in terms of areas.

\[
\int_{-9}^{9} \sqrt{81 - x^2} \, dx
\]
5.3 The Function $F(x) = \int_a^x f(t)dt$

**Theorem 1** Suppose that $f$ is continuous on $[a, b]$, and $P$ and $Q$ are partitions of $[a, b]$. If $Q \supseteq P$, then

$L_f(P) \leq L_f(Q) \quad \text{and} \quad U_f(Q) \leq U_f(P)$.

**Theorem 2** If $f$ is continuous on $[a, b]$ and $a < c < b$, then

$$\int_a^c f(t)dt + \int_c^b f(t)dt = \int_a^b f(t)dt.$$ 

$$\int_a^b f(t)dt = -\int_b^a f(t)dt, \quad \int_c^c f(t)dt = 0.$$ 

**Theorem 3** If $f$ is continuous on $[a, b]$ and let $c$ be any number in $[a, b]$. The function $F$ defined on $[a, b]$ by setting

$$F(x) = \int_c^x f(t)dt.$$ 

is continuous on $[a, b]$, differentiable on $(a, b)$, and has derivative

$$F'(x) = f(x) \quad \text{for all} \ x \in (a, b).$$
Example 1 Let \( \int_2^5 f(x) \, dx = 4 \), \( \int_2^3 f(x) \, dx = 13 \), \( \int_4^5 f(x) \, dx = 6 \). Evaluate \( \int_3^4 f(x) \, dx \) and \( \int_5^3 f(x) \, dx \).

Example 2 For all real \( x \), define
\[
F(x) = \int_0^x \sin \pi t \, dt
\]

Find \( F'(\frac{3}{4}) \) and \( F'(\frac{-1}{2}) \).
Example 3 Let
\[ F(x) = \int_x^{14} \sin(t^5) \, dt, \]
find \( F'((\pi/4)^{1/5}) \).

Example 4 Set
\[ F(x) = \int_0^x \frac{1}{1 + t^2} \, dt \text{ for all real numbers } x. \]

(a) Find the critical points of \( F \) and determine the intervals on which \( F \) increases and the intervals on which \( F \) decreases.

(b) Determine the concavity of the graph of \( F \) and find the points of inflection (in any).

(c) Sketch the graph of \( F \).
5.8 Additional Properties of The Definite Integral

Properties:

(a) If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x) \, dx \geq 0$.

(b) If $f(x) > 0$ for all $x \in [a, b]$, then $\int_a^b f(x) \, dx > 0$.

(c) If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$.

(d) If $f(x) < g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) \, dx < \int_a^b g(x) \, dx$.

(e) $\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx$.

(f) $m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$.

(g) $\frac{d}{dx} \left( \int_a^{u(x)} f(t) \, dt \right) = f(u(x))u'(x)$.

(h) If $f$ is odd on $[-a, a]$, then $\int_{-a}^a f(x) \, dx = 0$.

(i) If $f$ is even on $[-a, a]$, then $\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$. 
Example 1 Given that $4 \leq f(x) \leq 7$ for $-7 \leq x \leq 7$, find the best possible estimate of the value of $\int_{-7}^{7} f(x) \, dx$.

Example 2 Find \( \frac{d}{dx} \left( \int_{-2}^{x^3} \frac{1}{1+t} \, dt \right) \).

Example 3 Find \( \frac{d}{dx} \left( \int_{2}^{x} \frac{1}{1+t^2} \, dt \right) \).

Example 4 \( \int_{-\pi}^{\pi} (\sin x - x \cos x)^3 \, dx \).
5.4 The Fundamental Theorem of Integral Calculus

**Definition 3** Let $f$ be continuous on $[a, b]$. A function $G$ is called an antiderivative for $f$ on $[a, b]$ if

$G$ is continuous on $[a, b]$ and $G'(x) = f(x)$ for all $x \in (a, b)$.

