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AN APPROXIMATION TO THE DISTRIBUTION OF THE PRODUCT OF TWO DEPENDENT CORRELATION COEFFICIENTS

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Abstract

Many methodological studies depend on the product of two dependent correlation coefficients. However, the behavior of the distribution of the product of two dependent correlation coefficients is not well known. The distribution of sets of correlation coefficients has been well studied, but not the distribution of the product of two dependent correlation coefficients. The present study derives an approximation to the distribution of the product of two dependent correlation coefficients with a closed form, resulting in a Pearson Type I distribution. A simulation study is also conducted to assess the accuracy of the approximation.

Keywords: Approximate distribution; Correlation coefficient; Moments; Pearson distributions

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1. INTRODUCTION

Many regression- and path-based procedures depend on the product of two dependent correlation coefficients. For example, the product of two dependent correlation coefficients is needed to assess the impact of a confounding variable on a regression coefficient (Frank, 2000). Also, in path analysis the indirect effect is the product of path coefficients (Fox, 1980; Sobel, 1982, 1986). If we use standardized coefficients, the indirect effect can be expressed as a product of two dependent correlation coefficients. In each of these cases it would be valuable to understand the distribution of the product of two dependent correlations to aid in making statistical inferences. There is extensive research on the distribution of a single correlation coefficient (Konishi, 1978; Konishi, 1979; Kraemer, 1973; Olkin, 1967; Olkin & Siotani, 1976) and on the distribution of the difference of two correlation coefficients (Choi, 1977; Dunn & Clark, 1971; Meng, Rosenthal, & Rubin, 1992; Neill & Dunn, 1975; Olkin, 1967; Steiger, 1980; Wolfe, 1976), but little on the distribution of the product of two correlation coefficients. Thus, in this manuscript we develop an approximation to the distribution of the product of two dependent correlation coefficients.

We begin by recognizing the computational boundaries for the product of two dependent correlation coefficients. We then review extant techniques for approximating the distribution. We then develop a more direct and more accurate approximation based on the first four moments of the product of two dependent correlation coefficients and applying those moments to the Pearson distribution family (Pearson, 1895). We establish the accuracy of the approximation by comparing it with simulated data. We also graphically compare our results with those of Frank's (2000), and then identify possible extensions.

2. EXTANT KNOWLEDGE OF THE DISTRIBUTION OF THE PRODUCT OF TWO CORRELATIONS

Let *X* and *Y* be bivariate random variables both correlated with the third variable *C*; and r_{xy}, r_{xc}, and r_{yc} be the sample correlation coefficients between *X* and *Y*, between *X* and *C*, and between *Y* and *C*, respectively. From Cohen and Cohen (1983, p. 280), we know that the product of two dependent correlation coefficients, $r_{xc}r_{yc}$, is constrained by the upper and lower limits

$$r_{xy} - \sqrt{(1-r_{xc}^2)(1-r_{yc}^2)} < r_{xc}r_{yc} < r_{xy} + \sqrt{(1-r_{xc}^2)(1-r_{yc}^2)} \,. \tag{2.1}$$

However, beyond this constraint, little is known.

There are two extant approaches for obtaining the distribution of the product of two correlation coefficients. Mathai and Saxena (1969) express the product of two correlation coefficients as a special case of the product of two generalized Mellin-Barnes functions or *H*-functions (Mathai & Saxena, 1978). However, the expression obtained for the distribution function is quite unwieldy. In addition, in their study, the two correlation coefficients are assumed independent, while we are interested in two *dependent* correlation coefficients.

The other approach is described in Frank (2000). Frank transforms the two correlation coefficients into two asymptotically, normally distributed Fisher *z*'s, then uses Aroian and colleagues' (Aroian, 1947; Aroian, Taneja, & Cornwell, 1978; Cornwell, Aroian, & Taneja, 1978; Meeker, Cornwell, & Aroian, 1981; Meeker & Escobar, 1994) findings regarding the distribution of the product of two normal variables to obtain the distribution of the product of two Fisher *z*'s, instead of the two original correlation coefficients. In this approach, however, we do not have a closed form function for the distribution and we do not know how stable the Fisher *z* transformation is. Moreover, we are interested in the product of two *original* correlation coefficients, while the behavior of the distribution of the product of two Fisher *z*'s is detached from that of two original correlation coefficients. That is, Frank's approach relies on asymptotic theory for Fisher's *z*, compounded with the approximation error associated with Aroian's approach, which results in very slow convergence to an asymptotic result.

Following Hotelling's (1936, 1940) and Ghosh's (1966) approximations to the moments of the distribution of a correlation coefficient, the present study derives a more accurate approximation to the distribution of the product of two dependent correlation coefficients with a closed form, resulting in a Pearson Type I (*Beta*) distribution. A simulation study is also conducted to assess the accuracy of the approximation.

3. APPROXIMATION PROCEDURES

According to Pearson (1895), most distributions can be accurately characterized using the first four moments. Based on this principle, our approximation procedures have two steps. First, we obtain the first four moments of the product $r_{xc}r_{yc}$. Then, we apply these moments to the Pearson distribution family (Pearson, 1895), obtaining an approximate distribution of $r_{xc}r_{yc}$ as a Pearson Type I distribution.

3.1 Moments of $r_{xc}r_{yc}$

Let ρ_{xy}, ρ_{xc}, and ρ_{yc} be the population values of r_{xy}, r_{xc}, and r_{yc}, respectively. Let Δr_{xy}, Δr_{xc}, and Δr_{yc} be the deviations of the sample values from their population values. In particular, $\Delta r_{xy} = r_{xy} - \rho_{xy}$, $\Delta r_{xc} = r_{xc} - \rho_{xc}$, and $\Delta r_{yc} = r_{yc} - \rho_{yc}$. Then, we have

$$r_{xc}r_{yc} = \rho_{xc}\rho_{yc} + \rho_{xc}\Delta r_{yc} + \rho_{yc}\Delta r_{xc} + \Delta r_{xc}\Delta r_{yc};$$

$$\begin{aligned}(r_{xc}r_{yc})^2 = {} & \rho^2_{xc}\rho^2_{yc} + \rho^2_{xc}(\Delta r_{yc})^2 + \rho^2_{yc}(\Delta r_{xc})^2 + (\Delta r_{xc})^2(\Delta r_{yc})^2 + 2\rho^2_{xc}\rho_{yc}\Delta r_{yc} + 2\rho_{xc}\rho^2_{yc}\Delta r_{xc} \\ & + 2\rho_{yc}(\Delta r_{xc})^2\Delta r_{yc} + 2\rho_{xc}\Delta r_{xc}(\Delta r_{yc})^2 + 4\rho_{xc}\rho_{yc}\Delta r_{xc}\Delta r_{yc};\end{aligned}$$

$$\begin{aligned}(r_{xc}r_{yc})^3 = {} & \rho^3_{xc}\rho^3_{yc} + \rho^3_{xc}(\Delta r_{yc})^3 + \rho^3_{yc}(\Delta r_{xc})^3 + (\Delta r_{xc})^3(\Delta r_{yc})^3 + 3\rho^3_{xc}\rho^2_{yc}\Delta r_{yc} + 3\rho^3_{xc}\rho_{yc}(\Delta r_{yc})^2 \\ & + 3\rho^2_{xc}\rho^3_{yc}\Delta r_{xc} + 3\rho_{xc}\rho^3_{yc}(\Delta r_{xc})^2 + 3\rho^2_{xc}\Delta r_{xc}(\Delta r_{yc})^3 + 3\rho_{xc}(\Delta r_{xc})^2(\Delta r_{yc})^3 \\ & + 3\rho^2_{yc}(\Delta r_{xc})^3\Delta r_{yc} + 3\rho_{yc}(\Delta r_{xc})^3(\Delta r_{yc})^2 + 9\rho^2_{xc}\rho^2_{yc}\Delta r_{xc}\Delta r_{yc} + 9\rho^2_{xc}\rho_{yc}\Delta r_{xc}(\Delta r_{yc})^2 \\ & + 9\rho_{xc}\rho^2_{yc}(\Delta r_{xc})^2\Delta r_{yc} + 9\rho_{xc}\rho_{yc}(\Delta r_{xc})^2(\Delta r_{yc})^2;\end{aligned}$$

$$\begin{aligned}(r_{xc}r_{yc})^4 = {} & \rho^4_{xc}\rho^4_{yc} + \rho^4_{xc}(\Delta r_{yc})^4 + \rho^4_{yc}(\Delta r_{xc})^4 + (\Delta r_{xc})^4(\Delta r_{yc})^4 + 4\rho^4_{xc}\rho^3_{yc}\Delta r_{yc} + 4\rho^3_{xc}\rho^4_{yc}\Delta r_{xc} \\ & + 4\rho^3_{yc}(\Delta r_{xc})^4\Delta r_{yc} + 4\rho_{xc}(\Delta r_{xc})^3(\Delta r_{yc})^4 + 4\rho^4_{xc}\rho_{yc}(\Delta r_{yc})^3 + 4\rho^3_{xc}\Delta r_{xc}(\Delta r_{yc})^4 \\ & + 4\rho_{xc}\rho^4_{yc}(\Delta r_{xc})^3 + 4\rho_{yc}(\Delta r_{xc})^4(\Delta r_{yc})^3 + 6\rho^4_{xc}\rho^2_{yc}(\Delta r_{yc})^2 + 6\rho^2_{xc}\rho^4_{yc}(\Delta r_{xc})^2 \\ & + 6\rho^2_{xc}(\Delta r_{xc})^2(\Delta r_{yc})^4 + 6\rho^2_{yc}(\Delta r_{xc})^4(\Delta r_{yc})^2 + 16\rho^3_{xc}\rho^3_{yc}\Delta r_{xc}\Delta r_{yc} + 16\rho^3_{xc}\rho_{yc}\Delta r_{xc}(\Delta r_{yc})^3 \\ & + 16\rho_{xc}\rho^3_{yc}(\Delta r_{xc})^3\Delta r_{yc} + 16\rho_{xc}\rho_{yc}(\Delta r_{xc})^3(\Delta r_{yc})^3 + 24\rho^3_{xc}\rho^2_{yc}\Delta r_{xc}(\Delta r_{yc})^2 \\ & + 24\rho^2_{xc}\rho^3_{yc}(\Delta r_{xc})^2\Delta r_{yc} + 24\rho^2_{xc}\rho_{yc}(\Delta r_{xc})^2(\Delta r_{yc})^3 + 24\rho_{xc}\rho^2_{yc}(\Delta r_{xc})^3(\Delta r_{yc})^2 \\ & + 36\rho^2_{xc}\rho^2_{yc}(\Delta r_{xc})^2(\Delta r_{yc})^2.\end{aligned}$$

Dropping the terms of order higher than the fourth* and taking expectations of $(r_{xc}r_{yc})^i$, $i = 1, 2, 3, 4$, give us the approximate first four non-central moments as follows

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* Since we only want to obtain the approximate first four moments, the terms of order higher than the fourth will have little effect on the approximation.

$$
\left.\begin{aligned}
\mu'_1 &= E(r_{xc}r_{yc}) = \rho_{xc}\rho_{yc} + \rho_{xc}E(\Delta r_{yc}) + \rho_{yc}E(\Delta r_{xc}) + E(\Delta r_{xc}\Delta r_{yc});\\
\mu'_2 &= E[(r_{xc}r_{yc})^2] = \rho^2_{xc}\rho^2_{yc} + \rho^2_{xc}E[(\Delta r_{yc})^2] + \rho^2_{yc}E[(\Delta r_{xc})^2] + E[(\Delta r_{xc})^2(\Delta r_{yc})^2]\\
&\quad + 2\rho^2_{xc}\rho_{yc}E(\Delta r_{yc}) + 2\rho_{xc}\rho^2_{yc}E(\Delta r_{xc}) + 2\rho_{yc}E[(\Delta r_{xc})^2\Delta r_{yc}] + 2\rho_{xc}E[\Delta r_{xc}(\Delta r_{yc})^2]\\
&\quad + 4\rho_{xc}\rho_{yc}E(\Delta r_{xc}\Delta r_{yc});\\
\mu'_3 &= E[(r_{xc}r_{yc})^3] = \rho^3_{xc}\rho^3_{yc} + \rho^3_{xc}E[(\Delta r_{yc})^3] + \rho^3_{yc}E[(\Delta r_{xc})^3] + 3\rho^3_{xc}\rho^2_{yc}E(\Delta r_{yc})\\
&\quad + 3\rho^3_{xc}\rho_{yc}E[(\Delta r_{yc})^2] + 3\rho^2_{xc}\rho^3_{yc}E(\Delta r_{xc}) + 3\rho_{xc}\rho^3_{yc}E[(\Delta r_{xc})^2]\\
&\quad + 3\rho^2_{xc}E[\Delta r_{xc}(\Delta r_{yc})^3] + 3\rho^2_{yc}E[(\Delta r_{xc})^3\Delta r_{yc}] + 9\rho^2_{xc}\rho^2_{yc}E(\Delta r_{xc}\Delta r_{yc})\\
&\quad + 9\rho^2_{xc}\rho_{yc}E[\Delta r_{xc}(\Delta r_{yc})^2] + 9\rho_{xc}\rho^2_{yc}E[(\Delta r_{xc})^2\Delta r_{yc}] + 9\rho_{xc}\rho_{yc}E[(\Delta r_{xc})^2(\Delta r_{yc})^2];\\
\mu'_4 &= E[(r_{xc}r_{yc})^4] = \rho^4_{xc}\rho^4_{yc} + \rho^4_{xc}E[(\Delta r_{yc})^4] + \rho^4_{yc}E[(\Delta r_{xc})^4] + 4\rho^4_{xc}\rho^3_{yc}E(\Delta r_{yc})\\
&\quad + 4\rho^3_{xc}\rho^4_{yc}E(\Delta r_{xc}) + 4\rho^4_{xc}\rho_{yc}E[(\Delta r_{yc})^3] + 4\rho_{xc}\rho^4_{yc}E[(\Delta r_{xc})^3]\\
&\quad + 6\rho^4_{xc}\rho^2_{yc}E[(\Delta r_{yc})^2] + 6\rho^2_{xc}\rho^4_{yc}E[(\Delta r_{xc})^2] + 16\rho^3_{xc}\rho^3_{yc}E(\Delta r_{xc}\Delta r_{yc})\\
&\quad + 16\rho^3_{xc}\rho_{yc}E[\Delta r_{xc}(\Delta r_{yc})^3] + 16\rho_{xc}\rho^3_{yc}E[(\Delta r_{xc})^3\Delta r_{yc}] + 24\rho^3_{xc}\rho^2_{yc}E[\Delta r_{xc}(\Delta r_{yc})^2]\\
&\quad + 24\rho^2_{xc}\rho^3_{yc}E[(\Delta r_{xc})^2\Delta r_{yc}] + 36\rho^2_{xc}\rho^2_{yc}E[(\Delta r_{xc})^2(\Delta r_{yc})^2].
\end{aligned}\right\} \quad (3.1)
$$

In order to obtain closed form expressions for the first four moments, we need to express $E[(\Delta r_{xc})^i]$, $i = 1, 2, 3, 4$, $E[(\Delta r_{yc})^j]$, $j = 1, 2, 3, 4$, and $E[(\Delta r_{xc})^k(\Delta r_{yc})^l]$, $k, l = 1, 2,$ or 3, in terms of ρ_{xc}, ρ_{yc}, or ρ_{xy}.

