Optimal debt contracts and product market competition with replacement

by

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Abstract

We derive optimal debt contracts in a market with two incumbents, one levered and one unlevered. The market can support two firms; so entry occurs when an incumbent exits. Bondholders of the levered incumbent design an optimal contract to get truthful revelation of profits. Since the contract makes loan renewal monotonically increasing in reported profits, the unlevered competitor can increase its rival’s exit probability by lowering prices. Replacement, though, makes the competitor’s strategy dependent on the level of demand, and the relative efficiencies of the levered incumbent and replacement. When the levered incumbent is less efficient, the competitor attempts to insure its survival by raising prices. When it is more efficient, the competitor attempts to force its exit by lowering prices. Bondholders condition on the competitor’s expected behavior in determining the optimal profit-sensitivity of the contract. For a less efficient levered incumbent, bondholders may increase profit sensitivity of the debt contract, thereby exploiting competitor’s incentive to retain the incumbent. For an efficient incumbent, bondholders reduce the contract’s profit sensitivity inducing the competitor to be less aggressive. Resulting optimal contracts do not necessarily resemble simple debt contracts, as they may require only partial payment of profits in default and renewal probabilities in the shape of step functions. Efficiency can be a liability in that efficient rivals are targeted and efficient entrants experience endogenously increased barriers to entry.

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We derive optimal debt contracts for a levered firm that is competing against an unlevered competitor with deep pockets. The market can support two firms; so if the levered firm exits it is immediately replaced by a new entrant. The presence of a replacement affects how aggressively incumbents’ compete, the optimal debt contract for the levered incumbent, and the pricing equilibrium. As in Bolton and Scharfstein (1990), leverage emerges endogenously in response to an agency problem between bondholders and equityholders of the levered firm. A debt contract satisfying truthful revelation imposes financial constraints on the levered firm which can be exploited by the competitor to force its exit through lower prices, especially in low demand states. Bondholders take the competitor’s response into account when designing an optimal contract.

When the potential entrant is more efficient than the levered incumbent, the competitor has an incentive to prevent the levered firm from exiting and therefore competes less aggressively. Bondholders anticipate this action by the competitor and offer a contract with a high profit-sensitivity, further reducing competition.¹ There is a parameter range where an optimal debt contract is similar to a binary option: no payment upon default (and zero probability of contract renewal), and full payment (and certain renewal) otherwise. Both debt and the presence of an efficient replacement serve to make markets less competitive, which creates an endogenous barrier to entry and allows bondholders and the incumbent firm to gain at the expense of consumers. Unlike papers like Brander and Lewis (1986), Maksimovic (1988) and Dasgupta and Titman (1998), where firm aggressiveness depends on how debt affects the convexity of equity holders’ payoffs, here debt affects aggressiveness through the interdependence of competing firms’ profit functions that results because the unlevered firm’s future profits depend on whether the levered firm survives.

When the potential entrant is less efficient than the levered incumbent, the unlevered competitor and bondholders are antagonists. Bondholders reduce the competitor’s incentive to force exit by making the contract less profit sensitive, and are better off despite leaving the limited liability constraint nonbinding (that is, bondholders do not require the firm to pay them the entire profit upon default) because diminished competition increases profits in each demand state. Markets are less competitive (higher prices and profits) and loan renewal is more likely than under a standard debt contract, which requires a defaulting levered firm to pay its entire profit to bondholders²; nevertheless, markets are more competitive than if both incumbents were unlevered. Interestingly, the optimal contract resembles an equity stake in the firm, with bondholders receiving only a fixed portion of profits.

We extend the optimal contracting problem of Bolton and Scharfstein (BS) by introducing replacement and by allowing the unlevered competitor’s profit to depend on whether its rival’s loan is renewed. This dependence affects the incentive to compete. We model spatial competition between incumbents and solve for equilibrium profits, prices and market shares (as a function of demand) conditional on the terms of the financial contract and the payoff to the competitor if the levered firm’s loan is renewed. This allows us to derive contracts that maximize bondholders’ expected payoffs after conditioning on the impact of the contracts on equilibrium competition and profits.³

