Deconvolution of Scattered Acoustic Fields
for Ultrasonic Imaging

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1 Introduction

1.1 Background

Ultrasound imaging of human tissue has found widespread clinical application. However, the diagnostic usefulness of such images is limited by poor spatial resolution, low contrast between different tissue structures, sensor noise, and image artifacts (such as blur and speckle introduced by emitted pulse). Physical limitations of the transducer employed (such as the effective bandwidth of the pulse) coupled with the dispersive and attenuating properties of human tissue lead to less than ideal data. Also, noise inherent in the system and incomplete data sets (due to the geometry of receive arrays) can further limit the usefulness of imaging systems. Therefore, image processing algorithms have recently been developed to restore the degraded ultrasound data prior to the application of the inversion algorithm used to image the tissue.

Imaging of human tissue can be performed by applying inverse scattering theory to the measured scattered pressure field [1]. A rough approximation of the “soft-obstacle” problem (e.g., tumorous tissue buried in fatty tissue) models the received far-field data as a convolution of the interrogating pulse and the impulse response of the medium. The ideal imaging data would be produced by interrogating the medium with a Dirac-delta function; the quality of the measured data decreases as the effective bandwidth of the interrogating pulse decreases. The physical limitations of transducers limits the pass-band of the interrogating pulse and hence the effective bandwidth of the measured scattered field. Therefore, various deconvolution methods have been employed to extend the bandwidth beyond the passband of pulse emitted by the transducer. In particular, the use of deconvolution algorithms in the quantitative imaging of human tissue could lead to increased resolution in medical ultrasound imaging systems. By first applying one of many deconvolution methods to the measured scattering data, the effective spatial resolution of the reconstruction method (for instance, filtered backpropagation [2] or multiple-frequency time-domain methods [3] [4]) can be increased. Of particular interest is increasing the quality of reconstruction in the case of limited-angle data [5] [6] in synthetic aperture imaging.

Deconvolution methods have found wide application in such fields as speech analysis, seismic and geophysical applications, sonar, as well as biomedical imaging. In all these applications, the following signal model is employed: given some measured data $h[n]$ and possibly some “incident” signal $f[n]$ determine a signal $g[n]$ such that

$$h[n] = f[n] * g[n] + p[n]$$

(1)

where $*$ denotes the operation of convolution and $p[n]$ denotes some additive noise signal. The case where $f[n]$ is unknown is termed blind deconvolution and is generally far more difficult than the case where $f[n]$ is known. Due to the attenuating and dispersive properties of human tissue, the interrogating pulse $f[n]$ is not equal to the pulse emitted by the transducer; therefore, pulse estimation [7] [8] is often necessary for successful deconvolution of real data.
1.2 Ill-Posedness

As will be seen, the deconvolution problem is often \textit{ill-posed} \cite{9}, meaning one of the following criteria fails:

1. The solution \( g[n] \) exists.

2. The solution \( g[n] \) is unique.

3. The solution \( g[n] \) is stable (i.e. \( g[n] \) depends continuously upon the data \( h[n] \)).

In most cases, it is the instability of the deconvolution that prevents the application of a “simple-minded” deconvolution. To illustrate, let \( f[n], g[n] \in l^2 \) (square-integrable sequences). We can apply the well-known convolution theorem \cite{10} and Fourier transform (1) into the frequency domain, taking \( p[n] \equiv 0 \). Denoting \( f[n] \leftrightarrow \hat{f}(\omega), g[n] \leftrightarrow \hat{g}(\omega), \) and \( h[n] \leftrightarrow \hat{h}(\omega) \) yields:

\[
\hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega)
\]  

(2)

Solving for \( \hat{g}(\omega) \) and inverse transforming gives:

\[
g[n] = \mathbf{F}^{-1}\left( \frac{\hat{h}(\omega)}{\hat{f}(\omega)} \right)
\]  

(3)

where \( \mathbf{F}^{-1} \) denotes inverse Fourier transform.

The problem with (3) is that for most signals under consideration, the ratio \( \frac{\hat{h}(\omega)}{\hat{f}(\omega)} \not\to 0 \) as \( \omega \to \infty \), yielding undesired singularities in the recovered \( g[n] \). Our solution is hence unstable and the problem ill-posed. The addition of the noise term \( p[n] \) complicates the problem further. To circumvent this problem, a number of deconvolution algorithms have been devised, which will be the focus of this paper. The following methods, as applied to ultrasound data, will be surveyed: regularized inverse filtering, homomorphic processing, and alternating projections onto convex sets (POCS). Each of these methods provides unique enhancements to the scattering data, along with its own problems (both theoretical and numerical). However, these methods do not exhaust known deconvolution methods; other methods such as wavelet analysis, maximum-likelihood deconvolution, and various adaptive blind deconvolution methods exist and have been successfully applied in a multitude of fields \cite{11} \cite{12} \cite{13}.
2 Regularized Inverse Filtering

2.1 Pulse Estimation Algorithm

Although the incident waveform $f(t)$ is known a priori, in general a more realistic estimate for the pulse is necessary that takes into account the attenuation and dispersion of the medium [14]. Since all data sets employed here had far-field measurements in multiple directions, the pulse could be estimated by averaging the magnitude spectrum over all received directions. That is, the pulse $f(t)$ was estimated in the frequency domain via $\hat{f}(\omega)$ by

$$\hat{f}(\omega) \approx \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{p}_s(\theta, \omega)|^2 d\theta} \quad (4)$$

In the case of limited-aperture data, the integral is evaluated over the aperture interval. Note that our pulse estimate does not require any phase information, and therefore our estimate is unique only up to a phase factor. We therefore make the reasonable assumption that the pulse is minimum-phase, which holds for the class of pulses considered.

