A Multiple Order Conformability Model for Uniform Cross-Section Piston Rings

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ABSTRACT

This paper examines the conformability of elastic piston rings to a distorted cylinder bore. Several bounds are available in the literature to help estimate the maximum allowable Fourier coefficient in a Fourier expansion of bore distortion: the analytically derived bounds in [7] and [8], and the semi-empirically derived bounds discussed in [9]. The underlying assumptions for each set of analytic bounds are examined and a multiple order algorithm is derived. The proposed algorithm takes account of multiple orders of distortion at once. It is tested with finite element (FE) data and compared to the classical bound approach. The results indicate that the bounds in [7] are compatible with linear elasticity theory (LET), whereas the bounds in [8] are not. Furthermore, numerical evidence indicates that the present multiple order algorithm can predict seal breaches more accurately than either of the other analytic bounds.

INTRODUCTION

Compression piston rings in internal combustion engines are designed to minimize gas and oil leakage out of compression chambers during engine operation. Rings must seal at the ring-bore interface as well as at the ring-piston-groove interface. Ideally they should maintain this seal even during the normal pressure and temperature variation that occurs in a functioning engine. Robust finite element based tools [1, 2] exist for comprehensively addressing these and other ring design and performance issues. Nevertheless, there is a need for the comparatively simple bore-ring seal assessment algorithm proposed here.

When a block is subjected to operating loads, the piston bores of the block will deform. Any significant changes to the bore wall thickness, bore length, water jacket length, head bolt length and/or bore spacing. To assess a proposed design change, a finite element model assumes a standardized set of loads and computes expected deflection of the bore walls within the block. Once the projected distortion pattern is known, Algorithm 1 below provides a relatively simple screening procedure for assessing ring-bore seal. A design that passes initial screening by Algorithm 1 for ring-bore seal may warrant further, more computationally expensive and detailed investigations using finite element based tools such as those described in [2].

Since the piston ring acts like an elastic beam [3], the ring should successfully conform to a slightly distorted bore. However, if these distortions become too large, the ring may be unable to conform and may lose contact with the cylinder wall and piston. In an actual ring/cylinder assembly, mechanical, thermal and gas forces are acting simultaneously; to simplify the analysis, this paper considers a stationary ring in thermal equilibrium, neglecting the complicated dynamics in a real engine.

The contact problem considered in this paper can be summarized as follows: given distortions in the cylinder bore and a set of piston ring specifications, determine if the ring forms a seal with the bore. (Ring-piston-groove seals are not considered here.) In the following analysis, a positive pressure value at each point between cylinder and piston indicates full contact, or a "light-tight" seal. A negative pressure value at any point, while never physically realizable, indicates a tendency for the ring to pull away from the bore, and hence indicates a breach, or loss of contact. Therefore, the presence or absence of a "light-tight" seal can be determined by calculating the pressure distribution at each point between ring and bore. If the pressure becomes negative, a breach occurs. A seal breach may result in increased oil consumption and leakage, i.e., blowby, and engine wear; also, fuel efficiency may decrease [4, 5].
Beginning with the work of Gintsburg [6], many researchers have analyzed bore deformations by approximating distortions from the nominal radius \( r_0 \) with a Fourier series. Letting \( \xi(\phi) \) denote the measured bore cross section (where \( \phi \) is the polar angle), \( \xi(\phi) \) is approximated by

\[
\xi(\phi) = r + \sum_{k=1}^{N} A_k \cos(k(\phi + \delta_k)), \quad (1)
\]

where \( k \) is the order, \( \delta_k \) is a phase angle, and \( A_k \geq 0 \).

Reference [7] indicates a different set of bounds for Fourier coefficients \( A_k \), \( k \geq 2 \), than does [8]. (\( A_1 \) and \( \delta_1 \) indicate rigid body motion of the cross-section’s centroid, which is not important to conformability analysis). Both papers show that Fourier coefficients larger than the threshold quantities will likely lead to seal breach.

Although the bounds discussed in [7] and [8] agree for elliptical distortions (\( k = 2 \)), they deviate significantly at higher orders of distortion. Which set of bounds better assess the likelihood of a seal breach? We address this question by assessing compatibility of the two sets of bounds with implications of linear elasticity theory (LET).