¹ One might expect bondholders to make their debt contract less profit sensitive in order to increase the probability of repayment. But bondholders exploit the competitor’s incentive to have the levered firm survive; a more sensitive contract induces the competitor to raise prices to keep the levered firm from exiting, which then increases the payoff to bondholders.
² If entry is expected to occur with delay, the incentive to prey should increase. However preying on a more efficient competitor is still preferred.
³ The model also extend Hart and Moore (1998) by deriving optimal debt contracts that result in truthful revelation of firm profits in a product market with dynamic competition.
Some results are as follows. When the levered firm is less efficient than the potential entrant, the competitor benefits from increasing the levered incumbent’s probability of survival by increasing prices and transferring some of its market share to the levered firm. This is costly; so, whether it decides to prop up the levered rival depends on the relative difference in the efficiency levels of the rival and its replacement, as well as the level of demand. If the relative efficiency difference is "small," there are three interesting ranges: “low demand”, “moderate demand” and “high demand”. In the low demand states, the competitor increase his price, allowing the rival to raise his. With a higher price, the rival’s survival is more likely, though remaining less than certain. Similar results apply for moderate demand, except that the competitor raises prices high enough that renewal of the loan is certain. When demand is high, the competitor has no incentive to improve the rival’s profits as profits are high enough to ensure renewal. Thus, competition occurs as a regular Bertrand game. When the relative difference in efficiencies is large, the competitor ensures certain renewal of rival’s loan (the low demand region becomes void) as it is more profitable to ensure the weaker firm is the competitor next period. For some states, the competitor may charge high enough prices to give its rival the whole market in the first period. In all these ranges, the rival survives with a higher probability because of the accommodation (non-collusive) provided by the competitor.

When levered firm is incumbent is less efficient than its replacement, the unlevered competitor and bondholders share the same objective of ensuring survival of the levered rival. With a large efficiency difference, the unlevered competitor raises prices sufficiently to ensure certain loan renewal. Bondholders cannot improve on certain renewal, nor can they capture additional surplus because of the self-enforcing nature of the equilibrium. For intermediate differences in relative efficiency, bondholders benefit from endogenously steepening the contract and forcing the competitor into charging higher prices.

When the levered incumbent is more efficient than its replacement, the unlevered competitor would like to force its exit even though bondholders would like to see it survive. If demand is low and the levered rival much more efficient than the replacement, the competitor may lower its price in the first period to even below cost. When demand is high, on the other hand, the equilibrium is the same as if the rival were not levered. For this region, the competitor is unable to recover the cost of reducing rival’s loan renewal from the expected benefit of competing against the less efficient firm next period.

When the levered incumbent is only somewhat more efficient, the limited liability constraint binds, precluding bondholders from capturing any more surplus by conditioning on unlevered competitor’s response (by steepening the loan renewal schedule). However, when the efficiency difference is large, bondholders can improve their welfare by endogenously flattening the contract. By making the bond contract less sensitive to profits, profitability is increased as the unlevered competitor becomes less aggressive. Also, the flatter schedule allows the levered firm to keep a larger portion of first period profit after meeting the truthful revelation constraint. This makes the limited liability constraint non-binding allowing bondholders to increase their surplus. Surprisingly, optimizing the contract with an endogenous profit function (which anticipates the competitor’s actions) benefits not only bondholders, but also both incumbents. Consumers foot the bill through higher prices for every demand state in this region.

Recent empirical work on product market competition is consistent with the conclusions of this paper. Opler and Titman (1994), Chevalier (1995), Phillips (1995), Zingales (1998), and Khanna and Tice (2000) all document that financial leverage makes firms less aggressive and more vulnerable to less levered rivals. Zingales (1998) conditions on firm efficiency and shows that not only are highly levered firms more likely to exit during times of exogenous shocks, but the exiting firms can also
be efficient. Khanna and Tice (2005) document that markets with a larger proportion of high debt firms have higher prices during non-recessions but lower prices during recessions when more firms exit and levered exiting firms are more likely to be efficient.\(^4\) Since eliminating efficient rivals is more rewarding though also more difficult, competitors appear to target constrained, efficient firms at a time when the constraints are particularly binding.

The paper is organized as follows. Section 1 describes the model. Section 2 derives the equilibrium under a simple loan-renewal schedule, and Section 3 derives the optimal loan-renewal schedule (that is, the optimal debt contract). This is followed by the conclusion. The appendix contains most of the proofs.

1 The model

We assume a market large enough to accommodate two firms. In the first period these are \(L\) (levered firm) and \(U\) (the unlevered competitor). If \(L\) does not exit at the end of the first period, the same two firms compete also in the second/final period. If \(L\) exits, competition in the second period is between \(U\) and a new entrant. At the beginning of each period, each firm pays a fixed cost, \(F\). \(U\) has “deep pockets” and can finance \(F\) through existing internal funds, while \(L\) is financially constrained and must borrow the needed funds from bond investors. Investors lend for only one period and renew the loan depending on \(L\)’s reported profit at the end of the first period. If \(L\)’s loan is not renewed, it is replaced by a new entrant which is known to be either more or less efficient than \(L\).