Equation (4) can be justified by the following reasoning: denote the pulse, as it propagates through the medium, by $u(t) \leftrightarrow \hat{u}(\omega)$, and let the (ideal) measured impulse response be denoted by $g(\theta, t) \leftrightarrow \hat{g}(\theta, \omega)$. Taking into account an additive Gaussian noise term $n(t) \leftrightarrow \hat{n}(\omega)$, the frequency response of the measured pressure can be expressed as:

$$\hat{p}_s(\theta, \omega) = \hat{u}(\omega)\hat{g}(\theta, \omega) + \hat{n}(\omega) \quad (5)$$

Inserting equation (5) into (4) with the assumptions that $n(t)$ is small with respect to $u(t)$, with a real spectrum yields:

$$\hat{f}(\omega) \approx \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} (|\hat{u}(\omega)\hat{g}(\theta, \omega)|^2 + 2\hat{n}(\omega)\text{Re}(\hat{u}(\omega)\hat{g}(\theta, \omega)))} \, d\theta \quad (6)$$

The frequency response $\hat{g}(\theta, \omega)$ fluctuates randomly in frequency; however, when averaged over many angular measurements, we can make the approximation $\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{g}(\theta, \omega)| d\theta \approx 1$ (up to some multiplicative constant). That is, the averaged frequency response has an all-pass spectrum within the bandwidth considered. Although this approximation is crude and does not account for low-frequency attenuation, for sufficiently random medium with weak scattering characteristics, it should suffice. With these simplifications, we have

$$|\hat{f}(\omega)|^2 \approx |\hat{u}(\omega)|^2 + 2|\hat{n}(\omega)||\hat{u}(\omega)| \quad (7)$$

A simple binomial approximation yields

$$|\hat{f}(\omega)| \approx |\hat{u}(\omega)| \left(1 + \frac{|\hat{n}(\omega)|}{|\hat{u}(\omega)|}\right) \quad (8)$$

so that, for a small noise spectrum, the averaged pulse $f(t)$ has approximately the same magnitude spectrum as the true pulse $u(t)$. Also, we can take into account certain a priori information about the pulse such as effective bandwidth, central frequency, and arrival times. Combined, equation (4) provides a good pulse estimation for the construction of a stable inverse filter. Moreover, it can be shown that the inverse filter constructed with the pulse estimate in (4) is equivalent to the regularized inverse filter of the true pulse $u(t)$. 

3
2.2 The Inverse Filter

From the pulse estimate in equation (4), we can construct an inverse filter via

\[ I(\omega) = \frac{1}{f'(\omega)} \]  \hspace{1cm} (9)

If we assume the noise spectrum to be white, the approximation of the pulse in equation (8) shows that the inverse filter \( I(\omega) \) is equivalent to a regularized inverse filter constructed from the true pulse \( u(t) \). Thus, the filter in equation (9) can be used to execute a stable deconvolution on the data \( p_s(\theta, t) \). A sample inverse filter created from a pulse estimate is shown in figure 2.

2.3 Construction of the Wiener Filter

The pulse estimate achieved by equation (4) can now be used to filter each recorded signal in the frequency domain; however, artifacts will build up outside the effective bandwidth of the pulse where noise predominates. To alleviate the problem, we can construct a Wiener filter that takes into account the corruption of noise. Denoting the noise-to-signal ratio by \( q \), the Wiener filter \( W(\omega) \) is given by:

\[ W(\omega) = \frac{\hat{f}^*(\omega)}{|f(\omega)|^2 + q^2} \]  \hspace{1cm} (10)

From equation (8), we see that the noise-to-signal ratio \( q \approx \frac{n(\omega)}{|u(\omega)|} \) is included in our pulse estimate (4). As stated earlier, the pulse estimation method destroys the phase information, so the Wiener filter implemented is simply:

\[ W(\omega) = \frac{|\hat{f}(\omega)|}{|f(\omega)|^2 + q} \]  \hspace{1cm} (11)

The filter \( W(\omega) \) thus regularizes the ill-posed problem and gives a stable deconvolution. Figure 1 is shows a Wiener filter constructed from the same pulse estimate (using a noise to signal ratio of .01).
Figure 1: A Wiener filter constructed from a sample pulse estimate with $q=0.01$. The filter amplifies frequencies in the lower and upper passband of the pulse (passband 1.5 MHz, centered at 2.5 MHz) while attenuating lower frequencies.
Figure 2: An inverse filter constructed from a sample pulse estimate. The filter dramatically amplifies frequencies below the passband of the pulse (passband 1.5 MHz, centered at 2.5 MHz) and those above the passband.
3 Cepstral Analysis and Homomorphic Filtering

3.1 Definitions and Formalism

A homomorphic system \((K, \ast)\) satisfies a generalized superposition property; that is, there exists some function (a homomorphism) \(\psi : K \rightarrow H\) such that \(\psi(f \ast g) = \psi(f) + \psi(g)\) for all \(f, g \in K\). The most commonly considered homomorphic system is the convolutional system; the homomorphism for this system is:

\[
\psi(f) = F^{-1}[\log(F[f])]
\]  

(12)

Usually, we denote \(\hat{f} = \psi(f)\) as the complex cepstrum. The cepstrum was introduced by Bogert [15] for the problem of echo detection; the word “cepstrum” was chosen via interchanging the letters in “spectrum” to show that the cepstrum lay somewhere between the time-domain and the frequency-domain. By similar logic, Bogert termed the units in the cepstral domain “quefrency” by interchanging the letters in “frequency.” Since most cepstrum calculations are done numerically, the forward and inverse Fourier transforms in (12) are replaced with forward and inverse \(z\)-transforms respectively [16]. Once the signal \(f\) is transformed into the cepstral domain, ideal low-pass or high-pass filtering can be applied to get an estimate for either the incident pulse or impulse response of the system.

