On critical Fujita exponents for the porous medium equation with a nonlinear boundary condition

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Abstract  
We establish the critical Fujita exponents for the solution of the porous medium equation \( u_t = \Delta u^m \), \( x \in \mathbb{R}^N_+ \), \( t > 0 \), subject to the nonlinear boundary condition \(-\frac{\partial u^m}{\partial x_1} = u^p\), \( x_1 = 0\), \( t > 0\), in multi-dimension.  
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1. Introduction  
In this paper we determine the critical Fujita exponent concerned with the following initial-boundary value problem:  

\[
\begin{align*}
    u_t &= \Delta u^m, \quad x \in \mathbb{R}^N_+, \ t > 0, \\
    u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^N, \\
    -\frac{\partial u^m}{\partial x_1} &= u^p, \quad x_1 = 0, \ t > 0,
\end{align*}
\]  

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where \( R_N^+ = \{(x_1, x') | x' \in R^{N-1}, \, x_1 > 0\} \), \( m > 1 \), \( p > 0 \), and \( u_0(x) \) is a nonnegative bounded function satisfying the compatibility condition

\[
-\frac{\partial u_0^m(x)}{\partial x_1} = u_0^p(x), \quad x_1 = 0,
\]

and is locally supported near some point, namely, for some \( x^0 \in R_N^+ \), \( \text{supp} \, u_0 \subset B_R(x^0) \cap R_N^+ \) and \( u_0(x) \neq 0 \). However, the last assumption is not a real requirement for deriving our results. What we want to show is that even for the initial datum \( u_0(x) \) vanishing except for a small ball, the solutions may still blow up in a finite time.

The concept of critical Fujita exponents was proposed by Fujita before 1970’s in discussing the heat conduction equation with nonlinear source; see, for example, [1]. Following the idea of Fujita, we may define similar concepts for problem (1.1)–(1.3). We call \( p_0 \) the critical global existence exponent if it has the following property: if \( p > p_0 \), there always exist nonglobal solutions of problem (1.1)–(1.3) while if \( 0 < p < p_0 \), every solution of problem (1.1)–(1.3) is global. \( p_c \) is called the critical Fujita exponent if for \( p_0 < p < p_c \) any nontrivial solutions of problem (1.1)–(1.3) blows up in a finite time; for \( p > p_c \) small data solutions exist globally in time while large data solutions are nonglobal.

The problem of determining critical Fujita exponent is an interesting one in the general theory of blowing-up solutions to different nonlinear evolution equations of mathematical physics. Over the past few years there have been a number of extensions of Fujita result in various directions; see [2–6]. Recently, it was Galaktionov and Levine [2] who first studied the one-dimensional case for the nonlinear boundary-value problem (1.1)–(1.3) with \( u_0(x) \) having compact support. They showed that \( p_0 = (m + 1)/2 \), \( p_c = m + 1 \). As for the similar questions with positive initial data \( u_0(x) \), we refer to [7–9].

This paper can be thought of as a natural continuation of [2] to multi-dimensional case. Because we are interested in the phenomena that local initial perturbation may cause blow up of solutions, we need not to consider the domain rather than the half-space \( R_N^+ \). In fact, for a general domain, we may localize and flatten the boundary, and then make small modification for the arguments presented in this paper.

The main result of this paper is that \( p_0 = (m + 1)/2 \), \( p_c = m + 1/N \). The idea of the proof is to construct super-solutions and sub-solutions inspired by [2]. However, the sub-solutions are quite different from those adopted in [2], since they should be chosen to have compact support in any spatial direction. For the process of verifying such kind of sub-solutions, we use the discriminant of cubic algebraic equations.

2. The main results and their proofs

We need the following simple lemma.

**Lemma 2.1** (The discriminant of cubic algebraic equations). For the cubic equation

\[
x^3 + px + q = 0,
\]

(2.1)

there exist three roots

\[
x_1 = A + B, \quad x_{2,3} = -\frac{A + B}{2} \pm i \frac{A - B}{2} \sqrt{3},
\]

where \( A = \sqrt{\frac{2}{3} q} \) and \( B = \frac{p}{2} \).
where

\[ A = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}}, \quad B = \sqrt[3]{-\frac{q}{2} - \sqrt{Q}}, \]

and \( Q = (p/3)^3 + (q/2)^2 \).

- If \( Q = 0 \), (2.1) has three real roots and \( x_2 = x_3 \);
- If \( Q > 0 \), (2.1) has one real root and two conjugate complex roots;
- If \( Q < 0 \), (2.1) has three unequal real roots.