As in BS, bondholders have all the bargaining power and design a loan contract at the beginning of the first period to maximize their expected profits (subject to \(L\)’s individual rationality constraint). Firm \(L\)’s profit in the first period depends on uncertain demand, and is not observable or contractible; therefore bondholders cannot force payment. However, bondholders can design an optimal contract that results in truthful revelation by making the probability of receiving financing in the second period increasing in the reported profit and associated payment. Firms compete in the product market after observing demand, as in models of spatial competition like Hotelling (1929) and d’Aspremont et al. (1979). Consumers are uniformly distributed and incur quadratic transportation costs to shop at either \(L\) or \(U\). Competition is Bertrand and consumers determine where to shop on the basis of lower total cost. The firm offering the lower price gets a larger share of the market. Firms set prices taking the terms of the contract and their competitor’s price as given. Invoking the revelation principle, we consider only debt contracts that satisfy the incentive compatibility (truthful revelation) constraint, which is accomplished by designing the loan-renewal-probability schedule so that equity holders receive (in expectation) an extra dollar payout for each additional reported dollar profit. The levered-firm’s equity holders therefore face a simple problem: maximize profits independently period by period. Firm \(U\) faces a more complicated problem. Even taking the pricing strategy of the levered firm as given, \(U\)’s first-period pricing strategy affects the probability of facing a new competitor in period two. Therefore \(U\) must optimize over both first and second period profits jointly, taking into account the impact of its price on the probability of \(L\)’s return to the market, and the characteristics of the potential replacement.

Having financed the fixed cost, firms \(U\) and \(L\) compete on price, generating period-one profits \(\pi^U(s)\) and \(\pi^L(s)\), where \(s\) represents the random state of demand, assumed distributed according

\(^4\)Chevalier also provides evidence consistent with low debt firms actively targeting high debt LBO firms to encourage their exit. Prices are lower in markets with a mix of high debt (LBO) firms and low debt firms. While she does not explicitly condition on efficiency, the targeted LBO firms in her sample could well be efficient.
to the density function $\theta()$, whose support is the interval $[0, \tilde{s}]$. Though bondholders know the state’s distribution, only firms $L$ and $U$ observe the outcome and resulting profits, prices and market shares. The contract is composed of two functions of reported profit: $R : [0, \pi^L(\tilde{s})] \rightarrow [0, \pi^L(\tilde{s})]$, which specifies the transfer from the stockholders to bondholders in period one; and $\beta : [0, \pi^L(\tilde{s})] \rightarrow [0, 1]$, which specifies the probability of second-period loan renewal. If the loan is renewed, bondholders pay the second-period fixed cost $F$, firm $L$ receives $X^L \in (2F, \infty)$, and firm $U$ receives $X^U \in (-\infty, \infty)$. The benefit $X^L$ can be interpreted as $L$’s expected profit to be earned in the second period if the loan is renewed, and $X^U$ as the additional expected second-period profit to $U$ if $L$’s loan is renewed; $X^U > 0$ if $L$ is less efficient than its potential replacement, resulting in weaker second-period competition and higher profits for $U$, and $X^U < 0$ if $L$ is relatively efficient, resulting in lower profits for $U$.

Given the contract $(R, \beta)$, the ex-post total profit functions of the firms and bondholders, $B$, are therefore

$$
\Pi^U(s) = \pi^U(s) + X^U \beta(\pi^U(s)),
\Pi^L(s) = \pi^L(s) - R(\pi^L(s)) + X^L \beta(\pi^L(s)),
\Pi^B(s) = R(\pi^L(s)) - F\{1 + \beta(\pi^L(s))\}.
$$

Two key features distinguish our model from those of Povel and Raith (2004, a,b), and Faure-Grimaud (2000): 1) Loan renewal affects not only the levered firm’s payoff, but that of its competitor; 2) Competition-driving variables (price here, and quantities in the others) are chosen by the firms after the state is revealed. The first feature gives $U$ a stake in the profitability of $L$, and therefore an incentive to increase or decrease competition depending on whether $L$ is more or less efficient than the potential replacement. The second causes the firms’ pricing strategies and profit functions to depend on loan-renewal function $\beta$ (though this dependence is suppressed in the notation). For example, given a relatively inefficient $L$, the incentive for $U$ to raise prices is stronger the more sensitive the loan renewal schedule.

1.1 The Contract

Under the assumption that bondholders have all the leverage in designing the contract, and assuming risk neutrality and omitting discounting (for simplicity), bondholders design the contract to maximize their expected payoff subject to the constraints of truthful reporting, shareholder limited liability (bondholder’s payoff cannot exceed profit), and individual rationality (the net present value

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As in BS, allowing a second-period transfer adds no generality.

For example, if we model spatial competition in the second-period, then $X^U$ is increasing in the cost advantage of the potential replacement versus $L$.