3.2 Implementation Problems

Formally, we define the complex cepstrum of a data sequence \(f[n]\) as the inverse \(z\)-transform of the complex logarithm of the \(z\)-transform of the sequence (denoted by \(F(z)\)) via:

\[
\hat{f}(n) = \frac{1}{2\pi i} \oint_C \log(F(z))z^{n-1}dz
\]  

(13)

where the contour of integration lies within a region where the integrand is analytic and single-valued [17]. Moreover, it can be shown that a finite, stable sequence possess a stable cepstrum if and only if both its \(z\)-transform is non-zero on the unit circle and its phase is continuous on the unit circle [18]. Obviously, this restriction raises problems for cepstrum computations with band-limited signals. To circumvent this difficulty, band-limited signals can be exponentially weighted via multiplication by the sequence \(\beta^n\) prior to the computation to the cepstrum.

Another difficulty with cepstrum calculations are phase-wrapping artifacts. If we denote the Fourier transform of \(f[n]\) (i.e. the \(z\)-transform evaluated along the unit circle) in terms of its phase and magnitude:

\[
F(e^{j\omega}) = |F(e^{j\omega})|e^{i\phi(\omega)}
\]  

(14)

where \(\phi(\omega)\) denotes the wrapped phase. The log spectrum is thus given by:

\[
\hat{F}(e^{j\omega}) = \log|F(e^{j\omega})| + i\phi(\omega)
\]  

(15)

Since the \(z\)-transform must be analytic on the unit circle, it follows that the the function \(\phi(\omega)\) must be continuous. However, digital phase computations are performed modulo \(2\pi\),...
so the computed phase function $\phi_w(\omega)$ must be unwrapped in order to get an accurate cepstrum calculation. The problem of phase unwrapping is non-trivial and, in the presence of noise, ill-posed [19]. The phase unwrapping problem has motivated several novel methods to calculate cepstra, such as the real cepstrum and complex cepstrum, which are discussed below.

### 3.3 Real Cepstrum and Generalized Cepstrum

#### 3.3.1 Real Cepstrum

The real cepstrum of a sequence $f[n]$ is simply the inverse $z$-transform of the logarithm of the modulus of the $z$-transform:

$$\hat{f}(n) = \frac{1}{2\pi i} \oint_C \operatorname{Log}|F(z)|z^{-n-1}dz$$

which drastically simplifies cepstral calculations by removing phase-unwrapping problems. However, all phase information of the signal is lost in a real cepstrum calculation, making the operation non-invertible in all cases where the signal is not minimum phase. Therefore, the real cepstrum is only applicable to a homomorphic filtering scheme in the case where the input is minimum phase (which, in the case of ultrasound data, cannot be guaranteed a priori).

#### 3.3.2 Generalized Cepstrum

The generalized cepstrum [20] generalizes the complex cepstrum by using the mapping $s_\gamma(w)$ defined by

$$s_\gamma(w) = \frac{1}{\gamma} (w^\gamma - 1)$$

instead of the complex logarithm. Since $s_\gamma(w) \to \operatorname{Log}(w)$ as $\gamma \to 0$, the generalized cepstrum approaches the complex cepstrum for small $\gamma$. Similar to the inverse cepstrum, we can define an inverse mapping $s_\gamma^{-1}$ via:

$$s_\gamma^{-1}(w) = (1 + \gamma w)^{\frac{1}{\gamma}}$$

The generalized cepstrum effectively “smoothes” the data in the cepstral domain by adding a convolved term. That is, the generalized cepstrum of a convolution becomes

$$\bar{f} * g = \tilde{f} + \tilde{g} + \gamma \tilde{f} * \tilde{g}$$

thus reducing artifacts caused by cepstral filtering.

### 3.4 An Analytic Calculation

#### 3.4.1 Motivation and Definition

To give some insight into cepstral analysis, we present a simple “pencil and paper” deconvolution in the non-discrete case. While, in practice cepstral calculations are always performed
digitally, we attempt here to define a cepstral operator for real and complex-valued functions (in general, for generalized functions or distributions). For clarity, we summarize here some basic definitions and facts from Fourier Analysis. For functions \( f(t) \in S \) (good functions), more generally, \( f(t) \in L^2 \) (square-integrable functions) we can define the forward and inverse Fourier transforms as follows:

\[
\hat{f}(\omega) = \mathbf{F}[f(t)] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} \, dt
\]

(20)

\[
f(t) = \mathbf{F}^{-1}[\hat{f}(\omega)] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} \, d\omega
\]

(21)

In particular, it can be shown that the operators \( F : S \rightarrow S \) and \( F : L^2 \rightarrow L^2 \) are bijective (along with their inverses) [21]. However, our calculations will often involve transforming functions which are not good (indeed, unbounded). Therefore, we need to generalize our space of functions to \( S' \), or the space of tempered distributions [10]. On the space \( S' \) we can define the generalized Fourier transform (and its associated inverse) \( \mathbf{F}' : S' \rightarrow S' \) and \( \mathbf{F}'^{-1} : S' \rightarrow S' \) defined as the adjoint of \( f(t) \in S' \). That is, the generalized transform \( \hat{f}(\omega) \) is defined by

\[
< \hat{f}, \psi > = < f, \hat{\psi} >
\]

(22)

for any function \( \psi \in S \). Of course, the generalized Fourier transform of a good function is given by the classical formulas (20) and (21). Using this machinery, we can define a cepstral operator \( C : S' \rightarrow S' \) defined by

\[
\tilde{f}(t) = C[f(t)] = \mathbf{F}'^{-1}[	ext{Log}(\mathbf{F}'[f(t)])]
\]