**Theorem 2.2.** If \( p > m + 1/N \), then any nontrivial nonnegative solution of the problem (1.1)–(1.3) blows up in finite time for “large” \( u_0 \).

**Proof.** We begin with the construction of a nonglobal sub-solution of the self-similar form

\[ u(x, t) = (T - t)^{-\frac{1}{2p - m + 1}} \theta(\eta), \tag{2.2} \]

where \( T > 0 \) is a given constant,

\[ \eta = |\zeta|, \quad \zeta = \frac{x}{(T - t)^{\frac{p - m}{2p - m + 1}}}. \]

We can see that \( u(x, t) \) is a sub-solution of (1.1)–(1.3) if the function \( \theta(\eta) \) satisfies

\[ \frac{1}{\eta^{N-1}} (\eta^{m-1} (\theta^m)' - p - m - 2p - (m + 1)) \eta\theta' \geq \frac{1}{2p - m + 1} \theta \geq 0 \tag{2.3} \]

for \( \eta \in [\eta > 0 \mid \theta(\eta) > 0] \) and

\[ -\frac{\partial \theta^m}{\partial \zeta_1} \leq \theta^p, \quad \zeta_1 = 0. \tag{2.4} \]

We claim that (2.3) and (2.4) admits a solution of the form

\[ \theta(\eta) = A \left( a - \eta \right)^{-\frac{b}{m}} \left( \eta - b \right)^{-\frac{b}{m}}, \quad 0 < b < \eta < a, \]

for some positive constants \( A, a, b \) specified later. First, such \( \theta(\eta) \) satisfies (2.4), since \( x_1 = 0 \) implies \( \zeta_1 = 0 \) and

\[ -\frac{\partial \theta^m}{\partial \zeta_1} = -\frac{\partial \theta^m}{\partial \eta} \frac{\partial \eta}{\partial \zeta_1} = -\frac{\partial \theta^m}{\partial \eta} \zeta_1 \]

implies

\[ -\frac{\partial \theta^m}{\partial \zeta_1} \bigg|_{\zeta_1=0} = 0 \leq \theta^p. \]

To check (2.1), by a direct calculation we get

\[ \theta' = -A \frac{1}{m - 1} (a - \eta)^{-\frac{b}{m}} (\eta - b)^{-\frac{b}{m}} + A \frac{1}{m - 1} (a - \eta)^{-\frac{b}{m} - 1} (\eta - b)^{-\frac{b}{m} - 1}, \]
Set $(\theta^m)'' = A_m^m \frac{m}{m-1} \left[ \frac{1}{m-1} (a - \eta)^{\frac{1}{m-1} - 1} (\eta - b)^{\frac{m}{m-1}} - \frac{m}{m-1} (a - \eta)^{\frac{1}{m-1} - 1} (\eta - b)^{\frac{1}{m-1}} - \frac{m}{m-1} (a - \eta)^{\frac{1}{m-1} - 1} (\eta - b)^{\frac{1}{m-1}} + \frac{1}{m-1} (a - \eta)^{\frac{m}{m-1} - 1} (\eta - b)^{\frac{1}{m-1} - 1} \right]$. Substituting those into (2.3) and multiplying (2.3) by $(a - \eta)^{-1/(m-1)} (\eta - b)^{-1/(m-1)}$, we have

\[
\begin{align*}
\frac{A_m^m}{(m-1)^2} (\eta - b)^2_+ &- \frac{2A_m^m m^2}{(m-1)^2} (a - \eta)^+ (\eta - b)^+ + \frac{A_m^m}{(m-1)^2} (a - \eta)^2_+ \\
&- \frac{N-1}{\eta} \frac{A_m^m}{(m-1)^2} (a - \eta)(\eta - b)^2_+ + \frac{N-1}{\eta} \frac{A_m^m}{(m-1)^2} (a - \eta)^2_+ (\eta - b)^+ \\
&- \frac{p-m}{2p-(m+1)} \eta \left[ - \frac{A}{m-1} (a - \eta)^+ + \frac{A}{m-1} (a - \eta)^+ \right] \\
&- \frac{A}{2p-(m+1)} (a - \eta)^+ (\eta - b)^+ \geq 0.
\end{align*}
\]