The omission of the constant term from $\Pi^U(s)$ representing $U$’s expected second-period profit if $L$’s loan is not renewed changes none of our results.
of the equity must be at least zero). Letting $\Pi^B$ denote this maximum, we have

$$\Pi^B = \max_{\{\beta, R\}} -F + \int_0^s \theta(s) \left\{ R(\pi^L(s)) - F \beta(\pi^L(s)) \right\} ds$$

subject to

$$R(\pi) - \beta(\pi) X^L \geq R(\bar{\pi}) - \beta(\bar{\pi}) X^L, \quad \text{all } \pi, \bar{\pi} \in [0, \pi(s)]$$  

(incentive compatibility)

$$\pi \geq R(\pi), \quad \text{all } \pi \in [0, \pi(s)]$$  

(limited liability)

$$0 \leq \beta(\pi) \leq 1, \quad \text{all } \pi \in [0, \pi(s)]$$

It is easy to show that the incentive compatibility constraint always binds,

$$R(\pi) - \beta(\pi) X^L = R(0) - \beta(0) X^L, \quad \text{all } \pi \in [0, \pi(s)], \quad (2)$$

and, by limited liability, $R(0) = 0$. With these substitutions we obtain the equivalent problem

$$\Pi^B = \max_{\{\beta, R\}} (1 + \beta(0)) F + \left( X^L - F \right) \int_0^s \theta(s) \left\{ \beta(\pi^L(s)) - \beta(0) \right\} ds \quad (3)$$

subject to

$$\beta(\pi^L(s)) - \beta(0) \leq \min \left( \frac{\pi^L(s)}{X^L} , 1 - \beta(0) \right), \quad (4)$$

$$\beta(\pi^L(s)) \geq 0, \quad \text{all } s \in [0, \bar{s}]$$

If the optimal $\beta$ is a nondecreasing function of profits and, for every $s$, $\pi^L(s)$ depends only on $\beta - \beta(0)$ (both conditions hold in all the solutions below), then it follows immediately that $\beta(0) = 0$ is optimal, and bondholders choose the contract that maximizes the probability of renewal subject to incentive compatibility and limited liability.

From (1), the ex-post total payout to the equity holders of the levered firm reduces to

$$\Pi^L(s) = \pi^L(s) + \beta(0) X^L, \quad s \in [0, \bar{s}], \quad (5)$$

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*The shareholder individual rationality constraint,

$$\int \theta(s) \left\{ \pi^L(s) - R(\pi^L(s)) + \beta(\pi^L(s)) X^L \right\} ds \geq 0,$$

is nonbinding because of the limited liability constraint, the incentive compatibility constraint (2), and the assumption $X^L > 0$.

*The same conclusions hold even if negative transfers are allowed and the incentive compatibility constraint is relaxed to

$$R(\pi) - \beta(\pi) X^L \geq R(\bar{\pi}) - \beta(\bar{\pi}) \bar{\pi}^L, \quad \text{all } \pi, \bar{\pi} \in [0, \pi(s)] \text{ s.t. } R(\bar{\pi}) \leq \pi;$$

that is, only affordable lies are considered.
That is, to induce truthful revelation, equity holders of $L$ must receive an additional dollar payout for each additional dollar of reported profit. The steeper the loan-renewal contract, the shallower the transfer function $R$; because limited-liability requires $R$ to be nonnegative, the steepness of the contract is constrained.

The expected payoff to bondholders is the residual, $[\beta \left( \pi^L(s) - \beta(0) \right) X^L - F - |1 + \beta(0)| F]$. For each unit increase in the probability of second-period renewal, bondholders extract the entire profit net of fixed cost. The threat of nonrenewal is the only leverage available to bondholders to extract a payoff.

If investors were to take the profit function $\pi^L$ as exogenous (that is, not affected by the contract), then the upper-bound constraint on $\beta$ would be met with equality (either the limited liability or the $\beta \leq 1$ constraint would bind) and the optimal contract would be

$$\beta \left( \pi^L(s) \right) = \min \left( \frac{\pi^L(s)}{X^L}, 1 \right), \quad R \left( \pi^L(s) \right) = \beta \left( \pi^L(s) \right) X^L, \quad s \in [0, \bar{s}].$$

This is a traditional debt contract with face value $X^L$, with all profits below $X^L$ paid to the bondholders. We show below that when investors anticipate the impact of the loan renewal schedule on competition and therefore profits; when $X^U < 0$ ($L$ is more efficient than its potential replacement) it may be optimal to flatten the schedule, allowing equity holders to share in profits even when profits fall short of the face value.