* Table of Antiderivatives

<table>
<thead>
<tr>
<th>Function</th>
<th>Antiderivative</th>
<th>Function</th>
<th>Antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin x$</td>
<td>$\cos x$</td>
<td>$\sec^2 x$</td>
<td>$\csc^2 x$</td>
</tr>
<tr>
<td>$\sec x \tan x$</td>
<td>$\csc x \cot x$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x^r (r \neq -1)$</td>
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</tbody>
</table>

**Theorem 1** Let $f$ be continuous on $[a, b]$. If $G$ is any antiderivative for $f$ on $[a, b]$, then

$$\int_a^b f(t)dt = G(b) - G(a).$$

* The Linearity of the Integral

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

**Example 1** Evaluate

(a) $\int_1^2 \frac{dx}{x^3}$

(b) $\int_{-\pi/4}^{\pi/3} \sec^2 t \ dt$
Example 2 Evaluate
\[ \int_2^6 (5x^2 - 12x + 6) \, dx. \]

Example 3 Evaluate
\[ \int_1^2 \frac{x^4 + 1}{x^2} \, dx. \]

Example 4 Evaluate
\[ \int_{-1}^1 (x - 1)(x + 2) \, dx. \]

Example 5 Evaluate
\[ \int_0^{23} |x - 17| \, dx. \]
5.5 Some Area Problems

* Recall: Let \( f(x) \) be continuous and nonnegative on \([a, b]\), what is the area of \( \Omega \) which is under the graph of \( y = f(x) \) above \( x \)-axis between \( x = a \) and \( x = b \)?

\[
\text{area of } \Omega = \int_a^b f(x) \, dx
\]

* Question: Let \( f(x) \) and \( g(x) \) be continuous on \([a, b]\), what is the area of \( \Omega \) which is under the graph of \( y = f(x) \) above the graph of \( y = g(x) \) between \( x = a \) and \( x = b \)?

\[
\text{area of } \Omega = \int_a^b [f(x) - g(x)] \, dx
\]

* Question: Let \( f(x) \) be continuous on \([a, b]\), what is the total area of the region between the graph of \( y = f(x) \) and \( x \)-axis between \( x = a \) and \( x = b \)?
Example 1  Find the area of the region bounded above by the curve $y = 4 - x^2$ and below by $x$-axis.

Example 2  Find the area of the region bounded above by the line $y = x + 2$ and below by the parabola $y = x^2$. 
Example 3 Find the total area between $y = x^2 - 2x$ and $x$-axis from $x = -1$ to $x = 3$.

Example 4 Find the total area between $y = 4x$ and $y = x^3$ from $x = -2$ to $x = 2$. 
6.1 More on Area

*Question:* Let \( x = f(y) \) and \( x = g(y) \) be continuous on \([c, d]\), what is the area of \( \Omega \) bounded on the left by \( x = F(y) \) and bounded right by \( x = G(y) \) between \( y = c \) and \( y = d \)? between \( x = a \) and \( x = b \)?

Example 1 Find the area of the region bounded on the left by the curve \( x = y^2 \) and bounded on the right by the curve \( x = 3 - 2y^2 \).
Example 2 Calculate the area of the region bounded by the curves $x = y^2$ and $x - y = 2$ by
(a) integrating with respect to $x$.

(b) integrating with respect to $y$. 
5.6 Indefinite Integrals

Recall: If $F(x)$ is an antiderivative for $f(x)$ on $[a, b]$, so is $F(x) + C$, where $C$ is an arbitrary constant. Thus we have a family of antiderivative functions for $f(x)$.

**Definition 4** Let $f$ be a continuous function defined on a closed interval $[a, b]$. The **indefinite integral**, denoted by $\int f(x) \, dx$, is to represent the family of antiderivative functions of $f(x)$ on $[a, b]$.

**Table of Indefinite Integrals**

<table>
<thead>
<tr>
<th>Function</th>
<th>Integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin x$</td>
<td>$\int \sin x , dx = -\cos x + C$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$\int \cos x , dx = \sin x + C$</td>
</tr>
<tr>
<td>$\sec^2 x$</td>
<td>$\int \sec^2 x , dx = \tan x + C$</td>
</tr>
<tr>
<td>$\csc^2 x$</td>
<td>$\int \csc^2 x , dx = -\cot x + C$</td>
</tr>
<tr>
<td>$\sec x \tan x$</td>
<td>$\int \sec x \tan x , dx = \sec x + C$</td>
</tr>
<tr>
<td>$\csc x \cot x$</td>
<td>$\int \csc x \cot x , dx = -\csc x + C$</td>
</tr>
<tr>
<td>$x^r$</td>
<td>$\int x^r , dx = \frac{x^{r+1}}{r+1} + C$, $(r \neq 1)$</td>
</tr>
</tbody>
</table>