(23)

where \( \text{Log}(z) \) denotes the complex logarithm. Note that a necessary condition for the existence of \( \tilde{f}(t) \) is \( \hat{f}(\omega) \neq 0 \) for all \( \omega \). Likewise, we can define the inverse cepstral operator via

\[
f(t) = C^{-1}[\tilde{f}(t)] = \mathbf{F}'^{-1}[\exp(\mathbf{F}'[\tilde{f}(t)])]
\]

(24)

Now let \( f(t) \) and \( g(t) \) be of class \( S' \) such that \( \tilde{f}(t) \) and \( \tilde{g}(t) \) exist. If we let \( h(t) := f(t) * g(t) \), then it follows (using the convolution theorem and the properties of the logarithm) that

\[
\tilde{h}(t) = \tilde{f}(t) + \tilde{g}(t)
\]

(25)

Thus, the cepstral operator transforms convolution in the time-domain into addition in the cepstral domain. The function \( \text{Log}(z) \) in equation (23) homomorphically maps a product of Fourier transforms into a superposition of log spectra. We will use this generalized superposition of the cepstral operator to separate two signals convolved in the time-domain with an appropriate filter in the cepstral domain.

### 3.4.2 Sample Calculation

Consider an incident pulse modeled by the function

\[
f(t) = e^{-t^2/(2\sigma^2)}
\]

(26)
and an impulse response of some medium given by an impulse train with $N + 1$ terms

$$g(t) = \sum_{j=0}^{N} a_j \delta(t - t_j)$$

(27)

where we take $a_{j+1} < a_j \forall j$. Then the scattered field $h(t) = f(t) \ast g(t)$, or

$$h(t) := \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)d\tau$$

(28)

is given by

$$h(t) = \sum_{j=0}^{N} a_j e^{-(t-t_j)^2/(2\sigma^2)}$$

(29)

Our task now is, given equation (29), find the incident pulse $f(t)$ and/or the impulse response $g(t)$. In many cases, we may have a priori knowledge of either $f(t)$ or $g(t)$. For instance, we may know $f(t)$ to have a Gaussian envelope (with known bandwidth parameter $\sigma$) or $g(t)$ to have compact support. In the ultrasound model, we can safely assume the interrogating pulse $f(t)$ to be smooth (n-times continuously differentiable) while $g(t)$ is random-like and hence non-differentiable. In this calculation, we take $f(t) \in S$ and $g(t)$ to have a sinusoidal spectrum.

This information allows us to choose an appropriate filter in the cepstral domain. Just as the Fourier transform indicates frequency content of a function, the support of the cepstrum of a function measures the smoothness of the function. In fact, for the subset of good functions which have a polynomial log spectrum, the cepstrum will have support only at the origin. This fact follows from the Fourier pair $\sum_{j=0}^{N} a_j e^{j} \leftrightarrow \sum_{i=0}^{N} a_j \delta^{(i)}$.

We can easily calculate the Fourier transform of (29), given by

$$\hat{h}(\omega) = \frac{\sigma}{\sqrt{2\pi}} e^{-\sigma^2 \omega^2/2} \sum_{j=0}^{N} a_j e^{i\omega j}$$

(30)

which can also be obtained via transforming (26) and (27) and applying the convolution theorem. Note that $\hat{f}(\omega)$ is non-zero for all real $\omega$, allowing the complex logarithm to be taken. After some manipulation, we find the complex cepstrum to be

$$\tilde{h}(t) = (\log(\frac{\sigma}{\sqrt{2\pi}} + a_0))\delta(t) + \frac{\sigma^2 \sqrt{2\pi} \delta''(t)}{2} + \phi(t)$$

(31)

where

$$\phi(t) = \tilde{g}(t)/a_0$$

(32)

Now $\phi(t)$ is a series of impulses, with no support at the origin. Taking into account that $\text{supp} \tilde{f}(t) = \{0\}$, we can filter $\tilde{h}(t)$ with the function

$$\tilde{h}_c(t) = h(t) \text{ if } t \neq 0$$

$$= 0 \text{ if } t = 0$$

10
giving  
\[ \tilde{h}_c(t) = \phi(t) \]  
(33)

Inversion of \( \tilde{h}_c(t) \) is straightforward via application of (24), yielding the final result  
\[ h_c(t) = \sum_{j=0}^{N} (a_j)/(a_0) \delta(t - t_j) \]  
(34)

The above calculation can be specialized to an impulse response with only two spikes, providing insight into the cepstral filtering. The following calculations result (taking \( t_0 = 0 \) and \( a_1 < a_0 \)):

\[ h(t) = a_0 e^{-t^2/(2\sigma^2)} + a_1 e^{-(t-t_1)^2/(2\sigma^2)} \]
\[ \hat{h}(\omega) = \frac{\sigma}{\sqrt{2\pi}} e^{-\sigma^2 \omega^2/2} (a_0 + a_1 e^{i\omega t_1}) \]
\[ \tilde{h}(t) = (\log(\frac{\sigma}{\sqrt{2\pi}} + a_0)) \delta(t) + \frac{\sigma^2 \sqrt{2\pi} e^{-\sigma^2 \omega^2/2}}{2} \delta^\prime(t) + \phi(t) \]
\[ \phi(t) = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} (a_1/a_0)^k \delta(t - t_1 k) \]

Application of the filter \( \tilde{h}_c(t) \) yields \( \phi(t) \), which clearly has no support at the origin. This can be inverted back into the time domain to give  
\[ h_c(t) = \delta(t) + (a_1/a_0) \delta(t - t_1) \]  
(35)

Note that the recovered impulse response \( h_c(t) \) differs from the original by a multiplicative constant \( a_0 \); that is, we have lost the “DC-component” of the impulse response during the filtering process. Otherwise, this calculation completely separates the incident pulse from the response using the assumption that the pulse has limited support in the cepstral domain.