In addition, we propose a simple numerical technique that incorporates multiple orders of distortion in the conformability analysis.

Before embarking on a discussion of existing bounds for conformability, the authors wish to recognize that a private correspondence with an anonymous reviewer of this paper suggests that the bounds found in [8] were motivated by unpublished empirical conformability studies. This reviewer indicates that the bounds in [8] have been successfully applied to predict conformability of piston rings of a certain type of diesel engine. The authors wish to acknowledge that this may well be the case. Furthermore, the authors acknowledge that the critique offered in this paper is made solely based on analytical conformity with LET and the assumptions indicated below; unlike the bounds discussed in [9], empirical measurements were not considered in the present study.

DERIVATION AND COMPARISON OF BOUNDS

Consider a curved, prismatic member (or piston ring) with height \( h \) and thickness \( t \). For the present analysis, assume a constant and rectangular cross-section \( A = ht \). Assuming the ring is nearly circular, impose a polar coordinate system \((\phi, r)\) about the center of the ring. The split ring has homogenous density with modulus of elasticity \( E \). When the ring is compressed, this gap is reduced to a small space to allow sealing. (The gap is assumed infinitesimally small for our derivations.) Figure 1 displays the geometry.

In the following derivation, the pressure distribution \( p(\phi) \) between ring and bore is expressed in terms of curvature. Both [7] and [8] assume that a “light-tight” ring seal is lost when the pressure \( p(\phi) \) becomes negative for any polar angle \( \phi \). In addition, both derivations implicitly or explicitly assume the following:

1. The ring is made of a linear elastic material.
2. The Thin Rod approximation \( (t<<r) \) holds, where \( t \) is the thickness of the ring and \( r \) is the radius of the ring.
3. The bore (and hence the ring) distorts only in the radial direction. Axial distortions are neglected.
4. The ring, when inserted into a circular bore, exerts a uniform pressure at every point. (Note that actual ring pressure profiles need not be constant. For example, pressure near the gap may be targeted to be larger or smaller than pressure at cross sections distant from the gap.)
5. The bore distortion and its first derivative is small compared to the undistorted radius.
6. The “light-tight” seal between ring and piston exists when the pressure distribution \( p(\phi) \) is positive for each polar angle; loss of conformability, or a breach, occurs when pressure becomes negative at some angle.
7. The change of radius from free to compressed state is small relative to the installed radius. (See [10].)
8. The ring gap is infinitesimally small.

By combining the first three assumptions [10], we arrive at the fundamental elastic relation
where $\kappa$ is the curvature of the distorted ring, $\kappa_f$ is the curvature of the free, uncompressed ring, $M(\phi)$ is the induced bending moment, and the product $EI$ is the flexural rigidity (or "stiffness") of the ring. To determine the curvature of the free ring, invoke assumption 3 to derive the following expression for bending moment:

$$M(\phi) = p_o h r^2 (1 + \cos \phi), \quad (3)$$

where $p_o$ is the uniform pressure exerted by the ring on the bore [3]. The bending moment given by (3) can be derived by considering the small moment $dM$ arising from a uniform pressure $p_o$ acting on a segment with area $rh d\phi$. Integrating $dM$ with respect to $\phi$ yields the desired expression. Combining (2) and (3), taking curvature $\kappa = 1/r$ for a circular bore, yields an exact, analytical expression for the free ring:

$$\kappa_f = \frac{1-K(1+\cos \phi)}{r}. \quad (4)$$

The above equation uses the dimensionless conformability coefficient given by

$$K = \frac{p_o h r^3}{EI}. \quad (5)$$

For most rings, $K$ takes values between 0.01 and 0.04. (See [9].)

In [10] it is shown that pressure can be related to bending moment $M(\phi)$ via:

$$p(\phi) = \frac{1}{hr^2} \left(M + \frac{d^2M}{d^2\phi}\right), \quad (6)$$

assuming that deflection is small.

When the ring is inserted into a perfect bore, associated with (6) are boundary conditions that govern bending moment and shear force at the ring gap. The moment at each end is zero, as can be demonstrated using equation (3). Moreover, the forces acting at each point along the ring are equal in magnitude (due to the constant pressure distribution). For each force, there exists a force of equal magnitude and opposite direction at the other side of the ring. Summing forces, we see that the shear (which is proportional to the derivative of moment with respect to polar angle) must be zero at the ring gap. A non-uniform pressure distribution or a finite gap may imply non-zero shear or moment at the ring ends.