First, we define some notation and obtain some preliminary results that will be used to simplify our final expressions of the fist four moments. Assume three initial variables X, Y, and C follow a trivariate normal distribution. Then, following Ghosh (1966) and Hotelling (1936, 1940), the moments and the covariance of r_{xc} and r_{yc} can be approximately expressed as follows

$$\left.\begin{aligned}
\mu_{r_\bullet} &= E(r_\bullet) = \rho_\bullet - \frac{\rho_\bullet(1-\rho_\bullet^2)}{2M}\Big\{1 + \frac{9}{4M}(3+\rho_\bullet^2) + \frac{3}{8M^2}(121 + 70\rho_\bullet^2 + 25\rho_\bullet^4) \\
&\quad + \frac{3}{64M^3}(6479 + 4923\rho_\bullet^2 + 2925\rho_\bullet^4 + 1225\rho_\bullet^6) + \frac{3}{128M^4}(86341 \\
&\quad + 77260\rho_\bullet^2 + 58270\rho_\bullet^4 + 38220\rho_\bullet^6 + 19845\rho_\bullet^8)\Big\}; \\
\sigma_{r_\bullet}^{(2)} &= Var(r_\bullet) = E[(r_\bullet - \mu_{r_\bullet})^2] = \frac{(1-\rho_\bullet^2)^2}{M}\Big\{1 + \frac{1}{2M}(14 + 11\rho_\bullet^2) + \frac{1}{2M^2}(98 + 130\rho_\bullet^2 \\
&\quad + 75\rho_\bullet^4) + \frac{1}{8M^3}(2744 + 4645\rho_\bullet^2 + 4422\rho_\bullet^4 + 2565\rho_\bullet^6) + \frac{1}{8M^4}(19208 \\
&\quad + 37165\rho_\bullet^2 + 44499\rho_\bullet^4 + 40299\rho_\bullet^6 + 26685\rho_\bullet^8)\Big\}; \\
\sigma_{r_\bullet}^{(3)} &= E[(r_\bullet - \mu_{r_\bullet})^3] = -\frac{\rho_\bullet(1-\rho_\bullet^2)^3}{M^2}\Big\{6 + \frac{1}{M}(69 + 88\rho_\bullet^2) + \frac{3}{4M^2}(797 + 1691\rho_\bullet^2 \\
&\quad + 1560\rho_\bullet^4) + \frac{3}{8M^3}(12325 + 33147\rho_\bullet^2 + 488099\rho_\bullet^4 + 44109\rho_\bullet^6)\Big\}; \\
\sigma_{r_\bullet}^{(4)} &= E[(r_\bullet - \mu_{r_\bullet})^4] = \frac{3(1-\rho_\bullet^2)^4}{M^2}\Big\{1 + \frac{1}{M}(12 + 35\rho_\bullet^2) + \frac{1}{4M^2}(436 + 2028\rho_\bullet^2 \\
&\quad + 3025\rho_\bullet^4) + \frac{1}{4M^3}(3552 + 20009\rho_\bullet^2 + 46462\rho_\bullet^4 + 59751\rho_\bullet^6)\Big\}; \\
\sigma_{r_{xc}, r_{yc}} &= Cov(r_{xc}, r_{yc}) = E[(r_{xc} - \mu_{r_{xc}})(r_{yc} - \mu_{r_{yc}})] \\
&= \frac{1}{M}\left[\rho_{xy}(1 - \rho_{xc}^2 - \rho_{yc}^2) - \frac{1}{2}\rho_{xc}\rho_{yc}(1 - \rho_{xc}^2 - \rho_{yc}^2 - \rho_{xy}^2)\right],
\end{aligned}\right\} \quad (3.2)$$

where $M = N + 6$, N is the sample size; the subscript "•" represents *xc* or *yc*; and the superscripts "(2)", "(3)", and "(4)" represent a variance, a third moment, and a fourth moment, respectively (as distinguished from quadratic, cubic, and quartic powers).

Next we can express $E[(\Delta r_\bullet)^i]$, $i = 1, 2, 3, 4$, and $E(\Delta r_{xc}\Delta r_{yc})$ as

$$\left.\begin{aligned}
&E(\Delta r_\bullet) = E(r_\bullet - \rho_\bullet) = \mu_{r_\bullet} - \rho_\bullet \text{, which will be referred to as } b_{r_\bullet};\\
&E[(\Delta r_\bullet)^2] = E[(r_\bullet - \rho_\bullet)^2] = E\{[(r_\bullet - \mu_{r_\bullet}) - (\mu_{r_\bullet} - \rho_\bullet)]^2\}\\
&\qquad = E[(r_\bullet - \mu_{r_\bullet})^2] + (\mu_{r_\bullet} - \rho_\bullet)^2 = \sigma_{r_\bullet}^{(2)} + b_{r_\bullet}^2;\\
&E[(\Delta r_\bullet)^3] = E[(r_\bullet - \rho_\bullet)^3] = E\{[(r_\bullet - \mu_{r_\bullet}) - (\mu_{r_\bullet} - \rho_\bullet)]^3\}\\
&\qquad = E[(r_\bullet - \mu_{r_\bullet})^3] - 3(\mu_{r_\bullet} - \rho_\bullet)E[(r_\bullet - \mu_{r_\bullet})^2] - (\mu_{r_\bullet} - \rho_\bullet)^3\\
&\qquad = \sigma_{r_\bullet}^{(3)} - 3\sigma_{r_\bullet}^{(2)} b_{r_\bullet} - b_\bullet^3;\\
&E[(\Delta r_\bullet)^4] = E[(r_\bullet - \rho_\bullet)^4] = E\{[(r_\bullet - \mu_{r_\bullet}) - (\mu_{r_\bullet} - \rho_\bullet)]^4\}\\
&\qquad = E[(r_\bullet - \mu_{r_\bullet})^4] - 4(\mu_{r_\bullet} - \rho_\bullet)E[(r_\bullet - \mu_{r_\bullet})^3]\\
&\qquad\quad + 6(\mu_{r_\bullet} - \rho_\bullet)^2 E[(r_\bullet - \mu_{r_\bullet})^2] + (\mu_{r_\bullet} - \rho_\bullet)^4\\
&\qquad = \sigma_{r_\bullet}^{(4)} - 4\sigma_{r_\bullet}^{(3)} b_{r_\bullet} + 6\sigma_{r_\bullet}^{(2)} b_{r_\bullet}^2 + b_{r_\bullet}^4;\\
&E(\Delta r_{xc}\Delta r_{yc}) = E[(r_{xc} - \rho_{xc})(r_{yc} - \rho_{yc})]\\
&\qquad = E\{[(r_{xc} - \mu_{r_{xc}}) - (\mu_{r_{xc}} - \rho_{xc})][(r_{yc} - \mu_{r_{yc}}) - (\mu_{r_{yc}} - \rho_{yc})]\}\\
&\qquad = E[(r_{xc} - \mu_{r_{xc}})(r_{yc} - \mu_{r_{yc}})] + (\mu_{r_{xc}} - \rho_{xc})(\mu_{r_{yc}} - \rho_{yc})\\
&\qquad = \sigma_{r_{xc}, r_{yc}} + b_{r_{xc}} b_{r_{yc}}.
\end{aligned}\right\} \tag{3.3}$$

We also need the expressions for the third and the fourth order product-moments, $E[(\Delta r_{xc})^s(\Delta r_{yc})^t]$, s, $t = 1, 2,$ or 3. After a few simplifications, we first have

$$\left.\begin{aligned}
E(\Delta r_i \Delta r_j \Delta r_k \Delta r_l) &= E[(r_i-\rho_i)(r_j-\rho_j)(r_k-\rho_k)(r_l-\rho_l)] \\
&= E\{[(r_i-\mu_{r_i})+(\mu_{r_i}-\rho_i)][(r_j-\mu_{r_j})+(\mu_{r_j}-\rho_j)] \\
&\quad \cdot[(r_k-\mu_{r_k})+(\mu_{r_k}-\rho_k)][(r_l-\mu_{r_l})+(\mu_{r_l}-\rho_l)]\} \\
&= E[(r_i-\mu_{r_i})(r_j-\mu_{r_j})(r_k-\mu_{r_k})(r_l-\mu_{r_l})] \\
&\quad + b_{r_l}E[(r_i-\mu_{r_i})(r_j-\mu_{r_j})(r_k-\mu_{r_k})] \\
&\quad + b_{r_k}E[(r_i-\mu_{r_i})(r_j-\mu_{r_j})(r_l-\mu_{r_l})] \\
&\quad + b_{r_j}E[(r_i-\mu_{r_i})(r_k-\mu_{r_k})(r_l-\mu_{r_l})] \\
&\quad + b_{r_i}E[(r_j-\mu_{r_j})(r_k-\mu_{r_k})(r_l-\mu_{r_l})] \\
&\quad + b_{r_k}b_{r_l}E[(r_i-\mu_{r_i})(r_j-\mu_{r_j})] + b_{r_j}b_{r_l}E[(r_i-\mu_{r_i})(r_k-\mu_{r_k})] \\
&\quad + b_{r_j}b_{r_k}E[(r_i-\mu_{r_i})(r_l-\mu_{r_l})] + b_{r_i}b_{r_l}E[(r_j-\mu_{r_j})(r_k-\mu_{r_k})] \\
&\quad + b_{r_i}b_{r_k}E[(r_j-\mu_{r_j})(r_l-\mu_{r_l})] + b_{r_i}b_{r_j}E[(r_k-\mu_{r_k})(r_l-\mu_{r_l})] \\
&\quad + b_{r_i}b_{r_j}b_{r_k}b_{r_l} \\
&= \sigma_{r_i,r_j}\sigma_{r_k,r_l} + \sigma_{r_i,r_k}\sigma_{r_j,r_l} + \sigma_{r_i,r_l}\sigma_{r_j,r_k} + \sigma_{r_i,r_j}b_{r_k}b_{r_l} + \sigma_{r_i,r_k}b_{r_j}b_{r_l} \\
&\quad + \sigma_{r_i,r_l}b_{r_j}b_{r_k} + \sigma_{r_j,r_k}b_{r_i}b_{r_l} + \sigma_{r_j,r_l}b_{r_i}b_{r_k} + \sigma_{r_k,r_l}b_{r_i}b_{r_j} \\
&\quad + b_{r_i}b_{r_j}b_{r_k}b_{r_l}\,,
\end{aligned}\right\} \quad (3.4)$$

where the indices i, j, k, and l can be xc or yc. Anderson's (1958, p.39, Equations 25 & 26) formulae were applied to the last equality in (3.4).* By the same fashion, we further have

*Anderson's equations were tactically used here only for obtaining the higher order product-moments, although Anderson's equations are based on the normal distribution.

$$E(\Delta r_i \Delta r_j \Delta r_k) = E[(r_i - \rho_i)(r_j - \rho_j)(r_k - \rho_k)]$$

$$= \sigma_{r_i, r_j} b_{r_k} + \sigma_{r_i, r_k} b_{r_j} + \sigma_{r_j, r_k} b_{r_i} + b_{r_i} b_{r_j} b_{r_k} \ . \tag{3.5}$$

Then, in (3.4) letting $i = j = k = xc$ and $l = yc$, letting $i = xc$ and $j = k = l = yc$, and letting $i = j = xc$ and $k = l = yc$, respectively, give us the desired expressions for the fourth order product-moments as follows

$$\left.\begin{aligned}
E[(\Delta r_{xc})^3(\Delta r_{yc})] &= 3\sigma^{(2)}_{r_{xc}}\sigma_{r_{xc}, r_{yc}} + 3\sigma^{(2)}_{r_{xc}} b_{r_{xc}} b_{r_{yc}} + 3\sigma_{r_{xc}, r_{yc}} b^2_{r_{xc}} + b^3_{r_{xc}} b_{r_{yc}} \ ; \\
E[(\Delta r_{xc})(\Delta r_{yc})^3] &= 3\sigma^{(2)}_{r_{yc}}\sigma_{r_{xc}, r_{yc}} + 3\sigma^{(2)}_{r_{yc}} b_{r_{xc}} b_{r_{yc}} + 3\sigma_{r_{xc}, r_{yc}} b^2_{r_{yc}} + b_{r_{xc}} b^3_{r_{yc}} \ ; \\
E[(\Delta r_{xc})^2(\Delta r_{yc})^2] &= \sigma^{(2)}_{r_{xc}}\sigma^{(2)}_{r_{yc}} + 2\sigma^2_{r_{xc}, r_{yc}} + \sigma^{(2)}_{r_{xc}} b^2_{r_{yc}} + \sigma^{(2)}_{r_{yc}} b^2_{r_{xc}} + 4\sigma_{r_{xc}, r_{yc}} b_{r_{xc}} b_{r_{yc}} + b^2_{r_{xc}} b^2_{r_{yc}} \ .
\end{aligned}\right\} \tag{3.6}$$

And, in (3.5) letting $i = j = xc$ and $k = yc$ and letting $i = xc$ and $j = k = yc$, respectively, give us the desired expressions for the third order product-moments as follows

$$\left.\begin{aligned}
E[(\Delta r_{xc})^2(\Delta r_{yc})] &= \sigma^{(2)}_{r_{xc}} b_{r_{yc}} + 2\sigma_{r_{xc}, r_{yc}} b_{r_{xc}} + b^2_{r_{xc}} b_{r_{yc}} \ ; \\
E[(\Delta r_{xc})(\Delta r_{yc})^2] &= \sigma^{(2)}_{r_{yc}} b_{r_{xc}} + 2\sigma_{r_{xc}, r_{yc}} b_{r_{yc}} + b_{r_{xc}} b^2_{r_{yc}} \ .
\end{aligned}\right\} \tag{3.7}$$