### 1.2 Price Competition

This section derives necessary conditions for a Bertrand equilibrium with any nondecreasing loan renewal schedule (satisfying certain technical restrictions). These conditions are applied in Section 2 to obtain the equilibrium for the case of a "simple" loan-renewal schedule (which is linear in profits in the default region), and in Section 3 to derive the optimal contract. We fix throughout some loan-renewal function $\beta$, which is assumed nondecreasing, continuous, and with left and right-hand derivatives existing for all positive profits (allowing for possible kinks). We consider first the case of interior solutions, in which both firms have strictly positive market shares, and then the case of a corner solution, in which $U$ prices sufficiently high that $L$ captures the entire market.

Customers are assumed uniformly distributed on the interval $[0, 1]$, with firm $L$ located at 0 and firm $U$ at 1. Each customer buys one unit, paying the price and incurring a (quadratic) transportation cost of $q$ times the squared distance to the store. The state $s$ determines the density of consumers; that is, for any $\delta \in [0, 1]$, there are $s\delta$ consumers located in the interval $[0, \delta]$. Demand is observed by both firms before they set prices. Firm $j \in \{L, U\}$ faces marginal per-unit production cost $c^j$, and given demand $s$, charges per-unit price $p^j(s)$, and obtains the market share $\lambda^j(s) \in [0, 1]$.

To obtain interior solutions (strictly positive market shares for both firms) in regions when the loan renewal schedule is flat (or, equivalently, when neither firm is levered), we assume that cost differences are not too large relative to transportation costs:

$$3q > |c^U - c^L| \quad (\text{standing assumption})$$

Given state-$s$ prices $p^L(s)$ and $p^U(s)$, the proportion of consumers shopping at $L$ is obtained by determining the location of the consumer indifferent between the stores:

$$p^L(s) + q\lambda^L(s)^2 = p^U(s) + q \left(1 - \lambda^L(s)\right)^2,$$
which results in the market share
\[
\lambda^L(s) = \begin{cases} 
0 & \text{if } p^U(s) - p^L(s) < -q; \\
\frac{1}{2q} (q + p^U(s) - p^L(s)) & \text{if } p^U(s) - p^L(s) \in [-q, q]; \\
1 & \text{if } p^U(s) - p^L(s) > q.
\end{cases}
\]

Assuming an interior equilibrium, firm $L$’s first-period profit function with demand $s$ is

\[
\pi^L(s) = s \left( \frac{q + p^U(s) - p^L(s)}{2q} \right) (p^L(s) - c^L).
\] (8)

Firm $L$’s shareholders set $p^L(s)$, taking $p^U(s)$ as given, to maximize the total equity payout, $\Pi^L(s)$, which by equation (5) is equivalent to maximizing $\pi^L(s)$ for each $s$. The first-order condition (necessary and sufficient) implies the following reaction function to firm $U$’s price:

\[
p^L(s) = \frac{q + p^U(s) + c^L}{2}.
\] (9)

Firm $U$ faces a more complicated problem because its first-period price affects not only period-one profits, but also its expected second-period payout, which depends on the probability of firm $L$’s loan renewal. Substituting the first-period profit formula (which is analogous to (8)) into (1) gives the ex-post total profit

\[
\Pi^U(s) = s \left( \frac{q + p^L(s) - p^U(s)}{2q} \right) (p^U(s) - c^U) + X^U \beta (\pi^L(s)).
\] (10)

Taking the renewal function $\beta$ as given\(^{10}\), and holding $p^L(s)$ fixed (in the expressions for both $\pi^U(s)$ and $\pi^L(s)$), the first-order necessary condition (also sufficient if $\Pi^U(s)$ is a concave function of $p^U(s)$) for an interior maximum for $\Pi^U(s)$ is

\[
c^U + c^L + q - 2p^U(s) + (1 + X^U \delta(s)) (p^U(s) - c^U) = 0, \quad \delta(s) \in \partial \beta (\pi^L(s)),
\]

where $\partial \beta (\pi^L(s))$ is defined as the set of elements of the closed interval bounded by the left and right hand derivatives of $\beta$ at $\pi^L(s)$ (if $\beta$ is differentiable at $\pi^L(s)$ then $\partial \beta (\pi^L(s))$ is the singleton $\beta' (\pi^L(s))$). Firm $U$’s reaction function is therefore

\[
p^U(s) = \frac{c^U + c^L + q + (1 + X^U \delta(s)) (p^L(s) - c^L)}{2}, \quad \delta(s) \in \partial \beta (\pi^L(s)).
\] (11)

Defining

\[Q = 3q + c^U - c^L,\]

an interior Bertrand equilibrium is characterized in the following lemma.

\(^{10}\)Though firm $U$ takes the function $\beta$ as given, it does, of course, account for the impact of its price on $\pi^U_L(s)$ and therefore $\beta (\pi^U_L(s))$, for each state $s$.  