By combining (2), (4), and (6), we arrive at the following result:

$$p(\phi) = \frac{EI}{hr^2} \left(\kappa + \frac{d^2\kappa}{d\phi^2} - \frac{1-K}{r}\right). \quad (7)$$

Given a bore profile $\xi(\phi)$ or Fourier coefficients $A_k$, one can calculate the curvature $\kappa$ as a function of polar angle; using (7), the pressure distribution can be determined. Using the definition of conformability in assumption 6 allows one to test for a breach by simply evaluating the pressure.

In the special case of single order distortion, we can derive an estimate for the critical curvature $\kappa^*$ at which a breach first occurs. For single order distortions, the bore profile given by (1) simplifies to

$$\xi(\phi) = r + A_k \cos(k(\phi + \delta_k)). \quad (8)$$

In polar coordinates, curvature is given by the following exact expression [11]:

$$\kappa(\xi) = \frac{\xi^2 + 2\xi'^2 - \xi''}{(\xi^2 + \xi'^2)^{3/2}}, \quad (9)$$

where the primes denote differentiation with respect to polar angle. By applying assumption 5, (9) can be rewritten as a binomial series. After discarding terms involving $\xi^{-1}(\xi'/\xi)^2j$ for $j = 1,2,...$, it follows that

$$\kappa(\xi) \approx \frac{1}{\xi} - \frac{\xi''}{\xi^2}. \quad (10)$$

By inserting (8) into (10), writing the result as a series, and discarding terms involving $r^{-3(j+1)}$ for $j = 0,1,2,...$, we express curvature in terms of the measured Fourier coefficient:

$$\kappa \approx \frac{1}{r} + \frac{(k^2 - 1)A_k \cos(k(\phi + \delta_k))}{r^2}. \quad (11)$$

By a similar series argument, one can show that the second derivative of curvature is well approximated by differentiating (11) twice. Back-substituting yields

$$\frac{d^2\kappa}{d\phi^2} \approx k^2\left(\frac{1}{r} - \kappa\right). \quad (12)$$
Substituting (12) into (7) and setting pressure equal to zero gives the desired critical curvature expression:

\[
\hat{k} \approx \frac{1}{r} + \frac{K}{r(k^2 - 1)}.
\] (13)

The first term in (13) gives the curvature of the undistorted bore, while the second term gives the curvature of the small distortions. Most importantly, the critical curvature is dependent upon the order of distortion \(k\).

In [8], the author determines a threshold level of curvature, induced by second order distortion, which is sufficient to cause a breach in ring conformability. He asserts that this threshold level of curvature, induced by second order distortion, gives a good estimate of the curvature that causes a breach even when other orders of distortion dominate. Equation (13) shows this modeling assumption is inconsistent with the implications of linear elasticity theory.

To elucidate the problematic nature of this assumption, we will "derive" the bounds appearing in [8] by invoking this plausible, but evidently inconsistent critical curvature assumption.

In [8] second-order distortions are analyzed. Then, by assuming that the curvature that leads to a breach, i.e., the critical curvature, is constant across several orders, the bounds in [8] are derived. Explicitly, evaluate curvature (11) for \(k = 2\) and take the maximum to find second-order critical curvature:

\[
\hat{k}(k = 2) = \frac{1}{r} + \frac{3A_k}{r^2}.
\] (14)

Set the right hand side of (14) equal to the right hand side of (13) and solve for the maximum radial distortion \(A_k\). This yields the bounds found in [8]:

\[
A_k < \frac{Kr}{3(k^2 - 1)}.
\] (15)

To recover the original form of the bounds in [8], evaluate the conformability coefficient \(K\) in (5) with the following relations for a uniform, rectangular cross-section ring:

\[
I = \frac{1}{12}ht^3,
\] (16)

\[
p_0 = 0.76\frac{Q}{2hr},
\] (17)

where \(I\) is the moment of inertia and \(Q\) is the diametrical load under which a ring is compressed up to working end clearance. These substitutions yield the final inequality

\[
A_k < \frac{1.52Qr^3}{ht^3E} \frac{1}{k^2 - 1},
\] (18)

which corresponds to (22) and (27) from [8].