Applying (3.3), (3.6), and (3.7) to (3.1), we obtain the approximate first four non-central moments of $r_{xc}r_{yc}$ as follows, in terms of ρ_{xc} and ρ_{yc} and the moments and the covariance of r_{xc} and r_{yc} that are the functions of ρ_{xc}, ρ_{yc}, and ρ_{xy} (cf. Equation 3.2),

$$\left.\begin{aligned}
\mu'_1 &= E(r_{xc}r_{yc}) = (b_{r_{xc}} + \rho_{xc})(b_{r_{yc}} + \rho_{yc}) + \sigma_{r_{xc},r_{yc}}\,; \\
\mu'_2 &= E(r_{xc}r_{yc})^2 = [(b_{r_{xc}} + \rho_{xc})^2 + \sigma^{(2)}_{r_{xc}}][(b_{r_{yc}} + \rho_{yc})^2 + \sigma^{(2)}_{r_{yc}}] \\
&\quad + 4(b_{r_{xc}} + \rho_{xc})(b_{r_{yc}} + \rho_{yc})\sigma_{r_{xc},r_{yc}} + 2\sigma^2_{r_{xc},r_{yc}}\,; \\
\mu'_3 &= E(r_{xc}r_{yc})^3 = -b^3_{r_{yc}}\rho^3_{xc} + b^3_{r_{xc}}(3b_{r_{yc}} - \rho_{yc})\rho^2_{yc} + \rho^3_{xc}\rho^3_{yc} + 3\rho_{xc}\rho^3_{yc}\sigma^{(2)}_{r_{xc}} + 3\rho^3_{xc}\rho_{yc}\sigma^{(2)}_{r_{yc}} \\
&\quad + 9\rho_{xc}\rho_{yc}\sigma^{(2)}_{r_{xc}}\sigma^{(2)}_{r_{yc}} + \rho^3_{yc}\sigma^{(3)}_{r_{xc}} + \rho^3_{xc}\sigma^{(3)}_{r_{yc}} + 9[\rho^2_{yc}\sigma^{(2)}_{r_{xc}} + \rho^2_{xc}(\rho^2_{yc} + \sigma^{(2)}_{r_{yc}})]\sigma_{r_{xc},r_{yc}} \\
&\quad + 18\rho_{xc}\rho_{yc}\sigma^2_{r_{xc},r_{yc}} + 3b^2_{r_{yc}}\rho_{xc}(\rho^2_{xc}\rho_{yc} + 3\rho_{yc}\sigma^{(2)}_{r_{xc}} + 3\rho_{xc}\sigma_{r_{xc},r_{yc}}) + 3b^2_{r_{xc}}\rho_{yc}[\rho_{xc}(3b^2_{r_{yc}} \\
&\quad + 3b_{r_{yc}}\rho_{yc} + \rho^2_{yc} + 3\sigma^{(2)}_{r_{yc}}) + 3\rho_{yc}\sigma_{r_{xc},r_{yc}}] + 3b_{r_{yc}}\rho_{xc}[3\rho^2_{yc}\sigma^{(2)}_{r_{xc}} + \rho^2_{xc}(\rho^2_{yc} - \sigma^{(2)}_{r_{yc}}) \\
&\quad + 6\rho_{xc}\rho_{yc}\sigma_{r_{xc},r_{yc}}] + 3b_{r_{xc}}\{b^3_{r_{yc}}\rho^2_{xc} + 3b^2_{r_{yc}}\rho^2_{xc}\rho_{yc} + 3b_{r_{yc}}[\rho^2_{yc}\sigma^{(2)}_{r_{xc}} + \rho^2_{xc}(\rho^2_{yc} + \sigma^{(2)}_{r_{yc}}) \\
&\quad + 4\rho_{xc}\rho_{yc}\sigma_{r_{xc},r_{yc}}] + \rho_{yc}[-\rho^2_{yc}\sigma^{(2)}_{r_{xc}} + \rho^2_{xc}(\rho^2_{yc} + 3\sigma^{(2)}_{r_{yc}}) + 6\rho_{xc}\rho_{yc}\sigma_{r_{xc},r_{yc}}]\}\,; \\
\mu'_4 &= E(r_{xc}r_{yc})^4 = b^4_{r_{yc}}\rho^4_{xc} + 4b^3_{r_{yc}}(4b_{r_{xc}} - \rho_{xc})\rho^3_{xc}\rho_{yc} + b^4_{r_{xc}}\rho^4_{yc} - 4b^3_{r_{xc}}\rho_{xc}\rho^4_{yc} \\
&\quad + \rho^4_{xc}\rho^4_{yc} + 6\rho^2_{xc}\rho^4_{yc}\sigma^{(2)}_{r_{xc}} + 6\rho^4_{xc}\rho^2_{yc}\sigma^{(2)}_{r_{yc}} + 36\rho^2_{xc}\rho^2_{yc}\sigma^{(2)}_{r_{xc}}\sigma^{(2)}_{r_{yc}} + 4\rho_{xc}\rho^4_{yc}\sigma^{(3)}_{r_{xc}} \\
&\quad + 4\rho^4_{xc}\rho_{yc}\sigma^{(3)}_{r_{yc}} + \rho^4_{yc}\sigma^{(4)}_{r_{xc}} + \rho^4_{xc}\sigma^{(4)}_{r_{yc}} + 16\rho_{xc}\rho_{yc}[3\rho^2_{yc}\sigma^{(2)}_{r_{xc}} + \rho^2_{xc}(\rho^2_{yc} + 3\sigma^{(2)}_{r_{yc}})]\sigma_{r_{xc},r_{yc}} \\
&\quad + 72\rho^2_{xc}\rho^2_{yc}\sigma^2_{r_{xc},r_{yc}} + 6b^2_{r_{yc}}\rho^2_{xc}[6b^2_{r_{xc}}\rho^2_{yc} + 4b_{r_{xc}}\rho_{xc}\rho^2_{yc} + 6\rho^2_{yc}\sigma^{(2)}_{r_{xc}} + \rho^2_{xc}(\rho^2_{yc} + \sigma^{(2)}_{r_{yc}}) \\
&\quad + 8\rho_{xc}\rho_{yc}\sigma_{r_{xc},r_{yc}}] + 6b^2_{r_{xc}}\rho^2_{yc}[\rho^2_{yc}\sigma^{(2)}_{r_{xc}} + \rho^2_{xc}(\rho^2_{yc} + 6\sigma^{(2)}_{r_{yc}}) + 8\rho_{xc}\rho_{yc}\sigma_{r_{xc},r_{yc}}] \\
&\quad + 4b_{r_{xc}}\rho^2_{yc}[-3\rho_{xc}\rho^2_{yc}\sigma^{(2)}_{r_{xc}} + \rho^3_{xc}(\rho^2_{yc} + 6\sigma^{(2)}_{r_{yc}}) - \rho^2_{yc}\sigma^{(3)}_{r_{xc}} + 12\rho^2_{xc}\rho_{yc}\sigma_{r_{xc},r_{yc}}] \\
&\quad + 4b_{r_{yc}}\rho_{xc}\{4b^3_{r_{xc}}\rho^3_{yc} + 6b^2_{r_{xc}}\rho_{xc}\rho^3_{yc} + 4b_{r_{xc}}\rho_{yc}[3\rho^2_{yc}\sigma^{(2)}_{r_{xc}} + \rho^2_{xc}(\rho^2_{yc} + 3\sigma^{(2)}_{r_{yc}}) \\
&\quad + 9\rho_{xc}\rho_{yc}\sigma_{r_{xc},r_{yc}}] + \rho_{xc}[6\rho^3_{yc}\sigma^{(2)}_{r_{xc}} + \rho^2_{xc}(\rho^3_{yc} - 3\rho_{yc}\sigma^{(2)}_{r_{yc}} - \sigma^{(3)}_{r_{yc}}) + 12\rho_{xc}\rho^2_{yc}\sigma_{r_{xc},r_{yc}}]\}\,.
\end{aligned}\right\} \quad (3.8)$$

The derivations for the formulae (3.8) were generated by *Mathematica* (Wolfram, 1999; cf. Appendix A). We could further substitute (3.2) into (3.8) and obtain direct expressions for the first four moments of $r_{xc}r_{yc}$ in terms of ρ_{xc}, ρ_{yc}, and ρ_{xy} as well as the sample size N, but this is not advisable because the formulae would be much more complex. For the central moments about the mean, substitution in Kendall and Stuart's equations (Kendall & Stuart, 1977, Equation 3.9, p. 58) gives (also see Appendix A for the *Mathematica* code)

$$\left.\begin{aligned}
\mu_2 &= \mu'_2 - \mu'^2_1 = (b_{r_{xc}} + \rho_{xc})^2 \sigma^{(2)}_{r_{yc}} + \sigma^{(2)}_{r_{xc}}[(b_{r_{yc}} + \rho_{yc})^2 + \sigma^{(2)}_{r_{yc}}] \\
&\quad + 2(b_{r_{xc}} + \rho_{xc})(b_{r_{yc}} + \rho_{yc})\sigma_{r_{xc},r_{yc}} + \sigma^2_{r_{xc},r_{yc}}; \\
\mu_3 &= \mu'_3 - 3\mu'_1\mu'_2 + 2\mu'^3_1 = -2b^3_{r_{yc}}\rho^3_{xc} - 3b^3_{r_{yc}}\rho_{xc}\sigma^{(2)}_{r_{xc}} - 6b_{r_{yc}}\rho^3_{xc}\sigma^{(2)}_{r_{yc}} - 3b_{r_{yc}}\rho_{xc}\sigma^{(2)}_{r_{xc}}\sigma^{(2)}_{r_{yc}} \\
&\quad + 6\rho_{xc}\rho_{yc}\sigma^{(2)}_{r_{xc}}\sigma^{(2)}_{r_{yc}} - b^3_{r_{xc}}[b^3_{r_{yc}} + 3b^2_{r_{yc}}\rho_{yc} + 2\rho^3_{yc} + 3(b_{r_{yc}} + \rho_{yc})\sigma^{(2)}_{r_{xc}}] + \rho^3_{yc}\sigma^{(3)}_{r_{xc}} + \rho^3_{xc}\sigma^{(3)}_{r_{yc}} \\
&\quad - 3[-2\rho^2_{xc}\sigma^{(2)}_{r_{yc}} + \sigma^{(2)}_{r_{xc}}(b^2_{r_{yc}} + 2b_{r_{yc}}\rho_{yc} - 2\rho^2_{yc} + \sigma^{(2)}_{r_{yc}})]\sigma_{r_{xc},r_{yc}} - 6\rho_{xc}(2b_{r_{yc}} - \rho_{yc})\sigma^2_{r_{xc},r_{yc}} \\
&\quad - 4\sigma^3_{r_{xc},r_{yc}} - 3b^2_{r_{xc}}\{b_{r_{yc}}\rho_{xc}(b^2_{r_{yc}} + 3\sigma^{(2)}_{r_{yc}}) + [3b_{r_{yc}}(b_{r_{yc}} + 2\rho_{yc}) + \sigma^{(2)}_{r_{yc}}]\sigma_{r_{xc},r_{yc}}\} \\
&\quad + 3b_{r_{xc}}\{-\sigma^{(2)}_{r_{xc}}[b^3_{r_{xc}} + 3b^2_{r_{xc}}b_{r_{yc}} + 2\rho^3_{yc} + (b_{r_{yc}} + \rho_{yc})\sigma^{(2)}_{r_{yc}}] - 2\rho_{xc}(3b^2_{r_{yc}} + \sigma^{(2)}_{r_{yc}})\sigma_{r_{xc},r_{yc}} \\
&\quad - 4(b_{r_{xc}} + \rho_{yc})\sigma^2_{r_{xc},r_{yc}}\}; \\
\mu_4 &= \mu'_4 - 4\mu'_1\mu'_3 + 6\mu'^2_1\mu'_2 - 3\mu'^4_1 = b^4_{r_{yc}}(8\rho^4_{xc} + 6\rho^2_{xc}\sigma^{(2)}_{r_{xc}}) + 6\rho^2_{xc}\rho^2_{yc}\sigma^{(2)}_{r_{xc}}\sigma^{(2)}_{r_{yc}} \\
&\quad + b^4_{r_{xc}}[3b^4_{r_{yc}} + 12b^3_{r_{yc}}\rho_{yc} + 6b^2_{r_{yc}}\rho^2_{yc} + 4b_{r_{yc}}\rho^3_{yc} + 8\rho^4_{yc} + 6(b_{r_{yc}} + \rho_{yc})^2\sigma^{(2)}_{r_{yc}}] \\
&\quad + \rho^4_{yc}\sigma^{(4)}_{r_{xc}} + \rho^4_{yc}\sigma^{(4)}_{r_{yc}} + 4[3\rho_{xc}\rho_{yc}\sigma^{(2)}_{r_{xc}}(\rho^2_{yc} - 2\sigma^{(2)}_{r_{yc}}) - \rho^3_{yc}\sigma^{(3)}_{r_{xc}} + \rho^3_{xc}(3\rho_{yc}\sigma^{(2)}_{r_{yc}} \\
&\quad - \sigma^{(3)}_{r_{yc}})]\sigma_{r_{xc},r_{yc}} + 6[\rho^2_{xc}(2\rho^2_{yc} - 5\sigma^{(2)}_{r_{yc}}) + \sigma^{(2)}_{r_{xc}}(-5\rho^2_{yc} + \sigma^{(2)}_{r_{yc}})]\sigma^2_{r_{xc},r_{yc}} - 36\rho_{xc}\rho_{yc}\sigma^3_{r_{xc},r_{yc}} \\
&\quad + 9\sigma^4_{r_{xc},r_{yc}} - 4b^3_{rr_{yc}}\rho_{xc}(3\rho_{xc}\rho_{yc}\sigma^{(2)}_{r_{xc}} + 2\rho^2_{xc}\sigma_{r_{xc},r_{yc}} - 3\sigma^{(2)}_{r_{xc}}\sigma_{r_{xc},r_{yc}}) + 4b^3_{r_{xc}}\{\rho_{xc}[3\rho^4_{yc}
\end{aligned}\right\} \quad (3.9)$$