Set

\[
\begin{align*}
e_1 &= \frac{2A_m^m}{(m-1)^2} + \frac{2A_m^m m^2}{(m-1)^2} + (N-1) \frac{2A_m^m}{m-1} + \frac{A}{m-1}, \\
e_2 &= -\frac{2A_m^m}{(m-1)^2} (a+b) - \frac{2A_m^m m^2}{(m-1)^2} (a+b) - (N-1) \frac{A_m^m}{m-1} 3(a+b) \\
&- \frac{p-m}{2p-(m+1)} \frac{A}{m-1} (a+b) - \frac{A}{2p-(m+1)} (a+b), \\
e_3 &= \frac{A_m^m}{(m-1)^2} (a^2 + b^2) + \frac{2A_m^m m^2}{(m-1)^2} ab + \frac{A}{2p-(m+1)} ab \\
&+ (N-1) \frac{A_m}{m-1} (a^2 + b^2 + 4ab), \\
e_4 &= -(N-1) \frac{A_m}{m-1} ab(a+b).
\end{align*}
\]

We observe that (2.3) holds if

\[
e_1 \eta^3 + e_2 \eta^2 + e_3 \eta + e_4 \geq 0, \quad 0 < b < \eta < a.
\]

If we choose $A$ large enough, it reduces to show

\[
e_1 \eta^3 + \tilde{e}_2 \eta^2 + \tilde{e}_3 \eta + \tilde{e}_4 \geq 0, \quad 0 < b < \eta < a,
\]

where
\[ \hat{e}_1 = \frac{2 + 2m}{m - 1} + 2(N - 1), \]
\[ \hat{e}_2 = -\frac{2 + 2m}{m - 1} (a + b) - 3(N - 1)(a + b), \]
\[ \hat{e}_3 = \frac{1}{m - 1} (a^2 + b^2) + \frac{2m}{m - 1}ab + (N - 1)(a^2 + b^2 + 4ab), \]
\[ \hat{e}_4 = -(N - 1)ab(a + b). \]

To do this, set
\[ y(\eta) = \hat{e}_1 \eta^3 + \hat{e}_2 \eta^2 + \hat{e}_3 \eta + \hat{e}_4. \]

Notice that \( \hat{e}_1 > 0 \) implies \( \lim_{\eta \to +\infty} y(\eta) = +\infty. \)

Letting \( a = cb, \ c > 1, \) we want to show that if \( c \to 1^+, \) (2.6) holds. From Lemma 2.1 we know that when \( c \to 1^+, \) the equation \( y(\eta) = 0 \) has one real root and two complex roots. We only need to show that if \( c \to 1^+, \ y(b) \geq 0, \) namely,
\[ y(b) = \left[ \frac{2 + 2m}{m - 1} + 2(N - 1) \right] b^3 + \left[ -\frac{(2 + 2m)(1 + c)}{m - 1} - 3(N - 1)(1 + c) \right] b^3 \]
\[ + \frac{1}{m - 1} (1 + c^2 + (N - 1)(c^2 + 1 + 4c) + \frac{2m}{m - 1} c) b^3 \]
\[ - [(N - 1)c(c + 1)] b^3 \geq 0. \]

Let
\[ g(c) = \left[ \frac{2 + 2m}{m - 1} + 2(N - 1) \right] + \left[ -\frac{(2 + 2m)(1 + c)}{m - 1} - 3(N - 1)(1 + c) \right] \]
\[ + \frac{1}{m - 1} (1 + c^2 + (N - 1)(c^2 + 1 + 4c) + \frac{2m}{m - 1} c) \]
\[ - [(N - 1)c(c + 1)]; \]
then \( y(b) = g(c)b^3, \) where \( g(c) = (1 - c)^2/(m - 1) \geq 0. \) So \( y(b) \geq 0. \)

Thus we have verified that \( u(x, t) \) is a weak sub-solution of (1.1)–(1.3) and that \( u(x, t) \)
blows up in a finite time.

If for any given \( T, A, a, b, \) and \( c \) satisfying (2.3) and (2.4), the initial function \( u_0 \) is
large enough such that
\[ u_0(x) \geq u(x, 0), \quad x \in \mathbb{R}^N_+, \]
then from the comparison principle (see [10]), \( u(x, t) \geq u(x, t) \) in \( \mathbb{R}^N_+ \times (0, T) \) and hence
\( u(x, t) \) blows up in a finite time which is not larger than \( T. \) The proof is complete. \( \square \)

**Theorem 2.3.** If \((m + 1)/2 < p < m + 1/N, \) then any nontrivial nonnegative solution of problem (1.1)–(1.3) blows up in finite time.