The Müller bounds may be derived by setting the right hand side of (11) equal to the right hand side of (13), and then setting the cosine term equal to unity. We find the following inequality:

\[
A_k < \frac{Kr}{(k^2 - 1)^2}.
\] (19)

To recover the Müller bounds as they appear in [9], again apply (16) and (17) in the evaluation of \(K\), yielding the inequality

\[
A_k < \frac{1.52Qr^3}{ht^3E} \frac{1}{(k^2 - 1)^2}.
\] (20)

From the above analysis, we must conclude that the Müller bounds, given by (20), can be derived without assuming that the critical curvature is independent of the order of distortion. On the other hand, the bounds in [8], given by (18), rely on this assumption. We expect therefore that the Müller bounds provide more realistic estimates of acceptable bore distortion levels because they are more consistent with the implications of linear elasticity theory.

**COMPUTATIONAL METHODS**

In a realistic bore, multiple orders of distortion are present. However, all bound methods implicitly assume that one order of distortion dominates. To overcome this limitation, the results presented above can form the basis for an algorithm that tests for the presence or absence of a seal using all available orders of distortion.

During engine operation, the ring rotates tangentially as the piston moves up and down. Therefore, the offset angle \(\phi\) between the ring gap and the origin continually changes, possibly assuming all values between 0 and 360 degrees. Under the current modeling assumptions, the minimum pressure is independent of offset angle. This fact is demonstrated in the Appendix. The proof rests on the fact that (7) defines a linear operator between pressure and curvature. Since the minimum of curvature is unchanged by rotation by a fixed angle \(\phi\), from the linearity of (7) it follows that the minimum of pressure should also remain unchanged.
### Algorithm 1. (MULTIPLE ORDER ALGORITHM)

**INPUT:** Ring specifications (radius \( r \), height \( h \), thickness \( t \), Young’s modulus \( E \), and equilibrium pressure \( p_0 \)), \( k \) Fourier coefficients of radial distortion \( A_k \), and \( k \) phase angles \( \delta_k \).

**OUTPUT:** Minimum pressure value \( p^* \). If pressure \( p^* \) is positive, a “light-tight” seal is predicted between ring and bore. If \( p^* \) is negative, a breach has developed.

Calculate conformability coefficient via (3).

1. Form a Fourier series approximation of the bore profile, given by (1), with the given Fourier coefficients \( A_k \) and phase angles \( \delta_k \).
2. Calculate curvature \( \kappa \) via (9). The derivatives can be computed either analytically from (1) or numerically via standard difference schemes. Similarly, determine the second derivative of curvature \( \frac{d^2\kappa}{d\phi^2} \).
3. Use (7) to determine pressure as a function of polar angle.
4. Set \( p^* = \min p(\phi) \).

When analyzing a distorted bore consisting of multiple decks, or levels, this method should be applied to each individual deck. A breach on any deck indicates a loss of “light-tight” seal. For experimental studies, the Fourier coefficients \( A_k \) and phase angles \( \delta_k \) can be measured via an incometer [12]. For analyzing synthetic FE data, the Fourier coefficients and phase angles must be numerically approximated from the bore profile.

In the case of single order distortions, the multiple order algorithm presented above reduces to the Müller bounds given by (19) and (20). Hence, this algorithm can be considered a quantitative generalization of the classical bound approach.

#### Table 1. Specifications for the piston ring.

<table>
<thead>
<tr>
<th>Specification</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diametrical Load (N)</td>
<td>13.2558</td>
</tr>
<tr>
<td>Height (mm)</td>
<td>1.48</td>
</tr>
<tr>
<td>Thickness (mm)</td>
<td>3.985</td>
</tr>
<tr>
<td>Middle Radius (mm)</td>
<td>51.181</td>
</tr>
<tr>
<td>Young’s Modulus (N/mm²)</td>
<td>152000</td>
</tr>
<tr>
<td>Closing Force (N)</td>
<td>6.1655</td>
</tr>
<tr>
<td>Moment of Inertia (mm⁴)</td>
<td>7.8049</td>
</tr>
<tr>
<td>Stiffness (mm²)</td>
<td>1186300</td>
</tr>
<tr>
<td>Conformability</td>
<td>0.0136</td>
</tr>
</tbody>
</table>

### NUMERICAL RESULTS

The current multiple order algorithm is implemented as a MATLAB program called RINGPACK. The code accepts FE data defined on a 3D grid as input and computes the Fourier coefficients \( A_k \) and phase angles \( \delta_k \). The pressure distribution is then calculated via (7), (9), and (12) using a high-order finite-difference approximation for the derivatives [13]. Code can be downloaded from [http://www.msu.edu/~bardzima/files/ringpack.zip](http://www.msu.edu/~bardzima/files/ringpack.zip).