$$\left.\begin{aligned}
&+3b_{r_{yc}}^{3}\rho_{yc}-3b_{r_{yc}}^{2}(\rho_{yc}^{2}-2\sigma_{r_{yc}}^{(2)})-3\rho_{yc}^{2}\sigma_{r_{yc}}^{(2)}+b_{r_{yc}}(2\rho_{yc}^{3}+3\rho_{yc}\sigma_{r_{yc}}^{(2)})]+[6b_{r_{yc}}^{3}+18b_{r_{yc}}^{2}\rho_{yc}\\
&-2\rho_{yc}^{3}+3\rho_{yc}\sigma_{r_{yc}}^{(2)}+3b_{r_{yc}}(2\rho_{yc}^{2}+\sigma_{r_{yc}}^{(2)})]\sigma_{r_{xc},r_{yc}}\}+6b_{r_{yc}}^{2}[\rho_{xc}^{2}(4\rho_{xc}^{2}+\sigma_{r_{xc}}^{(2)})\sigma_{r_{yc}}^{(2)}\\
&+(2\rho_{xc}^{2}+\sigma_{r_{xc}}^{(2)})\sigma_{r_{xc},r_{yc}}^{2}]+6b_{r_{xc}}^{2}[b_{r_{yc}}^{4}\rho_{xc}^{2}-2b_{r_{yc}}^{3}\rho_{xc}^{2}\rho_{yc}+b_{r_{xc}}^{4}\sigma_{r_{xc}}^{(2)}+4b_{r_{yc}}^{3}\rho_{yc}\sigma_{r_{xc}}^{(2)}\\
&+4\rho_{yc}^{4}\sigma_{r_{xc}}^{(2)}-6b_{r_{yc}}\rho_{xc}^{2}\rho_{yc}\sigma_{r_{yc}}^{(2)}+b_{r_{yc}}^{2}\sigma_{r_{xc}}^{(2)}\sigma_{r_{yc}}^{(2)}+2b_{r_{yc}}\rho_{yc}\sigma_{r_{xc}}^{(2)}\sigma_{r_{yc}}^{(2)}+\rho_{yc}^{2}\sigma_{r_{xc}}^{(2)}\sigma_{r_{yc}}^{(2)}\\
&+6b_{r_{yc}}\rho_{xc}(2\rho_{yc}^{2}+b_{r_{yc}}\rho_{yc}-2\rho_{yc}^{2}+\sigma_{r_{yc}}^{(2)})\sigma_{r_{xc},r_{yc}}+(8b_{r_{yc}}^{2}+16b_{r_{yc}}\rho_{yc}+2\rho_{yc}^{2}\\
&+\sigma_{r_{yc}}^{(2)})\sigma_{r_{xc},r_{yc}}^{2}]+4b_{r_{xc}}\{b_{r_{yc}}^{4}\rho_{xc}^{3}+2b_{r_{yc}}^{3}\rho_{xc}^{3}\rho_{yc}+3b_{r_{yc}}^{4}\rho_{xc}\sigma_{r_{xc}}^{(2)}+3b_{r_{yc}}^{3}\rho_{xc}\rho_{yc}\sigma_{r_{xc}}^{(2)}\\
&-9b_{r_{yc}}^{2}\rho_{xc}\rho_{yc}^{2}\sigma_{r_{xc}}^{(2)}+6b_{r_{yc}}\rho_{xc}\rho_{yc}^{3}\sigma_{r_{xc}}^{(2)}+6b_{r_{yc}}\rho_{xc}^{3}\rho_{yc}\sigma_{r_{yc}}^{(2)}+3b_{r_{yc}}^{2}\rho_{xc}\sigma_{r_{xc}}^{(2)}\sigma_{r_{yc}}^{(2)}\\
&-3b_{r_{yc}}\rho_{xc}\rho_{yc}\sigma_{r_{xc}}^{(2)}\sigma_{r_{yc}}^{(2)}-6\rho_{xc}\rho_{yc}^{2}\sigma_{r_{xc}}^{(2)}\sigma_{r_{yc}}^{(2)}-b_{r_{yc}}\rho_{yc}^{3}\sigma_{r_{xc}}^{(3)}-2\rho_{yc}^{4}\sigma_{r_{xc}}^{(3)}-b_{r_{yc}}\rho_{yc}^{3}\sigma_{r_{xc}}^{(3)}\\
&-\rho_{xc}^{3}\rho_{yc}\sigma_{r_{yc}}^{(3)}+3[2b_{r_{yc}}^{3}\rho_{xc}^{2}-6b_{r_{yc}}^{2}\rho_{xc}^{2}\rho_{yc}+b_{r_{yc}}^{3}\sigma_{r_{xc}}^{(2)}+3b_{r_{yc}}^{2}\rho_{yc}\sigma_{r_{xc}}^{(2)}-3b_{r_{yc}}\rho_{yc}^{2}\sigma_{r_{xc}}^{(2)}\\
&-\rho_{yc}^{3}\sigma_{r_{xc}}^{(2)}-(b_{r_{yc}}+\rho_{yc})(3\rho_{xc}^{2}-\sigma_{r_{xc}}^{(2)})\sigma_{r_{yc}}^{(2)}]\sigma_{r_{xc},r_{yc}}+3\rho_{xc}(8b_{r_{yc}}^{2}-2b_{r_{yc}}\rho_{yc}-4\rho_{yc}^{2}\\
&+\sigma_{r_{yc}}^{(2)})\sigma_{r_{xc},r_{yc}}^{2}+9(b_{r_{yc}}+\rho_{yc})\sigma_{r_{xc},r_{yc}}^{3}\}-4b_{r_{yc}}[2\rho_{xc}^{4}\sigma_{r_{yc}}^{(3)}+3\rho_{xc}^{3}\sigma_{r_{yc}}^{(2)}\sigma_{r_{xc},r_{yc}}\\
&-3\rho_{yc}\sigma_{r_{xc}}^{(2)}\sigma_{r_{xc},r_{yc}}^{2}+6\rho_{xc}^{2}\rho_{yc}(\sigma_{r_{xc}}^{(2)}\sigma_{r_{yc}}^{(2)}+2\sigma_{r_{xc},r_{yc}}^{2})+\rho_{xc}(\rho_{yc}^{3}\sigma_{r_{xc}}^{(3)}+9\rho_{yc}^{2}\sigma_{r_{xc}}^{(2)}\sigma_{r_{xc},r_{yc}}\\
&-3\sigma_{r_{xc}}^{(2)}\sigma_{r_{yc}}^{(2)}\sigma_{r_{xc},r_{yc}}-9\sigma_{r_{xc},r_{yc}}^{3})].
\end{aligned}\right\}\tag{3.9′}$$

Note that, by definition, the first central moment $\mu_1 = 0$.

3.2 Pearson Distributions

The four moments only give us a general idea about the characteristics of the distribution of the product of two dependent correlation coefficients. To better understand the distribution of the product of two dependent correlation coefficients, we need to explore the shape of the distribution. Since the Pearson distribution family (Pearson, 1895) provides approximations to a wide variety of observed distributions using only the first four moments, in the present study the Pearson distribution family is employed to obtain an approximate distribution for the product of two dependent correlation coefficients.

The types of frequency curves in the Pearson distribution family are characterized by β-coefficients, $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$. In particular, after obtaining β_1 and β_2 from the first four moments, we can plot the couplet (β_1, β_2) on the (β_1, β_2) plane illustrated in Pearson and Hartley (1972, p. 78). From the (β_1, β_2) plane we can identify the type of Pearson distributions to which the observed distribution belongs. Instead of referencing the (β_1, β_2) plane, we can also distinguish the types of Pearson distributions by evaluating the criterion κ (*kappa*) (Kendall & Stuart, 1977, Equation 6.10)

$$\kappa = \frac{\beta_1(\beta_2+3)^2}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)} \tag{3.10}$$

with the specifications illustrated in Figure 1 (cf. Elderton & Johnson, 1969, p.49).

Insert Figure 1 about here.

Type I distributions are *beta distributions* and type III distributions are *gamma distributions*. Pearson and Hartley (1972, p. 261-285) have tables for percentage points of Pearson curves for given β_1 and β_2. Elderton and Johnson (1969) provide mathematical expressions for Pearson distributions with formulae for the parameters in terms of the first four moments.

3.3 Approximate Distribution of $r_{xc}r_{yc}$

In order to apply the first four moments of $r_{xc}r_{yc}$ to Pearson distributions, we begin by noting that ρ_{xc} and ρ_{yc} are dependent, that ρ_{xy} is constrained by $\rho_{xc}\rho_{yc} \pm \sqrt{(1-\rho_{xc}^2)(1-\rho_{yc}^2)}$ (cf. Cohen & Cohen, 1983, p. 280), and that $\rho_{xc}\rho_{yc}\rho_{xy} > 0$ because $\rho_{xc}\rho_{yc}$ always has the same sign as ρ_{xy}. In addition, without loss of generality, we assume that ρ_{xc}, ρ_{yc}, and ρ_{xy} are not –1, 0, or 1, the extreme cases.

Under these conditions, we might be able to theoretically prove that the criterion $\kappa < 0$, but for simplicity, we only numerically evaluated the criterion κ. We evaluated κ value for all possible conditional triplets of ρ_{xc}, ρ_{yc}, and ρ_{xy} with an increment of .10 for each correlation coefficient and we found that the larger N, the more likely is κ to be negative. When $N > 300$,* all values of κ are negative under the conditions mentioned above. Note that the values of κ are approximate, since the β-coefficients are evaluated by the approximate moments. As N approaches infinity, the approximate value of κ will approach the true value which will be negative, as observed by examining the numerical trend of κ as N increases. Thus, by referring Figure 1, we can conclude that the distribution of the product of two dependent correlation coefficients, $r_{xc}r_{yc}$, can be approximated by a Pearson Type I distribution. Correspondingly, the density function is (cf. Elderton and Johnson, 1969; Kendall & Stuart, 1977)

* For $N < 300$, ρ_{xc} or ρ_{yc} must be greater than .10 for κ to be negative. For smaller correlations, e.g., ρ_{xc} and $\rho_{yc} \leq .10$, $0 < \kappa < 1$; and the distribution of the product of two dependent correlation coefficients, $r_{xc}r_{yc}$, can be approximated by a Pearson Type IV distribution (cf. Figure 1).

$$f(k) = f_0 \left(1 + \frac{k}{a_1}\right)^{m_1} \left(1 - \frac{k}{a_2}\right)^{m_2}, -a_1 \le k \le a_2; \ \frac{m_1}{a_1} = \frac{m_2}{a_2},$$

where $k = r_{xc} r_{yc}$, and

$$f_0 = \frac{a_1^{m_1} a_2^{m_2}}{(a_1 + a_2)^{m_1 + m_2 + 1}} \cdot \frac{1}{B(m_1 + 1, m_2 + 1)},$$

$$a_1 + a_2 = \frac{1}{2}\sqrt{\mu_2 [\beta_1 (s+2)^2 + 16(s+1)]},$$

$$s = \frac{6(\beta_2 - \beta_1 - 1)}{6 + 3\beta_1 - 2\beta_2},$$

and m_1 and m_2 are given by

$$\frac{s-2}{2} \pm \frac{s(s+2)}{2}\sqrt{\frac{\beta_1}{\beta_1 (s+2)^2 + 16(s+1)}},$$

with $m_1 < m_2$ if $\mu_3 > 0$ and $m_1 > m_2$ if $\mu_3 < 0$. Note that this provides an approximate distribution for the product of two dependent *zero-order* correlation coefficients. The results hold for the product of two dependent *partial* correlation coefficients. It is evidenced by Fisher's (1924) finding that the distribution of the sample partial correlation is that of the zero-order correlation coefficient.

4. ACCURACY OF THE APPROXIMATION

Equation (3.1) in Section 3.1 gives us expressions for the first four moments of $r_{xc}r_{yc}$ in terms of the moments and product-moments of the original correlation coefficients, r_{xc} and r_{yc}. Then, by applying approximations to the moments and product-moments of the original correlation coefficients, this current study obtains approximate expressions for the non-central moments and central moments of the product $r_{xc}r_{yc}$. However, we do not know how accurate the approximation is for practical purposes. Therefore, we now conduct a simulation study to assess the accuracy of the approximation.

4.1 Simulation Design

The parameters in this simulation study are the sample size N and the population correlations, ρ_{xc}, ρ_{yc}, and ρ_{xy}. For ρ_{xc}, ρ_{yc}, and ρ_{xy}, we choose .10, .30, and .50 as small, medium, and large correlations (see Cohen, 1988); and for the sample size, we select 28, 84, and 783, which correspond to a statistical power of .80 for small, medium, and large correlations in the social sciences (Cohen & Cohen, 1983). Table I shows the parameter specifications for this simulation study.

Insert Table I about here.

As can been seen in Table I, we do not need to include every possible combination of .10, .30, and .50, because ρ_{xc} and ρ_{yc} are symmetric in the mathematical expressions in Section 3.

Therefore, we have removed duplicate cases. In addition, under the condition $\rho_{xc}\rho_{yc}\rho_{xy} > 0$, without loss of generality we can change the sign of the relevant variable to have all three correlations positive. Thus, there are only 18 positive correlation triplets needed for this simulation study. Multiplied by the three magnitudes of N, we have $18 \times 3 = 54$ cells to simulate.

For each of 54 cells (i.e., for each set of N, ρ_{xc}, ρ_{yc}, ρ_{xy}), by Cholesky factorization, we generated X, Y, and C with the specified population correlations (cf. Browne, 1968) and with the sample size as N. For simplicity, we generated (X, Y, C) as a trivariate standard normal distribution given values of ρ_{xc}, ρ_{yc}, and ρ_{xy}. Then, we computed the pair of estimated correlation coefficients $\tilde{r}_{xc}$ and $\tilde{r}_{yc}$.* Within each cell, the same procedure is replicated 1000 times, resulting in 1000 pairs of $\tilde{r}_{xc}$ and $\tilde{r}_{yc}$ values. Directly multiplying $\tilde{r}_{xc}$ and $\tilde{r}_{yc}$ generates a simulated distribution of the product $r_{xc}r_{yc}$ which serves as the true distribution of $r_{xc}r_{yc}$, with which the approximation can be compared. The 1000 replications are sufficiently large to yield precise results of the moments for the simulated distribution.

4.2 Moments Comparison

Tables II and III show the simulation results for the first four moments that were computed using the following formulae:

$$\tilde{\mu}'_1 = \frac{1}{1000}\sum_{i=1}^{1000}(\tilde{r}_{xc}\tilde{r}_{yc})_i$$

$$\tilde{\mu}_2 = \frac{1}{1000}\sum_{i=1}^{1000}[(\tilde{r}_{xc}\tilde{r}_{yc})_i - \tilde{\mu}'_1]^2$$

* Here, and later, we use "~" to distinguish the simulated values from the corresponding observed values.

$$\tilde{\mu}_3 = \frac{1}{1000}\sum_{i=1}^{1000}[(\tilde{r}_{xc}\tilde{r}_{yc})_i - \tilde{\mu}'_1]^3$$

$$\tilde{\mu}_4 = \frac{1}{1000}\sum_{i=1}^{1000}[(\tilde{r}_{xc}\tilde{r}_{yc})_i - \tilde{\mu}'_1]^4$$

Insert Tables II and III about here.