**Proof.** We now use the idea used for a different problem in [11]. We first notice that (1.1)–(1.3) admits the following well-known self-similar solution
\[ u_B(x, t) = (\tau + t)^{-\frac{N}{N-p}} \theta(\eta), \quad (2.7) \]
where
\[ \eta = |\zeta|, \quad \zeta = \frac{x}{(\tau + t)^{\frac{1}{2}m-1+2}}. \]
with \( \tau > 0 \) an arbitrary constant. We can see that \( u(x, t) \) is a sub-solution of (1.1)–(1.3) if \( \theta(\eta) \) satisfies
\[ \frac{1}{\eta^{N-1}} (\eta^{N-1}(\theta^m)'')' + \frac{1}{N(m - 1) + 2} \eta \theta' + \frac{N}{N(m - 1) + 2} \theta = 0 \quad (2.8) \]
and
\[ \frac{\partial \theta^m}{\partial \zeta} \bigg|_{\zeta = 0} = 0. \quad (2.9) \]
Here \( \eta \in [\eta > 0 \mid \theta(\eta) \geq 0] \).

By a simple calculation, we see that \( \theta(\eta) = A(c^2 - \eta^2)^{1/(m-1)} \), \( 0 < \eta < c \), satisfies (2.8) and (2.9), where
\[ A = \left\{ \frac{m - 1}{2m[N(m - 1) + 2]} \right\}^{\frac{1}{m-1}}. \]
Thus \( u_B \) is a sub-solution to problem (1.1)–(1.3).

By using the properties of weak solutions of problem (1.1)–(1.3), we deduce that there exist \( t_0 \geq 0 \) such that
\[ u(0, t_0) > 0. \]
Since \( u(x, t_0) \) is a continuous function, there exist \( \tau > 0 \) large enough and small \( c > 0 \) such that
\[ u(x, t_0) \geq u_B(x, t_0), \quad x \in \mathbb{R}_+^N. \]
Then by comparison principle we deduce that
\[ u(x, t) \geq u_B(x, t), \quad t \geq t_0, \quad x \in \mathbb{R}_+^N. \quad (2.10) \]
We now prove that there exist \( t_0 \geq 0 \) and \( T \) large enough so that
\[ u_B(x, t_0) \geq u(x, 0), \quad x \in \mathbb{R}_+^N, \quad (2.11) \]
where \( u(x, t) \) is the sub-solution given by (2.2). By using the space–time structure of both functions \( u_B \) and \( u \), we choose suitable constants \( a, b \) such that \( 0 < a - b < 1 \). If
\[ (\tau + t_0)^{\frac{N}{2(m-1)+2}} \gg T^{-\frac{1}{2p-m}}, \quad (2.12) \]
and
\[ a^\frac{N}{2(m-1)+2} \gg T^{\frac{p-m}{2p-m+1}}, \quad (2.13) \]
are satisfied, (2.11) is valid. We can see from (2.12) and (2.13) that such \( t_0 \) and \( T \) exist if
\[ T \gg T^{\frac{N(p-m)}{2p-m+1}}. \]
for arbitrarily large $T$. This implies that
\[ \frac{1}{2p - (m + 1)} > \frac{N(p - m)}{2p - (m + 1)}, \]
namely,
\[ p < m + \frac{1}{N}. \]

Hence, from (2.10) and (2.11), using the comparison principle we have that if $p_0 < p < p_c$, $u(x, t)$ blows up in a finite time. The proof is complete. \endproof

**Theorem 2.4.** If $p > m + 1/N$, then any nontrivial nonnegative solution of problem (1.1)–(1.3) is global in time for “small” $u_0$.

**Proof.** We shall seek a global super-solution of the self-similar form
\[ \bar{u}(x, t) = (T + t)^{-\frac{1}{2m-2}} B g(\eta), \quad (2.14) \]
where
\[ \eta = |\xi|, \quad \xi_1 = \frac{x_1 + b}{(T + t)^{\frac{p-m}{2p-(m+1)}}, \quad \xi_i = \frac{x_i}{(T + t)^{\frac{p-m}{2p-(m+1)}}} (i = 2, \ldots, N), \]
\[ T > 0 \text{ is a given positive constant. We can see that } \bar{u}(x, t) \text{ is a super-solution of (1.1)–(1.3) if } g(\eta) \geq 0 \text{ satisfies} \]
\[ \frac{1}{\eta^{N-1}} (\eta^{N-1}(g^m)'')' + \frac{p - m}{2p - (m + 1)} \eta g' + \frac{1}{2p - (m + 1)} g \leq 0 \quad (2.15) \]
and
\[ -B m \frac{\partial g^m}{\partial \xi_1} \bigg|_{\xi_1 = b} \geq B^p g^p. \quad (2.16) \]
Here $\xi \in \{\eta > 0 \mid g(\eta) \geq 0\}$, $B > 0$.