7
Lemma 1 (Necessary interior FOC ) For each state \( s \), an interior (that is, \( \lambda^L_1(s) \in (0,1) \)) Bertrand equilibrium satisfies\(^\text{11}\)

\[
\begin{align*}
p^L(s) - c^L &= \frac{Q}{3 - X^U \delta(s)}, \\
p^U(s) - c^U &= \frac{2Q}{3 - X^U \delta(s)} + c^L - c^U - q, \\
\pi^L(s) &= \frac{s}{2q} \left( \frac{Q}{3 - X^U \delta(s)} \right)^2, \\
\pi^U(s) &= \frac{s}{2q} \left( 2q - \frac{Q}{3 - X^U \delta(s)} \right) \left( 2q - \frac{[1 - X^U \delta(s)]Q}{3 - X^U \delta(s)} \right), \\
\lambda^L(s) &= \frac{Q}{2q(3 - X^U \delta(s))}, \\
\delta(s) &\in \partial\beta(\pi^L(s)), \\
X^U \delta(s) &< \frac{3q + c^L - c^U}{2q}.
\end{align*}
\]

Proof. The price expressions are obtained at the intersection of the reaction functions (9) and (11). Substituting \( q + p^U(s) - p^L(s) = p^U(s) - c^L \) (from (9) into the profit function (8)) gives \( \pi^L(s) = (s/2q) \left( p^L(s) - c^L \right)^2 \), into which we substitute the equilibrium expression for \( p^L(s) - c^L \).

These expressions have intuitive appeal. If \( L \) is less efficient (has higher marginal costs) than the replacement (that is, \( X^U > 0 \)), then a steeper contract results in higher markups and profits. Greater renewal-sensitivity to profits increases \( U \)'s incentive to increase prices to increase \( L \)'s profits. On the other hand, if \( L \) is relatively efficient (that is, \( X^U < 0 \)), \( U \)'s incentive to increase the probability of \( L \)'s exit strengthens with a steeper contract; so it lowers prices and decreases markups and profits. When \( X^U = 0 \), firm \( U \) is unaffected if \( L \)'s loan is not renewed, and prices and profits are the same as if \( L \) were unlevered.

Another possibility we encounter in Section 2 is a corner solution, with \( U \) allowing \( L \) to capture the entire market by charging \( p^U(s) = p^L(s) + q \). Since firm \( U \)'s profit is zero at this price, its objective is to maximize the renewal probability \( \beta(\pi^L(s)) \). Define \( \hat{\pi} \) as the smallest firm-\( L \) profit that achieves certain loan renewal:

\[
\hat{\pi} = \min \{ p : \beta(p) = 1 \}.
\]

A higher profit cannot be supported in a self-enforcing equilibrium. If \( L \) were to make higher profits, \( U \) would lower its price to capture some market share, forcing \( L \) to lower its prices to where it’s profit is \( \hat{\pi} \). The following lemma gives necessary conditions for a corner solution when (for simplicity) \( U \)'s costs are not too high (the proof in the appendix considers the other case).

\(^{11}\)The last condition implies that \( \lambda^L < 1 \), and, together with the standing assumption (7), that \( X^U \delta(s) < 3 \) and therefore \( \lambda^L > 0 \).
Lemma 2 (Necessary corner FOC) Suppose $q + c^L \geq c^U$. Then necessary conditions for the corner solution $\lambda^L_1(s) = 1$ are

$$p^U(s) = p^L(s) + q, \quad p^L(s) - c^L = \frac{\hat{\pi}}{s}$$

and

$$s \leq \frac{\hat{\pi}}{q^L/c^L - c^U}, \quad \text{if} \quad X^U \beta'_- (\hat{\pi}) \geq 1 + \frac{q + c^L - c^U}{2q},$$

$$s \leq \frac{X^U}{X^U \beta'_- (\hat{\pi}) - 1}, \quad \text{if} \quad 1 \leq X^U \beta'_- (\hat{\pi}) < 1 + \frac{q + c^L - c^U}{2q},$$

where $\beta'_- (\pi^L(s))$ is the left-hand derivative of $\beta$ at $\pi^L(s)$.

An equilibrium in which $U$ will price high enough that $L$ obtains the entire market (which, when $q + c^L \geq c^U$, is sufficient to insure loan renewal) requires low demand, a relatively inefficient $U$ ($X^U > 0$ is a necessary condition, by assumption (7)), and a sufficiently steep renewal schedule.