Some comments about this calculation include:

1. Can we find sufficient conditions for any \( f \in S' \) to have a cepstrum \( \tilde{f} \in S'' \)? Can we extend the sufficient condition for finite \( l^2 \) sequences for function in \( S \), or more generally, \( S'' \)?

2. Since our data must be processed in a discrete fashion (and will invariably be contaminated by noise), the notion of “smoothness” becomes problematic. A more accurate model for the incident pulse defined by (26) would include a sinusoidal varying term; if the frequency of this term is high, the sampled version of \( f \) would lead to a non-smooth pulse and our deconvolution scheme would break down.
4 Projections onto Convex Sets

4.1 Background and the Fundamental Theorem

Projections onto Convex Sets provide a powerful framework for the construction of signal restoration and extrapolation algorithms. One of the first POCS algorithms was the Papoulis-Gershberg algorithm [22] used to extrapolate a known-segment of a band-limited signal. Later, Youla and others [23][24][25] extended various isolated algorithms into a general mathematical theory. All POCS methods utilize various a priori information about the input and desired output, allowing a wide variety of implementations.

The general theory of vector space projections requires some basic knowledge of functional analysis; a good introduction to the basics can be found in [26] and [27]. For conciseness, we summarize some of the basic definitions and results.

We will work within the Hilbert space of square-integrable functions $L^2$, which physically denotes the space of finite-energy signals. The $L^2$ norm is defined by

$$
\|f(x)\|_2 := \sqrt{\int_{-\infty}^{+\infty} |f(x)|^2 dx}
$$

The $L^2$ norm defines a functional mapping a space of functions into a scalar field. Many POCS problems can be formulated by considering the minimization of a functional. In particular, the deconvolution problem modeled by (1) can be posed by minimization of the functional $J : L^2 \rightarrow \mathbb{R}_+$ where

$$
J := \|h[n] - u[n] * v[n]\|_2^2
$$

where $h[n]$ is some known data. Minimization of $J$ entails finding the optimal pairs of functions $u[n]$ and $v[n]$. Obviously, as it stands, the problem as posed will have an infinite number of solutions. To restrict the solution set, we can define constraint sets $C_f$ and $C_q$ that $u$ and $v$ must satisfy. Associated with each constraint set, we can define an orthogonal projector that maps functions onto the constraint set. The following result tells us what types of constraint sets are admissible in a POCS algorithm.

Let $H$ be a Hilbert space and let $C_i$ with $1 \leq i \leq m$ be closed convex subsets of $H$ and let $C_0 = \bigcap_{i=1}^{m} C_i$ denote the finite intersection of these sets. Now define $P_i : H \rightarrow C_i$ be the projector onto $C_i$. Define the composition projector $P = P_m P_{m-1} \cdots P_1$ Then the following holds:

**Theorem 4.1** Fundamental Theorem of POCS Let $C_0$ be non-empty. Then for all $x \in H$, the sequence $P^n x$ converges weakly to a point $C_0$.

Thus theorem (4.1) assures the convergence of any POCS algorithm with well-defined constraint sets; however, unless $C_0$ is a singleton set, the algorithm will not automatically converge to a unique solution. Also, the rate of convergence may be exceedingly slow. To accelerate the POCS algorithm, several researchers [28][29] have refined POCS with the introduction of relaxed projectors and other such improvements.

To devise a POCS algorithm for the ultrasound model, we incorporate both the convolutional model and certain a priori information about the pulse $f[n]$ and the impulse
response \( g[n] \), which we use to construct the constraint sets \( C_f \) and \( C_g \). For acoustical data, more frequency-domain constraints are available, so the functional \( J \) in equation (37) can be Fourier transformed via Parseval’s identity, yielding:

\[
J = \frac{1}{2\pi} \| \hat{h}(\omega) - \hat{u}(\omega) \hat{v}(\omega) \|_2^2
\]  

(38)

where \( \hat{u}(\omega) \in C_f \) and \( \hat{v}(\omega) \in C_g \). Once the constraint sets are found (and there associated projectors), a POCS algorithm can be implemented which minimizes \( J \) in equation (38).

### 4.2 Constraints and Implementation

In order to achieve a stable deconvolution, we need to know maximum \textit{a priori} information about the data at hand. Far-field acoustic data can be especially problematic since it does not have the most commonly considered properties (\textit{e.g.}, finite duration, strict band-limitation, non-negativity, etc.). Therefore, we propose several novel constraint sets that could be useful in a POCS algorithm.

#### 4.2.1 Angle-Dependent Band-Limitation

Under the Born Approximation, the far-field frequency response of a 2D scatterer with a spatially-varying sound-speed contrast \( \gamma(r) \) is given by [30]

\[
\hat{p}_s(\alpha, \theta, \omega) = \frac{2\pi^2}{R} e^{ikR} k^2 \Gamma(k|\alpha - \theta|)
\]  

(39)

where \( R \) is the far-field measurement radius and \( \Gamma(k) \) is the spatial Fourier transform of the contrast \( \gamma(r) \). Since the object being imaged is of compact support (with radius \( a \)), the contrast function \( \gamma(r) \) will have an analytic Fourier transform. However, we can use an uncertainty principle to estimate the effective “\( k \)-width” \( \sigma_k \) of \( \Gamma(k) \). That is:

\[
\sigma_k a \sim 1
\]  

(40)

Therefore, \( \Gamma(k) \) is effectively zero for spatial frequencies greater that \( \sigma_k \). Equating this information with the argument in equation (39), we have

\[
k_c|\alpha - \theta| \sim 1
\]  

(41)

where \( k_c \) is the spatial frequency cutoff. From this, we can easily deduce an angle-dependent frequency cutoff given by

\[
f_c \approx \frac{c}{\pi a} \csc \left( \frac{\theta - \alpha}{2} \right)
\]  