In order to quantify the accuracy of the approximation to the first four moments, the relative differences (rel. diff. = (Approximated – Simulated)/Simulated) between the simulated values and the approximated values[*] for the four moments are also listed in Tables II and III. The rationale of using the relative difference, instead of the absolute difference as did Ghosh (1966) for his approximation study, is that the relative difference would provide information not only about whether the approximated value overestimates or underestimates the true or simulated value, and also about whether the difference is big or small relative to the true or simulated value.

As can be seen in Tables II the approximation method proposed in this present study yields very accurate approximated values for the mean and the variance. We acknowledge that the approximation is not very favorable to the third and the fourth moments for certain cells, which is evidenced by several large relative differences in Table III. But, if we examine the absolute differences for the third the and the fourth moments, the approximation would still very

[*] The approximated moments were calculated by *SPSS* (see Appendix B for the *SPSS* code). To make the code clearer, we computed Equations (3.3), (3.6), and (3.7) and substituted them into (3.1), instead of directly using the more complex equations (3.9) and (3.9′).

encouragingly yield values of the third moment correct to three decimal places when $N = 28$, four decimal places when $N = 84$, and five decimal places when $N = 783$; and the approximation yields values of the four moment correct to four decimal places when $N = 28$, five decimal places when $N = 84$, and seven decimal places when $N = 783$. Therefore, in general, the approximation is adequate and useful.

4.3 Distribution Comparison

We have compared the approximated distribution of the product of two dependent correlation coefficients with a simulated distribution in terms of the first four moments. However, we usually use *p*-values obtained from the shape of a distribution, rather than the moments, to make statistical inferences. Thus, it is more desirable to compare the shape of the approximated distribution with the simulated distribution to validate the statistical decisions that would be made based on the approximated distribution. After obtaining the distribution function for the approximated distribution of the product of two dependent correlation coefficients as a Pearson Type I distribution, the distribution comparison becomes achievable. A P-P plot is employed for this comparison.

A P-P plot, a probability plot, is a graphical tool for assessing the fit of data to a theoretical distribution (Rice, 1995, p. 321). Specifically, for a given sample data $X_1, \ldots, X_n$, we plot $X_{(i)}$ versus $F^{-1}(i/(n+1))$, where $X_{(i)}$, $i = 1, \ldots, n$, are the order statistics of $X_1, \ldots, X_n$, and F is the cumulative distribution function of the theoretical distribution.

Although we do not know the theoretical distribution of the product of two dependent correlation coefficients, we have the approximated Pearson Type I distribution. Then, we can compare the simulated distribution with the approximated Pearson Type I distribution to validate the approximation of the distribution of the product of two dependent correlation coefficients to the Pearson Type I distribution. Thus, here in the P-P plot, we will plot the quantiles of the Pearson Type I distribution against those of the simulated distribution for the product of two dependent correlation coefficients. Due to the complexity of the calculations of its mathematical function for obtaining every quantile of the Pearson Type I distribution, we used the common 15 percentiles, .25%, .5%, 1%, 2.5%, 5%, 10%, 25%, 50%, 75%, 90%, 95%, 97.5%, 99%, 99.5%, and 99.75%, whose quantiles can be looked up in Pearson and Hartley's table (Pearson & Hartley, 1972, Table 32). Setting $i/(n + 1)$ = .25%, .5%, 1%, 2.5%, 5%, 10%, 25%, 50%, 75%, 90%, 95%, 97.5%, 99%, 99.5%, or 99.75%, and noting that $n = 1000$ in this (simulation) study, we obtained the corresponding ith order statistic $X_{(i)}$ in the simulated distribution. Therefore, in our P-P plot there are only 15 points, instead of 1000 points, each corresponding to one of the 15 percentiles above.

Insert Figure 2 about here.

We argued in Section 3.3 that if the correlations ≤ .10 and $N \leq 300$, the distribution of the product of two correlations can be approximated as the Pearson Type IV distribution (cf. the footnote on page 16). Thus, in Figure 2, we only display two P-P plots that have medium (.30) or large (.50) correlations with the corresponding sample sizes (84 and 783, respectively) (cf. Cohen & Cohen, 1983). From the P-P plots in Figure 2, we can see that the approximated Pearson Type I distribution fit the simulated data very well. In addition, Figure 2 shows that when the correlations are larger and N becomes bigger, the approximated Pearson Type I distribution fit the simulated data considerably better.

Insert Figure 3 about here.

Figure 3 is a P-P plot comparing the approximated Pearson Type I distribution with Frank's approximation. By Frank's (2000) approach, Fisher z's, $z(r_{xc})$ and $z(r_{yc})$, were obtained from the simulated correlations, and then, the quantiles of the adjusted product $z(r_{xc})\times z(r_{yc})$ were obtained from Meeker, Cornwell, and Aroian's (1981) tables (see Frank, 2000, p.174 for the detailed steps). Figure 3 shows that the approximated Pearson Type I distribution fits much better to the simulated data than does Frank's distribution. In general, Frank's distribution is too conservative. For example, the quantile that corresponds to 95% percentile for the approximated Pearson Type I distribution only corresponds to 50% percentile for Frank's distribution. Therefore, this figure provides evidence that the approximation procedures using the Pearson Type I distribution are more accurate than Frank's approach for approximating the distribution of the product of two dependent correlation coefficients.

5. REMAKES AND EXTENSIONS

The current study provided a much more accurate approximation to the distribution of the product of two correlation coefficients with a closed form—Pearson Type I distribution than Frank's (cf. Figure 3). A simulation study was conducted to assess the accuracy of the approximation method in the present study. We also understand that the approximation is comparatively not very favorable with respect to the third and the fourth moments. The problem may come from the lower-order approximation to the covariance of r_{xc} and r_{yc} (cf. the expression of $\sigma_{r_{xc},r_{yc}}$ in Equation 3.2). Thus, it would be valuable to find a better approximation to the covariance of r_{xc} and r_{yc} through further study.

An alternative approach of dealing with the comparatively less accurate approximation for the third and four moments would be to borrow the principle of regression analysis in which we use predictors to explain as much variation in an outcome as possible through a linear model. In this light, we can model the deviations of the simulated values from the approximated values of the moments as a function of ρ_{xc}, ρ_{yc}, ρ_{xy} and N. We then can use the predicted value of the deviation, $\hat{D}$, to correct the inaccuracy of the approximation to the moments. The corrected or estimated value for the approximation value, denoted as $\widetilde{A} = A + \hat{D}$, is theoretically assumed to be more accurate than the approximated value, A. The virtue of this methodology is to utilize all the available information borne in the simulation data across ρ_{xc}, ρ_{yc}, ρ_{xy}, and N to correct the inaccuracy of the approximation. This method applies because the range of parameters in the experimental design is restricted by theory, constraining ρ's according to what are defined as small, medium, and large effects.

Pan (2002) shows that the estimated values via the regression correction for the approximated Pearson Type I distribution fit much better to the simulated data than the original values of the approximated Pearson Type I distribution. The improved results come from the better estimation for the third and the fourth moments that regulate the tails of the distribution. Thus, the better results suggest that the regression approach be a helpful methodology to correct the inaccuracy of the approximated values.

As mentioned in the introduction section, the approximation can be used to conduct more valid statistical inferences on measures of the impact of confounding variables than did Frank (2000); and we can establish a standard measure of the robustness of a causal inference to confounding variables. The approximation can also be used to assess indirect effects in path or structural equation models. The current techniques (e.g., Sobel, 1982) of statistical inferences about indirect effects have relied exclusively on asymptotic methods which assume that the limiting distribution of the estimator is normal, with a standard error derived from the delta method. However, the fact that the techniques are only large-sample approximations has been cause for some concern (Allison, 1995; Bollen & Stine, 1990). As Bollen (1987) warned, "the accuracy of the delta method for small sample is not known." By using the more accurate approximation described in the present study, we can obtain a better estimate for the indirect effects in path or structural equation models.

It would also be valuable to extend the simulation range for the correlation coefficients beyond .50. Although .50 is a large size for correlation coefficients in the social sciences (Cohen & Cohen, 1983), in the real world we often see correlation coefficients larger than .50. Thus, it would be a valuable exploration to see if the conclusions of the current study still hold for correlation coefficients larger than .50. That is, it is important to know whether the Pearson Type I distribution still can be a reasonable approximate distribution for the product of two dependent correlation coefficients when some population correlations are larger than .50.

Lastly, note that the approximation method in the present study is based on the assumption that the three initial variables, X, Y, and C, follow a trivariate normal distribution. It would be valuable to find an approximation to the distribution of the product of two dependent correlation coefficients for non-normal, categorical, or mixed initial variables, X, Y, and C.

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APPENDIX A: *MATHEMATICA* CODE

The derivations of the central moments and the non-central moments were conducted by *Mathematica*. See the following code and outputs for reference, where

pxc = ρ_{xc}; bxc = $b_{r_{xc}}$; s2xc = $\sigma^{(2)}_{r_{xc}}$; s3xc = $\sigma^{(3)}_{r_{xc}}$; s4xc = $\sigma^{(4)}_{r_{xc}}$;
pyc = ρ_{yc}; byc = $b_{r_{yc}}$; s2yc = $\sigma^{(2)}_{r_{yc}}$; s3yc = $\sigma^{(3)}_{r_{yc}}$; s4yc = $\sigma^{(4)}_{r_{yc}}$;
pxy = ρ_{xy}; sxcyc = $\sigma_{r_{xc},r_{yc}}$; e*kl* = $E[(\Delta r_{xc})^k(\Delta r_{yc})^l]$, k, l = 0, 1, 2, 3, or 4;
mu0*i* = μ'_i, mu*i* = μ_i, i = 1, 2, 3, or 4.

```
In[1]:= e10 = bxc
        e01 = byc
        e20 = s2xc + bxc ^ 2
        e02 = s2yc + byc ^ 2
        e30 = s3xc - 3 s2xc * bxc - bxc ^ 3
        e03 = s3yc - 3 s2yc * byc - byc ^ 3
        e40 = s4xc - 4 s3xc * bxc + 6 s2xc * bxc ^ 2 + bxc ^4
        e04 = s4yc - 4 s3yc * byc + 6 s2yc * byc ^ 2 + byc ^4
        e11 = sxcyc + bxc * byc
        e21 = s2xc * byc + 2 sxcyc * bxc + bxc ^ 2 * byc
        e12 = s2yc * bxc + 2 sxcyc * byc + byc ^ 2 * bxc
        e31 = 3 s2xc * sxcyc + 3 s2xc * bxc * byc + 3 sxcyc * bxc ^ 2 + bxc ^ 3 * byc
        e13 = 3 s2yc * sxcyc + 3 s2yc * bxc * byc + 3 sxcyc * byc ^ 2 + byc ^ 3 * bxc
        e22 = s2xc * s2yc + 2 sxcyc ^ 2 + s2xc * byc ^ 2 + s2yc * bxc ^ 2 + 4 sxcyc * bxc * byc + bxc ^ 2 * byc ^ 2

In[15]:= mu01 = Expand[pxc * pyc + pxc * e01 + pyc * e10 + e11]

Out[15]= bxc byc + byc pxc + bxc pyc + pxc pyc + sxcyc

In[16]:= mu02 = Expand[pxc ^ 2 * pyc ^ 2 + pxc ^ 2 * e02 + pyc ^ 2 * e20 + e22 + 2 pxc ^ 2 * pyc * e01 +
           2 pxc * pyc ^ 2 * e10 + 2 pyc * e21 + 2 pxc * e12 + 4 pxc * pyc * e11]

Out[16]= bxc^2 byc^2 + 2 bxc byc^2 pxc + byc^2 pxc^2 + 2 bxc^2 byc pyc + 4 bxc byc pxc pyc + 2 byc pxc^2 pyc + bxc^2 pyc^2 +
          2 bxc pxc pyc^2 + pxc^2 pyc^2 + byc^2 s2xc + 2 byc pyc s2xc + pyc^2 s2xc + bxc^2 s2yc + 2 bxc pxc s2yc +
          pxc^2 s2yc + s2xc s2yc + 4 bxc byc sxcyc + 4 byc pxc sxcyc + 4 bxc pyc sxcyc + 4 pxc pyc sxcyc + 2 sxcyc^2
```

```
In[17]:= mu03 = Expand[pxc ^ 3 * pyc ^ 3 + pxc ^ 3 * e03 + pyc ^ 3 * e30 + 3 pxc ^ 3 * pyc ^ 2 * e01 +
    3 pxc ^ 3 * pyc * e02 + 3 pxc ^ 2 * pyc ^ 3 * e10 + 3 pxc * pyc ^ 3 * e20 + 3 pxc ^ 2 * e13 +
    3 pyc ^ 2 * e31 + 9 pxc ^ 2 * pyc ^ 2 * e11 + 9 pxc ^ 2 * pyc * e12 + 9 pxc * pyc ^ 2 * e21 + 9 pxc * pyc * e22]
```

Out[17]=
$$3 bxc byc^3 pxc^2 - byc^3 pxc^3 + 9 bxc^2 byc^2 pxc pyc + 9 bxc byc^2 pxc^2 pyc + 3 byc^2 pxc^3 pyc + 3 bxc^3 byc pyc^2 + 9 bxc^2 byc pxc pyc^2 + 9 bxc byc pxc^2 pyc^2 + 3 byc pxc^3 pyc^2 - bxc^3 pyc^3 + 3 bxc^2 pxc pyc^3 + 3 bxc pxc^2 pyc^3 + pxc^3 pyc^3 + 9 byc^2 pxc pyc s2xc + 9 bxc byc pyc^2 s2xc + 9 byc pxc pyc^2 s2xc - 3 bxc pyc^3 s2xc + 3 pxc pyc^3 s2xc + 9 bxc byc pxc^2 s2yc - 3 byc pxc^3 s2yc + 9 bxc^2 pxc pyc s2yc + 9 bxc pxc^2 pyc s2yc + 3 pxc^3 pyc s2yc + 9 pxc pyc s2xc s2yc + pyc^3 s3xc + pxc^3 s3yc + 9 byc^2 pxc^2 sxcyc + 36 bxc byc pxc pyc sxcyc + 18 byc pxc^2 pyc sxcyc + 9 bxc^2 pyc^2 sxcyc + 18 bxc pxc pyc^2 sxcyc + 9 pxc^2 pyc^2 sxcyc + 9 pyc^2 s2xc sxcyc + 9 pxc^2 s2yc sxcyc + 18 pxc pyc sxcyc^2$$

```
In[18]:= mu04 = Expand[pxc ^4 * pyc ^4 + pxc ^ 4 * e04 + pyc ^4 * e40 + 4 pxc ^4 * pyc ^ 3 * e01 +
    4 pxc ^ 3 * pyc ^4 * e10 + 4 pxc ^4 * pyc * e03 + 4 pxc * pyc ^4 * e30 + 6 pxc ^ 4 * pyc ^ 2 * e02 +
    6 pxc ^ 2 * pyc ^4 * e20 + 16 pxc ^ 3 * pyc ^ 3 * e11 + 16 pxc ^ 3 * pyc * e13 + 16 pxc * pyc ^ 3 * e31 +
    24 pxc ^ 3 * pyc ^ 2 * e12 + 24 pxc ^ 2 * pyc ^ 3 * e21 + 36 pxc ^ 2 * pyc ^ 2 * e22]
```