2 Equilibrium With a Simple Loan-Renewal Schedule

We derive the Bertrand equilibria assuming a simple loan-renewal schedule, whose definition is similar to Povel and Raith’s (2004) simple debt contract:

Definition 3 The loan-renewal schedule $\beta$ is simple if it satisfies

$$\beta(\pi) = \min \left( \frac{\pi}{K}, 1 \right), \quad \text{all} \ \pi \geq 0, \quad \text{and some} \ K \geq X^L. \quad (13)$$

The lower bound on the kink-point $K$ is needed to satisfy limited liability (see (4)). For example, as stated above, a simple schedule with $K = X^L$ is optimal when bondholders take the profit function $\pi^L$ as given. We show in Section 3 that simple schedules are also optimal for certain parameter ranges when bondholders anticipate the impact of the contract on market competition, though the optimal $K$ may be strictly greater than $X^L$.

Define

$$x = X^U/K,$$

which represents loan-renewal benefit to $U$ weighted by the contract sensitivity $\beta' = 1/K$ in the region $\beta < 1$, and assume\(^{12}\)

$$q + c^L - c^U \geq 0.$$ 

We first consider the case when the levered firm is relatively inefficient (has higher marginal costs than its potential replacement). The results also apply if the replacement’s costs are identical to $L$’s (that is, $X^U = 0$), which is the same as the standard solution without leverage.

\(^{12}\)This assumption only affects the parameter range under which corner equilibria (with $\lambda^L_1(s) = 1$) hold. If $q + c^L - c^U < 0$, then the proof of Lemma 2 shows that there may be parameter ranges under which $L$’s profit falls short of $K$ even when $\lambda^L_1(s) = 1$. 

9
Proposition 4 Suppose $X^U > 0$, and the loan-renewal schedule is given by (13).

a) (High demand) If

$$s > \frac{18qK}{Q^2},$$

then there is a unique equilibrium with $\pi_L(s) > K$ (and therefore certain renewal) given by Lemma 1 after making the substitution $\delta(s) = 0$.

b) (Highly inefficient $L$ and low demand) If

$$x \geq \frac{3q + c^L - c^U}{2q}, \quad s \leq \frac{18qK}{Q^2},$$

there is a unique equilibrium with $\pi_L(s) = K$ (profit is just sufficient for certain renewal). If $s \leq K/2q$, the (corner) equilibrium is given by Lemma 2, after making the substitution $\bar{\pi} = K$. If $s \in (K/2q, 18qK/Q^2]$, the equilibrium is interior and is given by Lemma 1 (and $\delta(s)$ solves $\pi_L(s) = K$).

c) (Moderately inefficient $L$ and low demand) When

$$x \in \left[0, \frac{3q + c^L - c^U}{2q}\right], \quad s \leq \frac{18qK}{Q^2},$$

then we obtain an interior equilibrium given by Lemma 1. If $s \in [(1 - x/3)^2 18qK/Q^2, 18qK/Q^2]$, renewal is certain (and $\delta(s)$ solves $\pi_L(s) = K$). On the other hand, if demand is very low, $s \in [0, (1 - x/3)^2 18qK/Q^2)$, renewal is uncertain (and $\delta(s) = 1/K$).

A pure-strategy equilibrium always exists. When demand exceeds $18qK/Q^2$, firm $L$ earns sufficient profits to ensure renewal, and the equilibrium is the same as if $L$ were unlevered. When demand is below this quantity, however, the possibility of nonrenewal and replacement causes prices and $L$'s profits to be increasing in $X^U$. If $X^U$ is large enough, $L$ earns sufficient profits for certain renewal for any positive demand; for low demand, $U$ prices high enough for $L$ to attain all the period-one market in order achieve certain loan renewal.

The intuition is straightforward. When $L$ is relatively inefficient, $U$ prefers it survive into the next period. Compared to the case with no leverage (or, equivalently, $X^U = 0$) $U$ raises prices and surrenders market share to increase $L$’s probability of survival. When $L$ is sufficiently inefficient, $U$ is prepared to lose enough market share in first period to ensure its survival, but when $L$ only moderately less efficient, prices are not high enough to to ensure $L$’s renewal when demand is low.

Because $L$’s equilibrium profits are decreasing in $K$ (as the renewal schedule flattens), both the bondholder payout, $\Pi^B(s)$, and the levered firm’s equity-holder payout, $\Pi^L(s)$, are decreasing in

---

13Note that our standing assumption implies that $(3q + c^L - c^U)/2q < 3$.
14When $x \in [0, 1]$ the equilibrium is unique. However, when $x \in (1, (3q + c^L - c^U)/2q)$ and $s < K(x - 1)/(q + c^L - c^U)$ there is another equilibrium is which firm $L$ has all the market share and charges $p^L(s) = c^L + K/s$. This equilibrium is unstable in the sense that if we temporarily relax the market share constraint (say by allowing firm $U$ to buy from firm $L$), we converge to (interior) equilibrium in Lemma 1.
for each demand state. Therefore the optimal simple debt contract is obtained by choosing the smallest feasible kink point: \( K = X^L \).