To investigate the multiple order algorithm discussed in the previous section, Figure 2 considers single order distortions. For each Fourier coefficient \((k = 2 \text{ to } 6)\), the minimum radial distortion for which pressure becomes negative is calculated. Starting with \( A_k = 0 \), the Fourier coefficient is increased and the associated pressure calculated. When the pressure becomes negative for at least one polar angle, the loop terminates, yielding the “RINGPACK” bounds \( A_{k} \). The offset angle \( \phi_0 \) can be set equal to zero because minimum pressure value is independent of the gap offset (see appendix).

Fig. 2 compares these RINGPACK bounds to the bounds found in [7], [8], and [9] for the ring parameters in Table 1. Developed in [9], the Tomanik bounds are given by

\[
A_k < \frac{K_r}{20(k^2 - 1)}.
\]  

(21)

Note that the RINGPACK bounds are virtually identical to the Müller bounds, thus demonstrating how the present algorithm reduces to the Müller bounds for
single order distortions. Although the Tomanik bounds are more aggressive at order two, the agreement between the Tomanik and Müller bounds is reasonable (33% and 17% relative error) for orders three and four.

As a test case, distorted bore data generated via finite element (FE) analysis was considered. Figure 3 shows a wire-frame of the FE data, which contains 384 grid points arranged in 16 decks. Bore out-of-roundness is exaggerated by a multiplicative factor of 100 in the figure to make distortions easily visible. Bore profiles are then extracted from the FE data; the corresponding Fourier coefficients and phase angles are calculated by interpolating the resulting bore profile with a cubic spline and then using a fast Fourier transform (FFT) [13]. The pressure distribution associated with the bore/ring pair is calculated via the multiple order algorithm outlined in the previous section. Specifically, curvature is computed numerically via (9) using the interpolated profile, followed by evaluating (7) to recover pressure.

In the following discussion, sample computations are shown for deck 14 of the distorted bore, which exhibit the most severe distortion. In practice, the pressure distribution should be calculated for all decks in the given bore data. If any deck exhibits negative pressure values, the bore fails the proposed screening test.

Figure 4 shows the pressure distribution (in MPa) associated with deck 14 of the FE data under consideration. Distributions for offset angles 0 and 60 degrees are displayed. The bore profile and pressure distribution are approximated with the first four terms in the Fourier series given by (1). Note that the pressure drops below zero at approximately -15 and -75 degree angles, thus indicating a loss of seal. Negative contact pressure is not physically possible. It is interpreted here as a numerical indication of a tendency for the bore and ring to pull away from each other, and hence indicates a seal breach.

In Fig. 4, the minimum pressure is unchanged by the choice of different offset angles, which is predicted by the proof in the Appendix. The user only needs to calculate a single pressure distribution of a fixed offset angle (say, $\phi_0 = 0$ degrees). Any small variation can be attributed to numerical error.

Figure 5 compares the new multiple order algorithm against the Müller bounds approach. The Fourier coefficients calculated from the bore profile for Deck 14 are shown with the dashed line, while the Müller bounds are shown with the solid line. Note that the Fourier coefficients for this deck do not exceed the Müller bounds for the ring under consideration.

**DISCUSSION**

In Fig. 2, we see that the bounds obtained via the multiple order algorithm match the Müller bounds almost exactly. Although the bounds discussed in [7] and [8] and RINGPACK bounds match at second order, there is wide variation for higher orders. Since this algorithm calculates each variable with minimal approximation, this result suggests the Müller bounds are more accurate in the case of single order distortion. Moreover, the close agreement between the Müller bounds and RINGPACK bounds validates the series approximations used in deriving the critical curvature given by (13).