Out[18]=
$$byc^4 pxc^4 + 16 bxc byc^3 pxc^3 pyc - 4 byc^3 pxc^4 pyc + 36 bxc^2 byc^2 pxc^2 pyc^2 + 24 bxc byc^2 pxc^3 pyc^2 + 6 byc^2 pxc^4 pyc^2 + 16 bxc^3 byc pxc pyc^3 + 24 bxc^2 byc pxc^2 pyc^3 + 16 bxc byc pxc^3 pyc^3 + 4 byc pxc^4 pyc^3 + bxc^4 pyc^4 - 4 bxc^3 pxc pyc^4 + 6 bxc^2 pxc^2 pyc^4 + 4 bxc pxc^3 pyc^4 + pxc^4 pyc^4 + 36 byc^2 pxc^2 pyc^2 s2xc + 48 bxc byc pxc pyc^3 s2xc + 24 byc pxc^2 pyc^3 s2xc + 6 bxc^2 pyc^4 s2xc - 12 bxc pxc pyc^4 s2xc + 6 pxc^2 pyc^4 s2xc + 6 byc^2 pxc^4 s2yc + 48 bxc byc pxc^3 pyc s2yc - 12 byc pxc^4 pyc s2yc + 36 bxc^2 pxc^2 pyc^2 s2yc + 24 bxc pxc^3 pyc^2 s2yc + 6 pxc^4 pyc^2 s2yc + 36 pxc^2 pyc^2 s2xc s2yc - 4 bxc pyc^4 s3xc + 4 pxc pyc^4 s3xc - 4 byc pxc^4 s3yc + 4 pxc^4 pyc s3yc + pyc^4 s4xc + pxc^4 s4yc + 48 byc^2 pxc^3 pyc sxcyc + 144 bxc byc pxc^2 pyc^2 sxcyc + 48 byc pxc^3 pyc^2 sxcyc + 48 bxc^2 pxc pyc^3 sxcyc + 48 bxc pxc^2 pyc^3 sxcyc + 16 pxc^3 pyc^3 sxcyc + 48 pxc pyc^3 s2xc sxcyc + 48 pxc^3 pyc s2yc sxcyc + 72 pxc^2 pyc^2 sxcyc^2$$

```
In[19]:= FullSimplify[mu01]
```

Out[19]=
$$(bxc + pxc) (byc + pyc) + sxcyc$$

```
In[20]:= FullSimplify[mu02]
```

Out[20]=
$$((bxc + pxc)^2 + s2xc) ((byc + pyc)^2 + s2yc) + 4 (bxc + pxc) (byc + pyc) sxcyc + 2 sxcyc^2$$

```
In[21]:= FullSimplify[mu03]
```

Out[21]=
$$-byc^3 pxc^3 + bxc^3 (3 byc - pyc) pyc^2 + pxc^3 pyc^3 + 3 pxc pyc^3 s2xc + 3 pxc^3 pyc s2yc + 9 pxc pyc s2xc s2yc + pyc^3 s3xc + pxc^3 s3yc + 9 (pyc^2 s2xc + pxc^2 (pyc^2 + s2yc)) sxcyc + 18 pxc pyc sxcyc^2 + 3 byc^2 pxc (pxc^2 pyc + 3 pyc s2xc + 3 pxc sxcyc) + 3 bxc^2 pyc (pxc (3 byc^2 + 3 byc pyc + pyc^2 + 3 s2yc) + 3 pyc sxcyc) + 3 byc pxc (3 pyc^2 s2xc + pxc^2 (pyc^2 - s2yc) + 6 pxc pyc sxcyc) + 3 bxc (byc^3 pxc^2 + 3 byc^2 pxc^2 pyc + 3 byc (pyc^2 s2xc + pxc^2 (pyc^2 + s2yc) + 4 pxc pyc sxcyc) + pyc (-pyc^2 s2xc + pxc^2 (pyc^2 + 3 s2yc) + 6 pxc pyc sxcyc))$$

In[22]:= **FullSimplify[mu04]**

$$
\begin{aligned}
\text{Out[22]=}\ & byc^4 pxc^4 + 4\, byc^3\, (4\, bxc - pxc)\, pxc^3\, pyc + bxc^4 pyc^4 - 4\, bxc^3\, pxc\, pyc^4 + pxc^4 pyc^4 + 6\, pxc^2\, pyc^4\, s2xc + \\
& 6\, pxc^4 pyc^2\, s2yc + 36\, pxc^2\, pyc^2\, s2xc\, s2yc + 4\, pxc\, pyc^4\, s3xc + 4\, pxc^4\, pyc\, s3yc + pyc^4\, s4xc + \\
& pxc^4\, s4yc + 16\, pxc\, pyc\, (3\, pyc^2\, s2xc + pxc^2\, (pyc^2 + 3\, s2yc))\, sxcyc + 72\, pxc^2\, pyc^2\, sxcyc^2 + \\
& 6\, byc^2\, pxc^2\, (6\, bxc^2\, pyc^2 + 4\, bxc\, pxc\, pyc^2 + 6\, pyc^2\, s2xc + pxc^2\, (pyc^2 + s2yc) + 8\, pxc\, pyc\, sxcyc) + \\
& 6\, bxc^2\, pyc^2\, (pyc^2\, s2xc + pxc^2\, (pyc^2 + 6\, s2yc) + 8\, pxc\, pyc\, sxcyc) + \\
& 4\, bxc\, pyc^2\, (-3\, pxc\, pyc^2\, s2xc + pxc^3\, (pyc^2 + 6\, s2yc) - pyc^2\, s3xc + 12\, pxc^2\, pyc\, sxcyc) + 4\, byc\, pxc \\
& \quad (4\, bxc^3\, pyc^3 + 6\, bxc^2\, pxc\, pyc^3 + 4\, bxc\, pyc\, (3\, pyc^2\, s2xc + pxc^2\, (pyc^2 + 3\, s2yc) + 9\, pxc\, pyc\, sxcyc) + \\
& \qquad pxc\, (6\, pyc^3\, s2xc + pxc^2\, (pyc^3 - 3\, pyc\, s2yc - s3yc) + 12\, pxc\, pyc^2\, sxcyc))
\end{aligned}
$$

In[23]:= **mu2 = Expand[mu02 - mu01 ^ 2]**

$$
\begin{aligned}
\text{Out[23]=}\ & byc^2\, s2xc + 2\, byc\, pyc\, s2xc + pyc^2\, s2xc + bxc^2\, s2yc + 2\, bxc\, pxc\, s2yc + pxc^2\, s2yc + \\
& s2xc\, s2yc + 2\, bxc\, byc\, sxcyc + 2\, byc\, pxc\, sxcyc + 2\, bxc\, pyc\, sxcyc + 2\, pxc\, pyc\, sxcyc + sxcyc^2
\end{aligned}
$$

In[24]:= **mu3 = Expand[mu03 - 3 mu01 * mu02 + 2 mu01 ^ 3]**

$$
\begin{aligned}
\text{Out[24]=}\ & -bxc^3\, byc^3 - 3\, bxc^2\, byc^3\, pxc - 2\, byc^3\, pxc^3 - 3\, bxc^3\, byc^2\, pyc - 2\, bxc^3\, pyc^3 - 3\, bxc\, byc^3\, s2xc - \\
& 3\, byc^3\, pxc\, s2xc - 9\, bxc\, byc^2\, pyc\, s2xc - 6\, bxc\, pyc^3\, s2xc - 3\, bxc^3\, byc\, s2yc - 9\, bxc^2\, byc\, pxc\, s2yc - \\
& 6\, byc\, pxc^3\, s2yc - 3\, bxc^3\, pyc\, s2yc - 3\, bxc\, byc\, s2xc\, s2yc - 3\, byc\, pxc\, s2xc\, s2yc - 3\, bxc\, pyc\, s2xc\, s2yc + \\
& 6\, pxc\, pyc\, s2xc\, s2yc + pyc^3\, s3xc + pxc^3\, s3yc - 9\, bxc^2\, byc^2\, sxcyc - 18\, bxc\, byc^2\, pxc\, sxcyc - \\
& 18\, bxc^2\, byc\, pyc\, sxcyc - 3\, byc^2\, s2xc\, sxcyc - 6\, byc\, pyc\, s2xc\, sxcyc + 6\, pyc^2\, s2xc\, sxcyc - \\
& 3\, bxc^2\, s2yc\, sxcyc - 6\, bxc\, pxc\, s2yc\, sxcyc + 6\, pxc^2\, s2yc\, sxcyc - 3\, s2xc\, s2yc\, sxcyc - \\
& 12\, bxc\, byc\, sxcyc^2 - 12\, byc\, pxc\, sxcyc^2 - 12\, bxc\, pyc\, sxcyc^2 + 6\, pxc\, pyc\, sxcyc^2 - 4\, sxcyc^3
\end{aligned}
$$

```
In[25]:= mu4 = Expand[mu04 - 4 mu01 * mu03 + 6 mu01^2 * mu02 - 3 mu01^4]

Out[25]= 3 bxc^4 byc^4 + 12 bxc^3 byc^4 pxc + 6 bxc^2 byc^4 pxc^2 + 4 bxc byc^4 pxc^3 + 8 byc^4 pxc^4 + 12 bxc^4 byc^3 pyc +
  12 bxc^3 byc^3 pxc pyc - 12 bxc^2 byc^3 pxc^2 pyc + 8 bxc byc^3 pxc^3 pyc + 6 bxc^4 byc^2 pyc^2 -
  12 bxc^3 byc^2 pxc pyc^2 + 4 bxc^4 byc pyc^3 + 8 bxc^3 byc pxc pyc^3 + 8 bxc^4 pyc^4 + 6 bxc^2 byc^4 s2xc +
  12 bxc byc^4 pxc s2xc + 6 byc^4 pxc^2 s2xc + 24 bxc^2 byc^3 pyc s2xc + 12 bxc byc^3 pxc pyc s2xc -
  12 byc^3 pxc^2 pyc s2xc - 36 bxc byc^2 pxc pyc^2 s2xc + 24 bxc byc pxc pyc^3 s2xc + 24 bxc^2 pyc^4 s2xc +
  6 bxc^4 byc^2 s2yc + 24 bxc^3 byc^2 pxc s2yc + 24 byc^2 pxc^4 s2yc + 12 bxc^4 byc pyc s2yc +
  12 bxc^3 byc pxc pyc s2yc - 36 bxc^2 byc pxc^2 pyc s2yc + 24 bxc byc pxc^3 pyc s2yc + 6 bxc^4 pyc^2 s2yc -
  12 bxc^3 pxc pyc^2 s2yc + 6 bxc^2 byc^2 s2xc s2yc + 12 bxc byc^2 pxc s2xc s2yc + 6 byc^2 pxc^2 s2xc s2yc +
  12 bxc^2 byc pyc s2xc s2yc - 12 bxc byc pxc pyc s2xc s2yc - 24 byc pxc^2 pyc s2xc s2yc +
  6 bxc^2 pyc^2 s2xc s2yc - 24 bxc pxc pyc^2 s2xc s2yc + 6 pxc^2 pyc^2 s2xc s2yc - 4 bxc byc pyc^3 s3xc -
  4 byc pxc pyc^3 s3xc - 8 bxc pyc^4 s3xc - 4 bxc byc pxc^3 s3yc - 8 byc pxc^4 s3yc - 4 bxc pxc^3 pyc s3yc +
  pyc^4 s4xc + pxc^4 s4yc + 24 bxc^3 byc^3 sxcyc + 72 bxc^2 byc^3 pxc sxcyc + 24 bxc byc^3 pxc^2 sxcyc -
  8 byc^3 pxc^3 sxcyc + 72 bxc^3 byc^2 pyc sxcyc + 36 bxc^2 byc^2 pxc pyc sxcyc - 72 bxc byc^2 pxc^2 pyc sxcyc +
  24 bxc^3 byc pyc^2 sxcyc - 72 bxc^2 byc pxc pyc^2 sxcyc - 8 bxc^3 pyc^3 sxcyc + 12 bxc byc^3 s2xc sxcyc +
  12 byc^3 pxc s2xc sxcyc + 36 bxc byc^2 pyc s2xc sxcyc - 36 bxc byc pyc^2 s2xc sxcyc -
  36 byc pxc pyc^2 s2xc sxcyc - 12 bxc pyc^3 s2xc sxcyc + 12 pxc pyc^3 s2xc sxcyc +
  12 bxc^3 byc s2yc sxcyc + 36 bxc^2 byc pxc s2yc sxcyc - 36 bxc byc pxc^2 s2yc sxcyc -
  12 byc pxc^3 s2yc sxcyc + 12 bxc^3 pyc s2yc sxcyc - 36 bxc pxc^2 pyc s2yc sxcyc +
  12 pxc^3 pyc s2yc sxcyc + 12 bxc byc s2xc s2yc sxcyc + 12 byc pxc s2xc s2yc sxcyc +
  12 bxc pyc s2xc s2yc sxcyc - 24 pxc pyc s2xc s2yc sxcyc - 4 pyc^3 s3xc sxcyc - 4 pxc^3 s3yc sxcyc +
  48 bxc^2 byc^2 sxcyc^2 + 96 bxc byc^2 pxc sxcyc^2 + 12 byc^2 pxc^2 sxcyc^2 + 96 bxc^2 byc pyc sxcyc^2 -
  24 bxc byc pxc pyc sxcyc^2 - 48 byc pxc^2 pyc sxcyc^2 + 12 bxc^2 pyc^2 sxcyc^2 - 48 bxc pxc pyc^2 sxcyc^2 +
  12 pxc^2 pyc^2 sxcyc^2 + 6 byc^2 s2xc sxcyc^2 + 12 byc pyc s2xc sxcyc^2 - 30 pyc^2 s2xc sxcyc^2 +
  6 bxc^2 s2yc sxcyc^2 + 12 bxc pxc s2yc sxcyc^2 - 30 pxc^2 s2yc sxcyc^2 + 6 s2xc s2yc sxcyc^2 +
  36 bxc byc sxcyc^3 + 36 byc pxc sxcyc^3 + 36 bxc pyc sxcyc^3 - 36 pxc pyc sxcyc^3 + 9 sxcyc^4

In[26]:= FullSimplify[mu2]

Out[26]= (bxc + pxc)^2 s2yc + s2xc ((byc + pyc)^2 + s2yc) + 2 (bxc + pxc) (byc + pyc) sxcyc + sxcyc^2

In[27]:= FullSimplify[mu3]

Out[27]= -2 byc^3 pxc^3 - 3 byc^2 pxc s2xc - 6 byc pxc^2 s2yc - 3 byc pxc s2xc s2yc + 6 pxc pyc s2xc s2yc -
  bxc^3 (byc^3 + 3 byc^2 pyc + 2 pyc^3 + 3 (byc + pyc) s2yc) + pyc^3 s3xc + pxc^3 s3yc -
  3 (-2 pxc^2 s2yc + s2xc (byc^2 + 2 byc pyc - 2 pyc^2 + s2yc)) sxcyc - 6 pxc (2 byc - pyc) sxcyc^2 -
  4 sxcyc^3 - 3 bxc^2 (byc pxc (byc^2 + 3 s2yc) + (3 byc (byc + 2 pyc) + s2yc) sxcyc) +
  3 bxc (-s2xc (byc^3 + 3 byc^2 pyc + 2 pyc^3 + (byc + pyc) s2yc) -
    2 pxc (3 byc^2 + s2yc) sxcyc - 4 (byc + pyc) sxcyc^2)
```

```
In[28]:= FullSimplify[mu4]

Out[28]= byc^4 (8 pxc^4 + 6 pxc^2 s2xc) + 6 pxc^2 pyc^2 s2xc s2yc +
    bxc^4 (3 byc^4 + 12 byc^3 pyc + 6 byc^2 pyc^2 + 4 byc pyc^3 + 8 pyc^4 + 6 (byc + pyc)^2 s2yc) + pyc^4 s4xc +
    pxc^4 s4yc + 4 (3 pxc pyc s2xc (pyc^2 - 2 s2yc) - pyc^3 s3xc + pxc^3 (3 pyc s2yc - s3yc)) sxcyc +
    6 (pxc^2 (2 pyc^2 - 5 s2yc) + s2xc (-5 pyc^2 + s2yc)) sxcyc^2 - 36 pxc pyc sxcyc^3 +
    9 sxcyc^4 - 4 byc^3 pxc (3 pxc pyc s2xc + 2 pxc^2 sxcyc - 3 s2xc sxcyc) +
    4 bxc^3 (pxc (3 byc^4 + 3 byc^3 pyc - 3 byc^2 (pyc^2 - 2 s2yc) - 3 pyc^2 s2yc + byc (2 pyc^3 + 3 pyc s2yc)) +
        (6 byc^3 + 18 byc^2 pyc - 2 pyc^3 + 3 pyc s2yc + 3 byc (2 pyc^2 + s2yc)) sxcyc) +
    6 byc^2 (pxc^2 (4 pxc^2 + s2xc) s2yc + (2 pxc^2 + s2xc) sxcyc^2) +
    6 bxc^2 (byc^4 pxc^2 - 2 byc^3 pxc^2 pyc + byc^4 s2xc + 4 byc^3 pyc s2xc + 4 pyc^4 s2xc -
        6 byc pxc^2 pyc s2yc + byc^2 s2xc s2yc + 2 byc pyc s2xc s2yc + pyc^2 s2xc s2yc +
        6 byc pxc (2 byc^2 + byc pyc - 2 pyc^2 + s2yc) sxcyc + (8 byc^2 + 16 byc pyc + 2 pyc^2 + s2yc) sxcyc^2) +
    4 bxc (byc^4 pxc^3 + 2 byc^3 pxc^3 pyc + 3 byc^4 pxc s2xc + 3 byc^3 pxc pyc s2xc -
        9 byc^2 pxc pyc^2 s2xc + 6 byc pxc pyc^3 s2xc + 6 byc pxc^3 pyc s2yc + 3 byc^2 pxc s2xc s2yc -
        3 byc pxc pyc s2xc s2yc - 6 pxc pyc^2 s2xc s2yc - byc pyc^3 s3xc - 2 pyc^4 s3xc -
        byc pxc^3 s3yc - pxc^3 pyc s3yc + 3 (2 byc^3 pxc^2 - 6 byc^2 pxc^2 pyc + byc^3 s2xc +
            3 byc^2 pyc s2xc - 3 byc pyc^2 s2xc - pyc^3 s2xc - (byc + pyc) (3 pxc^2 - s2xc) s2yc) sxcyc +
        3 pxc (8 byc^2 - 2 byc pyc - 4 pyc^2 + s2yc) sxcyc^2 + 9 (byc + pyc) sxcyc^3) -
    4 byc (2 pxc^4 s3yc + 3 pxc^3 s2yc sxcyc - 3 pyc s2xc sxcyc^2 + 6 pxc^2 pyc (s2xc s2yc + 2 sxcyc^2) +
        pxc (pyc^3 s3xc + 9 pyc^2 s2xc sxcyc - 3 s2xc s2yc sxcyc - 9 sxcyc^3))
```