(42)

From this cutoff estimate, we see that the forward-scatter case gives a broadband response, while side and back scattering is band-limited. Equation (42) thus implicitly defines a band-limited set, which is known to be both closed and convex.
4.2.2 Low Bandwidth Envelope

Another possible frequency-domain constraint that could be applied is an upper bound on the spectrum. In general, the wideband response of a scatterer with contrast $\gamma(r)$ and support $\Omega$ is given by

$$p_s(\alpha, \theta, \omega) = k^2 \hat{f}(\omega) \int_{\Omega} G_0(R\theta - r_0, \omega) \gamma(r) p(r, \alpha, \omega) d\Omega_0$$  \hspace{1cm} (43)

where $G_0(\omega, r)$ is the frequency domain Green’s function for the reduced wave equation (Helmholtz equation). By applying the Born approximation and the far-field approximation for a cylinder of radius $a$, equation (43) becomes:

$$\hat{p}_s(\theta, \alpha, \omega) = \frac{i}{4} k^2 \hat{f}(\omega) \sqrt{\frac{2}{\pi k R}} \frac{e^{ikR \cos(\theta - \phi)}}{e^{-ikR}} \int_0^{2\pi} \int_0^a \gamma(r, \phi) e^{-ikr \cos(\theta - \phi)} rdrd\phi$$  \hspace{1cm} (44)

Taking the modulus of both sides of (44) and approximating the integral with a n upper bound $\Gamma = \max \gamma(r, \phi)$ yields the inequality

$$|p_s(\omega)| \leq \frac{1}{4} k^2 \hat{f}(\omega) \sqrt{\frac{2}{\pi k R}} a^2 \Gamma$$  \hspace{1cm} (45)

which may be simplified to give

$$|p_s(\omega)| \leq K \omega^{3/2} |\hat{f}(\omega)|$$  \hspace{1cm} (46)

where the constant $K$ is given by

$$K = \frac{1}{4} \sqrt{\frac{2\pi}{Rc_0}} \Gamma a^2$$  \hspace{1cm} (47)

Equation (46) yields some a priori information about the scattered wave-field. Consider the ideal case where the incident pulse $f(t) = \delta(t)$; then $|\hat{f}(\omega)| = 1$, implying that the frequency response $p_s$ is bounded by an envelope proportional to $\omega^{3/2}$. This bound allows us to enforce a frequency-domain constraint in the case of low-bandwidth extrapolation ($\omega \ll 2\pi f_0$), where $f_0$ is the central frequency of the incident pulse employed. Also, equation (46) coupled with Parseval’s identity allows us to find a bound on the energy of the signal.

4.2.3 Other A Priori Constraints

Since the basic characteristics of the pulse (e.g., effective bandwidth, central frequency, linear phase, etc.) are known a priori, constraint sets could be defined which reflect these parameters. For instance, a Gaussian envelope in the frequency domain could be used to bound the spectrum. Other possibilities include energy constraints (in either the time or the frequency domain) or angle-dependent amplitude limitation.
5 Computational Techniques and Numerical Results

5.1 Description of Data and Reconstruction Algorithm

We tested the deconvolution algorithms on two synthetic far-field time-domain scattering data sets computed via $k$-space method [31]. The first data set was generated from a cylinder with random internal structure with added white, Gaussian noise with an SNR value of 38.7 dB. The second data set was generated from a simulated breast model with sound speed contrast mimicking fat and tissue. No artificial noise was added to the breast data. The synthetic data was computed using $N_{\alpha}$ transmit angles and $N_{\theta}$ receive angles. The data sets are summarized in tables 1 and 2. Both data sets used the time-domain waveform

$$f(t) = \cos(2\pi f_0 t) e^{-t^2/(2\sigma^2)}$$

where $f_0$ is the central frequency and $\sigma$ is the temporal Gaussian parameter.

After the deconvolution algorithm was applied, a the time-domain reconstruction algorithm outlined in [3] reconstructed the scatterers. The scattering configuration is shown in figure 3. The far-field data $p_s(\alpha, \theta, t)$ was used to reconstruct the sound-speed contrast function $\gamma(r)$ via the formula

$$\gamma_M(r) = \frac{1}{N} \int \int \Phi(\alpha, \theta) \left( p_s(\alpha, \theta, \tau) + iH^{-1}[p_s(\alpha, \theta, \tau)] \right) d\alpha d\omega$$

where the delay term $\tau$ is given by

$$\tau = \frac{R}{c_0} + \frac{(\alpha - \theta) \cdot r}{c_0}$$

the coefficient $N$ is determined by a frequency weight incorporating the bandwidth of the incident pulse $f(t)$, $H^{-1}$ is the inverse Hilbert transform, and $\Phi(\alpha, \theta)$ is a “filter” dependent upon the scattering geometry. In the case of 2D scattering, the filter is given by:

$$\Phi(\alpha, \theta) = |\sin(\theta - \alpha)|$$

The method embodied in equation (49) synthetically delays and sums all transmit angles $\alpha$ and receive angles $\theta$ to focus an image at position $r$, using all available bandwidth of the incident pulse. Of special interest in deconvolving ultrasound data is its performance with limited-aperture data; that is, when only a limited number of transmit and receive angles are available. Therefore, we performed reconstructions at aperture width $\phi_{ap}$ where the integral in equation (49) was evaluated over

$$|\alpha| \leq \phi_{ap}/2 \text{ and } |\theta - \pi| \leq \phi_{ap}/2$$

For brevity, we call aperture widths $\phi_{ap} = 2\pi$ full, $\phi_{ap} = \pi$ half, and $\phi_{ap} = \pi/2$ quarter in the descriptions of the reconstructions.