Comparing Figs. 4 and 5, note that a small breach has developed at approximately -15 degrees using an offset angle of 0 degrees and -75 degrees using an offset of 60 degrees; however, the bore has not violated the Müller
bounds at any order. In particular, it appears that the fourth-order contribution has contributed enough distortion to break the seal. From this example, we see the advantage of analyzing multiple Fourier components at once. By limiting the analysis to an isolated Fourier coefficient, the bore satisfies the Müller bounds; yet Fig. 4 shows a negative pressure value, implying the presence of a breach. Thus, distortions at multiple orders constructively interfere to produce a seal breach.

This analysis and numerical evidence shows that assuming that the curvature at which a breach occurs is constant across several orders of distortion is incompatible with the implications of linear elasticity theory. Because of this assumption, the bounds in [8] overestimate conformability for orders three and beyond by over 200%. In particular, small distortions of high order can produce large pressure variations, while large distortions of low order may produce mild pressure variations.

CONCLUSION

This paper analyzes two alternate approaches, given in [7] and [8], currently used for predicting the seal between piston rings and cylinder bores. From this analysis, the Müller bounds emerge as consistent with elasticity theory, whereas the approach given in [8] contains a critical curvature assumption that is inconsistent with the implications of linear elasticity theory. To assess multiple orders of distortion present in a given bore profile, a new algorithm is derived which reduces to the Müller bounds in the limiting case of single order distortion. The current study assumes a highly simplified ring/bore model (uniform pressure distribution in a perfect bore, infinitesimal ring gap, and uniform cross-section). In future studies, some of these assumptions may be relaxed, yielding a more accurate conformability criterion. The proposed multiple-order conformability method serves as a screening tool for quickly assessing predicted bore distortion for a proposed engine block design. A design that passes initial screening by the multiple order algorithm may warrant further, more computationally expensive and detailed analysis such as is specified in [1] or [2]. The multiple order algorithm is easily implemented on standard PC hardware and can be used to analyze either FE data or incometer measurements; moreover, seal-breaches caused by the superposition of several orders of distortion can now be predicted, which is not possible with the existing single-order bounds given in [7], [8], and [9].

ACKNOWLEDGMENTS

The authors thank Boon-Keat Choi and Harold Schock from the Engine Research Lab, Michigan State University, for helpful discussion and insight. We thank the anonymous reviewers for bringing [2] to our attention. The authors also thank Charles R. MacCluer, Department of Mathematics, Michigan State University, for helpful criticisms on the manuscript, and William Resh and Sam Asirvatham of DaimlerChrysler for their insights and editorial comments.

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**APPENDIX**

Let \( p_1(\phi) \) be the pressure generated when the ring gap is at \( \phi = \phi_1 \). Let \( p_2(\phi) \) be the pressure generated when the ring gap is at \( \phi = \phi_2 \). Then

\[
p_1^\ast = \min p_1(\phi) = \min p_2(\phi) = p_2^\ast.
\]

**Proof:** Let \( \xi_j(\phi) \) be the bore profile associated with offset angle \( \phi_j \) for \( j = 1 \) or \( 2 \). Using (1), write

\[
\xi_j(\phi) = r + \sum_{k=2}^{N} A_k \cos(k(\phi - \phi_j + \delta_k)).
\]  

Calculate minimum pressure \( p_1^\ast \) as follows:

\[
p_1^\ast = \min p_1(\phi) = \min p_1(\phi - (\phi_2 - \phi_1)),
\]

since the pressure distribution is periodic with respect to \( \phi \). To evaluate \( p_1(\phi) \) in terms of curvature, apply (7). Then it follows that

\[
p_1^\ast = \min \frac{EI}{hr^2} (\kappa(\xi_1(\phi - (\phi_2 - \phi_1))) + \frac{d^2 \kappa}{d \phi^2} (\xi_1(\phi - (\phi_2 - \phi_1))) - \frac{1-K}{r})
\]

\[
= \min \frac{EI}{hr^2} (\kappa(\xi_2(\phi))) + \frac{d^2 \kappa}{d \phi^2} (\xi_2(\phi)) - \frac{1-K}{r})
\]

\[
= \min p_2(\phi) = p_2^\ast.
\]

We conclude that minimum pressure is independent of the orientation of the ring gap with respect to the origin.