APPENDIX B: *SPSS* CODE

Below is the *SPSS* code for calculating the approximate moments for $N = 28$ or $M = N + 6 = 34$ as an example, where

bxc = $b_{r_{xc}}$;	s2xc0 = $\sigma^{(2)}_{r_{xc}}$;	s3xc0 = $\sigma^{(3)}_{r_{xc}}$;	s4xc0 = $\sigma^{(4)}_{r_{xc}}$;
byc = $b_{r_{yc}}$;	s2yc0 = $\sigma^{(2)}_{r_{yc}}$;	s3yc0 = $\sigma^{(3)}_{r_{yc}}$;	s4yc0 = $\sigma^{(4)}_{r_{yc}}$;
pxc = ρ_{xc};	s2xc = $E[(\Delta r_{xc})^2]$;	s3xc = $E[(\Delta r_{xc})^3]$;	s4xc = $E[(\Delta r_{xc})^4]$;
pyc = ρ_{yc};	s2yc = $E[(\Delta r_{yc})^2]$;	s3yc = $E[(\Delta r_{yc})^3]$;	s4yc = $E[(\Delta r_{yc})^4]$;
pxy = ρ_{xy};	sxcyc0 = $\sigma_{r_{xc},r_{yc}}$;	sxcyc = $E(\Delta r_{xc}\Delta r_{yc})$;	
e*kl* = $E[(\Delta r_{xc})^k(\Delta r_{yc})^l]$,	k, l = 1, 2, 3, or 4;	m*i* = μ'_i, mu*i* = μ_i,	i = 1, 2, 3, or 4.

```
*******************************************************************.
*                                                                 *.
*  The SPSS program for calculating approximate moments using SPSS   *.
*  data PCM.sav.  (PCM = Population correlation matrix)          *.
*                                                                 *.
*******************************************************************.

GET
 FILE='C:\My Documents\Academics\Simulation\PCM.sav'.
EXECUTE .

**** N = 28 ****.

COMPUTE bxc = (-pxc*(1 - pxc ** 2)/(2*34))*(1 + 9*(3 + pxc**2)/(4*34)
 + 3*(121 + 70*pxc**2 + 25*pxc**4)/(8*34**2)
 + 3*(6479+4923*pxc**2+2925*pxc**4+1225*pxc**6)/(64*34**3)
 + 3*(86341+77260*pxc**2+58270*pxc**4+38220*pxc**6
 + 19845*pxc**8)/(128*34**4)) .
EXECUTE .

COMPUTE byc = (-pyc * (1 - pyc ** 2) /(2*34))* (1+9*(3+pyc**2)/(4*34)
 + 3*(121+70*pyc**2+25*pyc**4)/(8*34**2)
 + 3*(6479+4923*pyc**2+2925*pyc**4+1225*pyc**6)/(64*34**3)
 + 3*(86341+77260*pyc**2+58270*pyc**4+38220*pyc**6
 + 19845*pyc**8)/(128*34**4)) .
EXECUTE .

COMPUTE s2xc0 = ((1 - pxc ** 2) ** 2 / 34)* (1+(14+11*pxc**2)/(2*34)
 + (98+130*pxc**2+75*pxc**4)/(2*34**2)+(2744+4645*pxc**2+4422*pxc**4
 + 2565*pxc**6)/(8*34**3)+(19208+37165*pxc**2+44499*pxc**4+40299*pxc**6
 + 26685*pxc**8)/(8*34**4)) .
EXECUTE .

COMPUTE s2yc0 = ((1 - pyc ** 2) ** 2 / 34)* (1+(14+11*pyc**2)/(2*34)
 + (98+130*pyc**2+75*pyc**4)/(2*34**2)+(2744+4645*pyc**2+4422*pyc**4
 + 2565*pyc**6)/(8*34**3)+(19208+37165*pyc**2+44499*pyc**4+40299*pyc**6
```

```
 + 26685*pyc**8)/(8*34**4)) .
EXECUTE .
COMPUTE s3xc0 = (-pxc*(1 - pxc ** 2)**3/(34**2))*(6+(69+88*pxc**2)/34
 + 3*(797+1691*pxc**2+1560*pxc**4)/(4*34**2)+3*(12325+33147*pxc**2
 + 48099*pxc**4+44109*pxc**6)/(8*34**3)) .
EXECUTE .

COMPUTE s3yc0 = (-pyc*(1 - pyc ** 2)**3/(34**2))*(6+(69+88*pyc**2)/34
 + 3*(797+1691*pyc**2+1560*pyc**4)/(4*34**2)+3*(12325+33147*pyc**2
 +48099*pyc**4+44109*pyc**6)/(8*34**3)) .
EXECUTE .

COMPUTE s4xc0 =(3*(1 - pxc ** 2) ** 4 /(34**2))* (1+(12+35*pxc**2)/34
 + (436+2028*pxc**2+3025*pxc**4)/(4*34**2)+(3552+20009*pxc**2
 + 46462*pxc**4 +59751*pxc**6)/(4*34**3)) .
EXECUTE .

COMPUTE s4yc0 = (3*(1 - pyc ** 2) ** 4 /(34**2))*(1+(12+35*pyc**2)/34
 + (436+2028*pyc**2+3025*pyc**4)/(4*34**2)+(3552+20009*pyc**2
 + 46462*pyc**4+59751*pyc**6)/(4*34**3)) .
EXECUTE .

COMPUTE sxcyc0 = (pxy * (1 - pxc ** 2 - pyc ** 2)
 - pxc * pyc * (1 - pxc ** 2 - pyc ** 2 - pxy ** 2) / 2)/34 .
EXECUTE .

COMPUTE s2xc = s2xc0 + bxc**2.
EXECUTE .

COMPUTE s2yc = s2yc0 + byc**2.
EXECUTE .

COMPUTE s3xc = s3xc0 - 3*s2xc0*bxc - bxc**3.
EXECUTE .

COMPUTE s3yc = s3yc0 - 3*s2yc0*byc - byc**3.
EXECUTE .

COMPUTE s4xc = s4xc0 - 4*s3xc0*bxc + 6*s2xc0*bxc**2 + bxc**4.
EXECUTE .

COMPUTE s4yc = s4yc0 - 4*s3yc0*byc + 6*s2yc0*byc**2 + byc**4.
EXECUTE .

COMPUTE sxcyc = sxcyc0 + bxc*byc.
EXECUTE.

COMPUTE e21 = bxc**2*byc + s2xc0*byc + 2*sxcyc0*bxc.
EXECUTE.

Compute e12 = byc**2*bxc + s2yc0*bxc + 2*sxcyc0*byc.
Execute.

Compute e31 = 3*s2xc0*sxcyc0 + bxc**3*byc + 3*s2xc0*bxc*byc
 + 3*sxcyc0*bxc**2.
Execute.
```

```
Compute e13 = 3*s2yc0*sxcyc0 + byc**3*bxc + 3*s2yc0*byc*bxc
 + 3*sxcyc0*byc**2 .
Execute.

Compute e22 = s2xc0*s2yc0 + 2*sxcyc0**2 + bxc**2*byc**2 + s2xc0*byc**2
 + s2yc0*bxc**2 + 4*sxcyc0*bxc*byc.
Execute.

COMPUTE m1 = pxc*pyc + pxc*byc + pyc*bxc + sxcyc.
EXECUTE .

COMPUTE m2 = pxc**2*pyc**2 + pxc**2*s2yc + pyc**2*s2xc + e22
 + 2*pxc**2*pyc*byc + 2*pxc*pyc**2*bxc + 2*pyc*e21 + 2*pxc*e12
 + 4*pxc*pyc*sxcyc.
EXECUTE .

COMPUTE m3 = pxc**3*pyc**3 + pxc**3*s3yc + pyc**3*s3xc
 + 3*pxc**3*pyc**2*byc + 3*pxc**3*pyc*s2yc + 3*pxc**2*pyc**3*bxc
 + 3*pxc*pyc**3*s2xc + 3*pxc**2*e13 + 3*pyc**2*e31
 + 9*pxc**2*pyc**2*sxcyc + 9*pxc**2*pyc*e12 + 9*pxc*pyc**2*e21
 + 9*pxc*pyc*e22.
EXECUTE .

COMPUTE m4 = pxc**4*pyc**4 + pxc**4*s4yc + pyc**4*s4xc
 + 4*pxc**4*pyc**3*byc + 4*pxc**3*pyc**4*bxc +4*pxc**4*pyc*s3yc
 + 4*pxc*pyc**4*s3xc + 6*pxc**4*pyc**2*s2yc + 6*pxc**2*pyc**4*s2xc
 + 16*pxc**3*pyc**3*sxcyc + 16*pxc**3*pyc*e13 + 16*pxc*pyc**3*e31
 + 24*pxc**3*pyc**2*e12 + 24*pxc**2*pyc**3*e21 +36*pxc**2*pyc**2*e22.
EXECUTE .

COMPUTE mu2 = m2 -m1**2.
EXECUTE .

COMPUTE mu3 = m3-3*m1*m2+2*m1**3.
EXECUTE .

COMPUTE mu4 = m4-4*m1*m3+6*m1**2*m2-3*m1**4.
EXECUTE .

SAVE OUTFILE='C:\My Documents\Academics\Simulation\PCM and Approximated'+
'Moments (N = 28).sav'
 /keep=pxc pyc pxy m1 mu2 mu3 mu4
 /COMPRESSED .
```