For reference, figures 5 and 6 show the raw far-field data reconstructed with no deconvolution, while figure 4 shows the actual (non-imaged) models.
Figure 3: Illustration of the scattering configuration employed. An incident pulse $f(t-\alpha \cdot r/c)$ is scattered by some medium and the time-domain scattered pressure $p_s(\alpha, \theta, t)$ is measured in the far-field at radius $R$ and angle $\theta$.

<table>
<thead>
<tr>
<th></th>
<th>Central Frequency $f_0$</th>
<th>2.5 MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Effective Bandwidth</td>
<td>1.5 MHz</td>
</tr>
<tr>
<td>Gaussian Parameter $\sigma$</td>
<td>.25 $\mu$s</td>
<td></td>
</tr>
<tr>
<td>Sampling Frequency</td>
<td>9.144 MHz</td>
<td></td>
</tr>
<tr>
<td>Number of Samples</td>
<td>84</td>
<td></td>
</tr>
<tr>
<td>Number of Receive Angles</td>
<td>256</td>
<td></td>
</tr>
<tr>
<td>Number of Send Angles</td>
<td>64</td>
<td></td>
</tr>
<tr>
<td>Radius of Cylinder</td>
<td>3.0 mm</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Summary for synthetic cylinder with random internal structure data.

<table>
<thead>
<tr>
<th></th>
<th>Central Frequency</th>
<th>2.5 MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Effective Bandwidth</td>
<td>1.5 MHz</td>
</tr>
<tr>
<td>Gaussian Parameter $\sigma$</td>
<td>.25 $\mu$s</td>
<td></td>
</tr>
<tr>
<td>Sampling Frequency</td>
<td>9.153 MHz</td>
<td></td>
</tr>
<tr>
<td>Number of Samples</td>
<td>256</td>
<td></td>
</tr>
<tr>
<td>Number of Receive Angles</td>
<td>512</td>
<td></td>
</tr>
<tr>
<td>Number of Send Angles</td>
<td>128</td>
<td></td>
</tr>
<tr>
<td>Radius of Breast</td>
<td>8.5 mm</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Summary of simulated breast data.
Figure 4: The actual models: cylinder with random internal structure and the simulated cross-section of a breast.

Figure 5: Full, half, and quarter aperture reconstructions of cylinder data.

Figure 6: Full, half, and quarter aperture reconstructions of synthetic breast data.
Figure 7: Full, half, and quarter aperture reconstructions of cylinder data deconvolved using an inverse filter from a pulse estimate.

Figure 8: Full, half, and quarter aperture reconstructions of synthetic breast data deconvolved using an inverse filter from a pulse estimate.

5.2 Inverse Filtering Results

The inverse filtering algorithms are both the simplest to implement and the most widely used. We used both simple inverse filtering and a Wiener filter (i.e. regularized inverse filtering) constructed from a pulse estimation using equation (4). For the regularized algorithm, we use the parameter $q = .01$. Figures 7 and 8 show the inverse filtered cylinder and breast data (respectively) and figures 9 and 10 show Wiener filtered cylinder and breast data.

5.3 Homomorphic Filtering Results

As discussed above, homomorphic filtering is implemented in three steps:

1. The data (perhaps pre-processed via exponential weighing or the application of some window) is transformed into the cepstral domain via one of the algorithms discussed above.
Figure 9: Full, half, and quarter aperture reconstructions of cylinder data deconvolved using a Wiener filter created from a pulse estimate.

Figure 10: Full, half, and quarter aperture reconstructions of synthetic breast data deconvolved using a Wiener filter created from a pulse estimate.
Figure 11: Full, half, and quarter aperture reconstructions of cylinder data using a one-stage cepstral deconvolution.

Figure 12: Full, half, and quarter aperture reconstructions of simulated breast data using a one-stage cepstral deconvolution.

2. The cepstrum is either low-pass filtered (in order to recover the pulse) or high-pass filtered (in order to recover the impulse response).

3. The filtered cepstrum is inverted back into the time domain.

Step 2 requires an estimate for a cepstral cutoff $N_c$; that is, we must identify the quefrency at which to apply the filter. To recover the pulse, for instance, we need an estimate of the quefrency band which its cepstrum occupies.

### 5.3.1 One-Stage Deconvolution

Here we present a simple 1-stage deconvolution scheme based on long-pass filtering of the complex and generalized cepstrum. The data was exponentially weighted by a factor $\beta = .975$ and high-pass filtered above $N_c = 7$ for the cylindrical speckle data and $N_c = 21$ for the breast data. Instead of a hard cut in the cepstral domain, a cosine taper of width 4 was applied. Figures 11 and 12 show the deconvolved reconstructions.
Table 3: Parameters used in two-stage homomorphic deconvolution using the complex cepstrum.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cepstral Cutoff (speckle)</td>
<td>$N_c$ 7</td>
</tr>
<tr>
<td>Cepstral Cutoff (breast)</td>
<td>$N_c$ 21</td>
</tr>
<tr>
<td>Exp. Weighting Factor</td>
<td>$\beta$ .975</td>
</tr>
<tr>
<td>Noise-to-Signal Ratio</td>
<td>$q$ .001</td>
</tr>
</tbody>
</table>

Figure 13: Full, half, and quarter aperture reconstructions of cylinder data deconvolved using a two-stage homomorphic deconvolution.

5.3.2 Two-Stage Deconvolution

Here we present a 2-stage deconvolution method based on the complex and generalized cepstrum. First the cepstrum of the data is calculated, then short-pass filtering is performed to recover a pulse estimate. The short-passed cepstrum is then averaged over all receive angles to get an improved pulse estimate. This final pulse estimate is used to construct a Wiener filter. In the calculation of the complex cepstrum, exponential weighting with parameter $\beta$ was used to reduce phase wrapping. The exponential weighting changed the signal-to-noise ratio, so the parameter $q$ in the Wiener filter had to be adjusted. This method was repeated using the generalized cepstrum. Tables 3 and 4 show the parameters chosen. The reconstructions using a complex cepstrum implementation are shown in figures 13 and 14 while those using the generalized cepstrum are shown in figures 15 and 16.