Now we show that $g(\eta) = A(\eta^2 - \eta^2)^{1/(m-1)}$, where
\[ A = \left\{ \frac{m - 1}{2m[N(m - 1) + 2]} \right\}^{\frac{1}{m-1}}, \]
satisfies (2.15) and (2.16), where $b \in (0, c)$. Using
\[ \frac{1}{\eta^{N-1}} (\eta^{N-1}(g^m)')' = -\frac{1}{N(m - 1) + 2} \eta g' = \frac{N}{N(m - 1) + 2} g, \]
we see that $g$ satisfies
\[ B^{m-1} \left\{ -\frac{1}{N(m - 1) + 2} \eta g' - \frac{N}{N(m - 1) + 2} g \right\} + \frac{p - m}{2p - (m + 1)} \eta g' + \frac{1}{2p - (m + 1)} g \leq 0, \]
namely,
\[
\left\{ \begin{array}{c}
2 \frac{B^m - 1}{m - 1} - \frac{p - m}{2p - (m + 1)} \\
- \frac{1}{2p - (m + 1)} - \frac{NB^m - 1}{N(m - 1) + 2} \\
+ \frac{1}{2p - (m + 1)} - \frac{NB^m - 1}{N(m - 1) + 2}
\end{array} \right\} \eta^2 
\leq 0.
\]
Since \( p > m + 1/N \), we can choose a suitable constant \( B \) such that
\[
\frac{B^m - 1}{N(m - 1) + 2} < \frac{p - m}{2p - (m + 1)} \quad \frac{1}{2p - (m + 1)} < \frac{NB^m - 1}{N(m - 1) + 2}.
\]
Thus (2.15) is valid. Finally, we notice that, for \( g \), inequality (2.16) is equivalent to
\[
(BA)^{p-m}(c^2 - \eta^2) \frac{p-1}{m-1} \leq \frac{2mb}{m-1}, \quad b < \eta < c.
\]
If the inequality
\[
(BA)^{p-m}(c^2 - b^2) \frac{p-1}{m-1} \leq \frac{2mb}{m-1}, \quad b < \eta < c,
\]
holds, (2.17) is true. Setting \( c = ab, \alpha \geq 1 \), by choosing \( B \) small enough, we have
\[
(BA)^{p-m}(\alpha^2 - 1) \frac{p-1}{m-1}b^{2-p/(m+1)} \leq \frac{2m}{m-1},
\]
which implies that (2.17) is valid.

Thus, for \( p > p_c \), there exists a nontrivial global super-solution, and hence a class of small global solutions. We have thus completed the proof. \( \square \)

**Theorem 2.5.** If \( 0 < p < p_0 \), then any nontrivial nonnegative solution of problem (1.1)–(1.3) is global in time.

**Proof.** If \( p = p_0 = (m + 1)/2 \), we can construct a global super-solution of the self-similar form
\[
u^*(x_1, x', t) = u^*(x_1, 0, t) = e^{\alpha(T+t)}h(\zeta),
\]
where \( \zeta = x_1/e^{\alpha(T+t)} \), \( T > 0 \) is a given positive constant, and \( \alpha > 0 \). We can see that \( u^*(x, t) \) is a solution of (1.1)–(1.3) if \( h(\zeta) \) satisfies
\[
(h^m)''(\zeta) + \frac{\alpha(m-1)}{2} \zeta h'(\zeta) - \alpha h(\zeta) = 0 \quad (2.18)
\]
and
\[
-(h^m)'(0) = h^{p_0}(0). \quad (2.19)
\]
From [12] we see that there exists a unique solution \( h \neq 0 \) which has compact support on \( x_1 \) such that (2.18) and (2.19) hold. Thus, we can choose \( T \) large enough such that
\[
\nu_0(x) \leq u^*(x, 0), \quad x \in \mathbb{R}^N_+.
\]
Using comparison principle, we have
\[ u(x, t) \leq u^*(x, t). \]
From the global existence of \( u^*(x, t) \), we see that \( u(x, t) \) is also global in time.

If \( p < p_0 \), \( u^*(x, t) \) is a global super-solution of (1.1)–(1.3) whenever \( u^*(0, t) \geq 1 \). Hence, using comparison principle again, we can get the global existence of \( u(x, t) \). The proof is complete. \( \square \)

References