TABLE I Parameter Specifications for the Simulation Study

TABLE II Comparison between the Simulated Values and the Approximated Values for the Mean μ'_1 and the Variance μ_2 (Rel. Diff. = (Approximated – Simulated)/Simulated)

TABLE III Comparison between the Simulated Values and the Approximated Values for the Third Moment μ_3 and the Fourth Moment μ_4 (Rel. Diff. = (Approximated – Simulated)/Simulated)

TABLE I Parameter Specifications for the Simulation Study

ρ_{xc}	ρ_{yc}	ρ_{xy}	*N* 28	84	783
.10	.10	.10			
		.30			
		.50			
	.30	.10			
		.30			
		.50			
	.50	.10			
		.30			
		.50			
.30	.30	.10			
		.30			
		.50			
	.50	.10			
		.30			
		.50			
.50	.50	.10			
		.30			
		.50			

TABLE II Comparison between the Simulated Values and the Approximated Values for the Mean μ'_1 and the Variance μ_2 (Rel. Diff. = (Approximated – Simulated)/Simulated)

			$N = 28$		$N = 84$		$N = 783$	
ρ_{xc}	ρ_{yc}	ρ_{xy}	*Simulated*	*Rel. Diff.*	*Simulated*	*Rel. Diff.*	*Simulated*	*Rel. Diff.*
					Mean μ'_1			
.10	.10	.10	.01623754	-.052835	.01039807	.049849	.01003714	.006802
		.30	.02270585	-.068254	.01264837	.035598	.01028838	.006410
		.50	.02905664	-.072696	.01490883	.025251	.01054522	.005551
	.30	.10	.03578902	-.043999	.03020888	.009964	.03001761	.001436
		.30	.04185949	-.055322	.03231504	.006442	.03025353	.001218
		.50	.04786088	-.061688	.03449961	.001457	.03049761	.000785
	.50	.10	.05481586	-.032350	.05055207	-.009008	.04996783	.000945
		.30	.05994305	-.041518	.05235884	-.011373	.05016661	.000768
		.50	.06503751	-.047858	.05434098	-.016355	.05037164	.000520
.30	.30	.10	.09383205	-.026557	.08954873	-.000291	.08968169	.003026
		.30	.09950354	-.032501	.09130437	.000882	.08987289	.003255
		.50	.10507749	-.035903	.09306646	.002371	.09009245	.003219
	.50	.10	.15151819	-.017163	.14949675	-.005373	.14962453	.001592
		.30	.15634273	-.021531	.15092247	-.004609	.14978847	.001663
		.50	.16115526	-.024470	.15245153	-.004098	.14998773	.001549
.50	.50	.10	.24873705	-.012193	.24891337	-.005225	.24935434	.001570
		.30	.25291764	-.015729	.24998378	-.004595	.24940893	.001910
		.50	.25712597	-.018112	.25104010	-.003472	.24952249	.002064
					Variance μ_2			
.10	.10	.10	.00200730	.037670	.00038261	.031499	.00002809	.031684
		.30	.00219837	.027718	.00042955	.040782	.00003298	.033141
		.50	.00243679	.027167	.00048174	.056867	.00003797	.035028
	.30	.10	.00470786	.005337	.00135638	-.026224	.00012860	.014137
		.30	.00507615	.003536	.00147154	-.016190	.00014181	.017404
		.50	.00549659	.003344	.00159487	-.007034	.00015512	.020739
	.50	.10	.01002729	-.002493	.00331846	-.054253	.00032274	.016964
		.30	.01050633	-.004095	.00348151	-.049782	.00034106	.018205
		.50	.01102304	-.004795	.00366409	-.048634	.00035930	.020456
.30	.30	.10	.00682378	-.027794	.00206342	-.041535	.00019589	.031273
		.30	.00759283	-.008360	.00236165	-.019920	.00023303	.031095
		.50	.00836145	.015851	.00266361	.000857	.00027119	.030466
	.50	.10	.01045024	-.013348	.00349269	-.076511	.00032802	.026593
		.30	.01151447	-.000238	.00392298	-.060967	.00037966	.025120
		.50	.01260161	.015748	.00437117	-.046687	.00043340	.024451
.50	.50	.10	.01120022	-.023076	.00377413	-.099475	.00033973	.040055
		.30	.01272200	-.015668	.00435511	-.080283	.00040817	.036321
		.50	.01419075	.005668	.00495852	-.058024	.00048330	.032545

TABLE III Comparison between the Simulated Values and the Approximated Values for the Third Moment μ_3 and the Fourth Moment μ_4 (Rel. Diff. = (Approximated – Simulated)/Simulated)

			$N = 28$		$N = 84$		$N = 783$	
ρ_{xc}	ρ_{yc}	ρ_{xy}	*Simulated*	*Rel. Diff.*	*Simulated*	*Rel. Diff.*	*Simulated*	*Rel. Diff.*
					Third Moment μ_3			
.10	.10	.10	.000102612	-.203163	.000010500	-.099905	.000000108	.027778
		.30	.000158858	-.456691	.000015611	-.225482	.000000151	.026490
		.50	.000213329	-.585973	.000020994	-.287701	.000000197	.040609
	.30	.10	.000229896	.045403	.000032053	-.185755	.000000127	1.354331
		.30	.000342621	.013134	.000047655	-.143741	.000000288	.684028
		.50	.000460741	-.000677	.000064406	-.117567	.000000461	.501085
	.50	.10	.000312498	-.074032	.000051326	-.401726	-.000000203	-2.714286
		.30	.000528974	.018260	.000082326	-.245645	.000000128	4.718750
		.50	.000761741	.051549	.000116846	-.182702	.000000506	1.260870
.30	.30	.10	.000400153	.330803	.000052947	.039738	.000000605	.016529
		.30	.000514982	.373980	.000072472	.071917	.000000801	.117353
		.50	.000637833	.429043	.000093117	.122760	.000000992	.240927
	.50	.10	.000332125	.584187	.000054061	-.042508	.000000137	3.124088
		.30	.000484532	.606443	.000075321	.102521	.000000275	2.436364
		.50	.000656781	.633975	.000098774	.216727	.000000389	2.601542
.50	.50	.10	.000186486	1.628498	.000027162	.688499	.000000545	-.100917
		.30	.000223523	2.193215	.000031018	1.349313	.000000710	.146479
		.50	.000281911	2.528103	.000041201	1.608966	.000000774	.595607
					Fourth Moment μ_4			
.10	.10	.10	.000025395	-.963300	.000001057	-.848628	.000000003	-.333333
		.30	.000033559	-1.040466	.000001517	-.938036	.000000004	-.250000
		.50	.000043744	-1.111032	.000002080	-1.023077	.000000006	-.333333
	.30	.10	.000091691	-.583961	.000007925	-.457287	.000000053	-.056604
		.30	.000109363	-.667127	.000010005	-.518941	.000000064	-.046875
		.50	.000132794	-.787121	.000012543	-.595950	.000000077	-.064935
	.50	.10	.000332066	-.239353	.000037124	-.246283	.000000327	-.018349
		.30	.000366214	-.269681	.000042258	-.274410	.000000364	-.013736
		.50	.000417883	-.349947	.000048534	-.320600	.000000401	-.004988
.30	.30	.10	.000152673	-.309105	.000015310	-.288243	.000000116	.051724
		.30	.000188228	-.331141	.000020430	-.290651	.000000166	.030120
		.50	.000229483	-.385510	.000026042	-.295676	.000000227	.013216
	.50	.10	.000325797	-.077895	.000039028	-.213206	.000000332	.024096
		.30	.000378532	-.022585	.000049363	-.193161	.000000447	.013423
		.50	.000444161	-.008573	.000061049	-.178660	.000000583	.010292
.50	.50	.10	.000349379	.072314	.000043661	-.190811	.000000342	.096491
		.30	.000432846	.125086	.000057562	-.151263	.000000499	.078156
		.50	.000517932	.196458	.000073253	-.101743	.000000703	.064011

FIGURE 1 Specifications of κ for distinguishing the types of Pearson distributions

FIGURE 2 P-P plots for the selected cells

FIGURE 3 A P-P plot with Frank's distribution for $\rho_{xc} = \rho_{yc} = \rho_{xy} = .5$ and $N = 783$

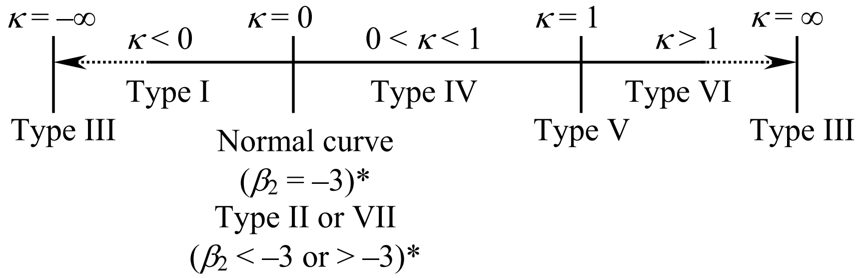
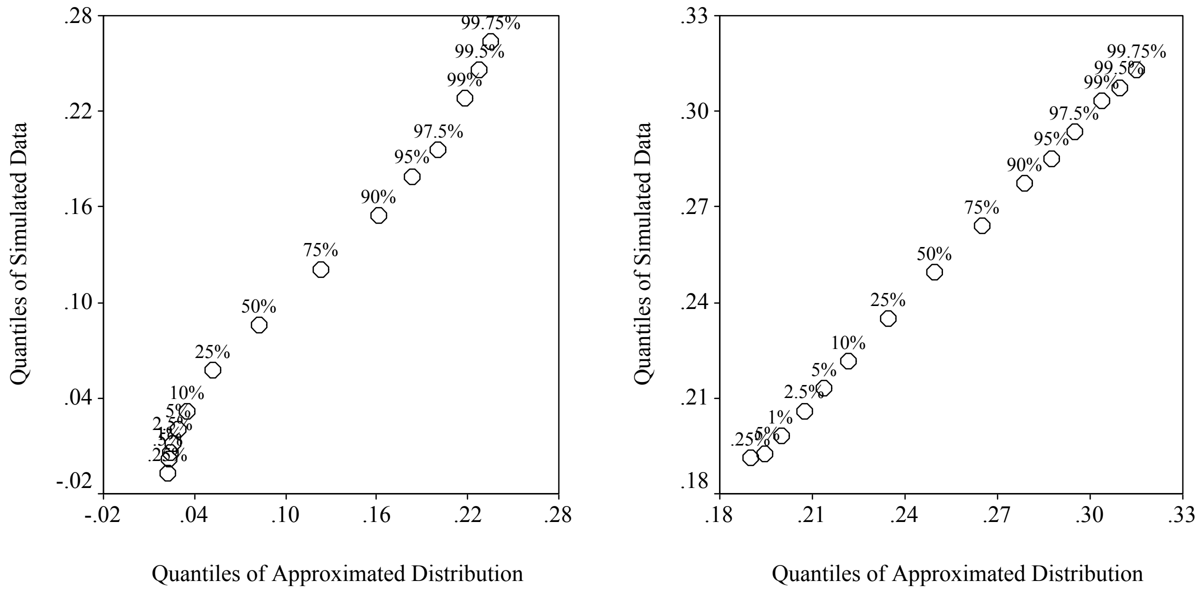


FIGURE 1 Specifications of κ for distinguishing the types of Pearson distributions. (*Originally, they were "$\beta_2 = 3$" and "$\beta_2 < 3$ or > 3" in Elderton and Johnson (1969, p.49), but according to the expression (3.2.2) above (from Kendall & Stuart, 1977, Equation 6.10) and the expression (4) in Elderton and Johnson (1969, p.41), they should be "$\beta_2 = -3$" and "$\beta_2 < -3$ or > -3", respectively.)



$\rho_{xc} = \rho_{yc} = \rho_{xy} = .30$ and $N = 84$

$\rho_{xc} = \rho_{yc} = \rho_{xy} = .50$ and $N = 783$

FIGURE 2 P-P plots for the selected cells.

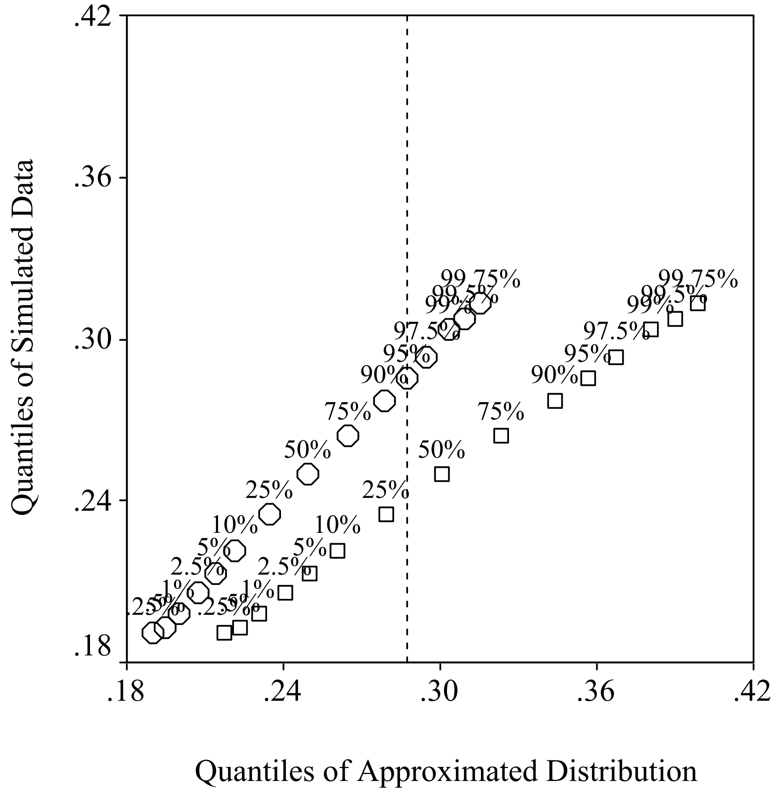


FIGURE 3 A P-P plot with Frank's distribution for $\rho_{xc} = \rho_{yc} = \rho_{xy} = .5$ and $N = 783$. (○ = Approximated Pearson Type I distribution; □ = Frank's distribution.)