Table 4: Parameters used in two-stage homomorphic deconvolution using the generalized cepstrum.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cepstral Cutoff (speckle)</td>
<td>$N_c$ 7</td>
</tr>
<tr>
<td>Cepstral Cutoff (breast)</td>
<td>$N_c$ 21</td>
</tr>
<tr>
<td>Cepstrum Exponent</td>
<td>$\gamma$ .125</td>
</tr>
<tr>
<td>Noise-to-Signal Ratio</td>
<td>$q$ .001</td>
</tr>
</tbody>
</table>
Figure 14: Full, half, and quarter aperture reconstructions of synthetic breast data deconvolved using a two-stage homomorphic deconvolution.

Figure 15: Full, half, and quarter aperture reconstructions of cylinder data deconvolved using a two-stage homomorphic deconvolution with the generalized cepstrum.

Figure 16: Full, half, and quarter aperture reconstructions of synthetic breast data deconvolved using a two-stage homomorphic deconvolution with the generalized cepstrum.
6 Conclusions and Remarks

6.1 Comparison of Methods

Inverse and Wiener filtering provided good results which can serve as a benchmark with which to compare other methods. For full aperture reconstruction, the Wiener filtered data (figures 9 and 10) provided a clean deconvolution, revealing small amounts of detail not present in the raw reconstruction. However, for half and quarter aperture sizes, the reconstructions became contaminated by lateral blurring and noisy artifacts. Although the unregularized inverse filtered data (figures 7 and 8) lost spatial resolution as the aperture size decreased, the artifact build-up was not as severe as in the case of Wiener filtering. Minimal internal features were still distinguishable even at quarter aperture. In fact, the inverse filtered reconstruction provided the best limited-aperture reconstruction, giving the most information about the imaged object. Perhaps the increased resolution at low aperture width can be attributed to the low frequency amplification provided by the inverse filter (figure 2), which is not present in the Wiener filter (figure 1).

The one-stage cepstral deconvolution (figures 11 and 12) attempted to separate the cepstrum of the impulse response from that of the pulse via high-pass filtering (lifting) in the cepstral domain. Most internal detail was lost at limited-aperture, although the boundaries remained distinct. From the results we achieved, high-pass filtering of the cepstrum failed as a viable deconvolution method.

From the numerical results presented above, the two-stage cepstral deconvolution (figures 13 and 14) does not give a degraded reconstruction at full aperture (unlike the other methods). It is similar in quality to the raw reconstruction. However, in the limited aperture reconstructions, broadband noise and severe lateral blurring builds up, making most internal features indistinguishable. As the aperture shrinks to \( \pi/2 \) radians, all internal features are lost with only the boundary remaining distinct. The effect of using both the complex and generalized cepstra in two-stage homomorphic filtering makes only a minimal difference in reconstruction quality. More detail can be found on competing cepstral realizations in [14].

The cepstral methods suffer from a need to estimate parameters such as minimum phase cutoff \( N_c \) and exponential weighting parameter \( \beta \). It seems, however, that these parameters are dependent only upon the scattering configuration and the pulse waveform employed, thus independent of the medium being investigated.

Of the methods presented above, we must conclude that inverse filtering is the most viable deconvolution method. None of the methods provided significant improvement over no deconvolution at full aperture, yet inverse filtering preserved useful detail at half and quarter aperture, whereas all other methods failed. We attribute this to the pulse estimation algorithm, which incorporates both noise regularization and amplification at low frequencies. The Wiener filter apparently does not provide enough low frequency amplification to usable limited-aperture reconstructions. Likewise, the homomorphic filtering schemes do not give a strong enough deconvolution to provide a useful limited-aperture reconstruction.
6.2 Problems and Further Work

As we have seen, the acoustic deconvolution problem is highly non-trivial and there is much work to be done before any one algorithm can be used in a clinical installation. Inverse and Wiener filtering provide good results while having a low computational cost and simple implementation. The easiest way to improve an inverse filtering scheme is provide a better pulse and noise-to-signal estimate. The accuracy of the inverse filtering scheme at limited aperture provides evidence for the pulse estimation algorithm embodies in equation (4). Perhaps advances in inverse filtering could therefore start with a more precise estimation.

In the case of homomorphic deconvolution, the problem of phase unwrapping and the computational realization of the cepstrum are persistent. Phase unwrapping is highly sensitive to noisy data and poses a severe problem to cepstrum calculations. Also, estimation of the parameters used in homomorphic filtering is largely ad hoc; although estimates from the rise-time of the log spectrum [7] and spatial extent of the pulse [14] have been proposed, there is presently no accurate way to estimate the cepstral cutoff $N_c$ short of trial and error. Also, both the generalized and continuous-time cepstrum are not well-understood.

POCS algorithms have been widely applied in such fields as optics, spectral analysis, and digital communications, but their introduction into acoustic signal processing has been slow. Perhaps this is due to the lack of a priori information available to construct constraint sets; although the analytical work done on acoustic scattering is enormous, the mathematical complexities involved do not lend themselves easily to simple constraints. Other difficulties with POCS include the large number iterations often necessary for convergence, which could limit real-time implementation.

In summary, the deconvolution of ultrasound data is not fully understood and much work is needed for an algorithm that gives good results from limited data. From our research, it appears that work can be done in two directions: designing better inverse filters and finding a viable POCS implementation. Both directions hope to provide exciting areas of research.
7 Acknowledgements

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References