**Lineararity of Indefinite Integrals**

$$\int [\alpha f(x) + \beta g(x)] \, dx = \alpha \int f(x) \, dx + \beta \int g(x) \, dx$$

**Differential Equations**

Given $f(x)$, consider the **differential equation**

$$\frac{dy}{dx} = f(x),$$

where $y$ is the unknown function. The **general solution** of this equation is given by

$$y = \int f(x) \, dx + C.$$
Given $f(x)$, $x_0$ and $y_0$, the initial value problem
\[ \frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \]
can be solved by finding the general solution first and then determine arbitrary constant $C$ by substituting $x_0, y_0$.

* Application to Motion

Recall: $x(t)$ means position; $v(t)$ means velocity; $a(t)$ means acceleration; $\nu(t)$ means speed. And $x'(t) = v(t)$, $v'(t) = a(t)$. Then

- Given the acceleration $a(t)$, the velocity is given by $v(t) = \int a(t) \, dt$.
- Given the velocity $v(t)$, the position $x(t) = \int v(t) \, dt$.
- Given the velocity $v(t)$, the distance traveled from time $t = a$ to $t = b$ is given by $\int_a^b |v(t)| \, dt$.

Example 1 Calculate $\int [5x^{3/2} - 2 \csc^2 x] \, dx$.

Example 2 Let $F$ be the antiderivative of
\[ f(x) = \frac{5}{x^2} - \frac{8}{x^6} \]
with $F(1) = 0$. Find a formula for $F(x)$.
Example 3  Solve the initial value problem

\[ \frac{dy}{dx} = \frac{4}{x^4} + 8x^7, \quad y(1) = 14. \]

Example 4  An object moves along the x-axis with acceleration \( a(t) = 2t - 2 \) units per second per second. Its initial position (position at time \( t = 0 \)) is 5 units to the right of the origin. One second later the object is moving left at the rate of 4 units per second.

(a) Find a formula for the velocity \( v(t) \).

(b) Find a formula for the position \( x(t) \) and find its position at \( t = 4 \) seconds.

(c) How far does the object travel during these 4 seconds?
5.7 Working Back From the Chain Rule; The $u$-Substitution

**Theorem 1** If $F' = f$, then
\[
\int f(u(x))u'(x) \, dx = F(u(x)) + C.
\]
Also
\[
\int_{a}^{b} f(u(x))u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du.
\]

**Example 1** Calculate

(a) $\int x^2 \sqrt{4 + x^3} \, dx$

(b) $\int \frac{2 \sin 3x}{(5 + \cos 3x)^4} \, dx$
Example 2 Calculate
(a) \( \int x^3 \sin(5x^4 + 6) \, dx \)

(b) \( \int 9 \sec^2(3\sqrt{x} + 5) \frac{1}{\sqrt{x}} \, dx \)

Example 3 Evaluate
(a) \( \int_{\sqrt{3}}^{3} x^5 \sqrt{x^2 + 1} \, dx \)

(b) \( \int_{0}^{1/2} \cos^3 \pi x \sin \pi x \, dx \)
5.9 Mean-Value Theorems for Integrals; Average Value of a Function

**Theorem 1 (The First Mean-Value Theorem for Integrals)** If $f$ is continuous on $[a, b]$, then there is at least one number $c$ in $(a, b)$ for which

$$\int_{a}^{b} f(x) \, dx = f(c)(b - a).$$

This number $f(c)$ is called the **average value or mean value** of $f$ on $[a, b]$.

* Geometric Interpretation of Average Value:

**Example 1** Find the average value of $f(x) = 2x^2 - 5x$ on the interval $[2, 6]$. Then find $c \in (2, 6)$ such that $f(c)$ equals this average value.
Example 2 Find the average value of \( f(x) = |11x| \) on the interval \([-5, 5]\). Then find \( c \in (-5, 5) \) such that \( f(c) \) equals this average value.

Example 3 An object moves along a coordinate line with acceleration \( a(t) = (t + 2)^3 \) units per second per second. Its initial velocity is 5 units per second. Find its average velocity from \( t = 5 \) seconds to \( t = 9 \) seconds.