We next consider the case when \( X^U < 0 \); that is, the levered firm is relatively efficient (has lower marginal costs than its potential replacement).

**Proposition 5** Suppose \( X^U < 0 \), and the loan-renewal schedule is given by (13).

a) (High demand) If

\[
x \in (-2, 0) \text{ and } s \geq \frac{18qK}{Q^2} \frac{1}{1 + x/4}, \quad \text{or} \quad x \leq -2 \text{ and } s \geq \frac{18qK}{Q^2} |x|,
\]

there is an equilibrium with \( \pi^L (s) \geq K \) (and therefore certain renewal) given by Lemma 1 after making the substitution \( \delta(s) = 0 \).

b) (Low demand) If

\[
s < \frac{18qK (1 - x/3)^2}{Q^2 (1 - x/4)}
\]

there is an equilibrium with \( \pi^L (s) < K \) (and therefore uncertain renewal) given by Lemma 1 after making the substitution \( \delta(s) = 1/K \).

It is too costly for \( U \) to prevent certain renewal of an efficient \( L \) in high demand states; though the size of this range is declining as the cost advantage of \( L \) over its potential replacement increases (that is, as \( |X^U| \) increases). In lower demand states \( U \) increases \( L \)'s probability of exit by lowering prices (compared to the case when the competitor is unlevered). Prices and profits fall as \( |X^U| \) increases, and the low demand range expands. In fact, if \( 3q + c^L - c^U < |x| (q + c^L - c^U) \) (for example, \( c^U \geq c^L \) and \( |x| \) is sufficiently large), then \( U \) prices below cost to increase the probability of \( L \)'s departure from the market. The proof of Proposition 5 shows that if \( x \) is sufficiently close to zero, the high and low demand ranges overlap, and two equilibria are supported (both are interior, but renewal is uncertain in one and certain in the other); on the other hand, if \( x \) is slightly larger than \(-2\) then there is a gap between these demand ranges in which no pure-strategy equilibrium exists. The complication arises because \( \pi^U (s) \) is not concave \( p^U (s) \), and therefore the necessary conditions of Lemma 1 are not sufficient.

Within each equilibrium, \( L \) profits are increasing in \( K \); that is, a flatter schedule inhibits predation by \( U \) and increases \( L \)'s profit for each demand state. On the other hand, the bondholder payout for a given reported profit is smaller due to the flatter renewal schedule. Section 3.2 shows that when we restrict attention to the low demand range (to bypass the issues of equilibrium existence and uniqueness), the optimal contract is a simple contract, and that when \( |X^U| \) is sufficiently large, the optimal kinkpoint satisfies \( K > X^U \); that is, the limited liability constraint is nonbinding.

### 3 The Optimal Contract

This section examines the problem of determining the optimal contract and the resulting product market equilibrium. The investors' optimization problem is again (3) subject to the constraints (4) as well as the necessary equilibrium conditions of Lemmas 1 or 2. We will find that bondholders
may make loan renewal more sensitive to profit changes (compared to the exogenous-profit case) when \( L \) is inefficient to induce \( U \) to further increase price, and may flatten \( \beta \) when \( L \) is efficient to deter predation. Therefore the limited liability constraint may not bind, allowing the levered firm’s equity holders to share in profits despite the fact that bondholders have not been fully paid. Though bondholders seek to maximize the ex-ante probability of renewal,\(^{15}\) they face a trade-off between maximizing \( \beta (\pi^L) \) for each \( \pi^L \), and maximizing \( \pi^L (s) \) for each \( s \).

### 3.1 The Case When \( X^U > 0 \)

We have already commented in Section 2 that an optimal simple contract when \( L \) is less efficient than its potential replacement is the steepest feasible contract, with kink point \( K = X^U \) and therefore a binding limited liability constraint (all profits are paid to bondholders) when renewal is uncertain. Bondholders and the unlevered firm share the same objective of ensuring \( L \)'s survival, and increasing the sensitivity of the contract induces higher prices and profits for \( L \).

When the \( L \) is sufficiently inefficient that \( \frac{X^U}{X^L} \geq \max \left( \frac{(3q + c^L - c^U)}{2q}, 1 \right) \), Proposition 4 (together with further results in the proof of Lemma 2) showed that a simple contract with \( K = X^U \) cannot be improved upon because renewal is certain (and the face value \( X^L \) fully is paid) in every positive demand state.\(^{16}\) Firm \( L \)'s profit function is then

\[
\pi^L (s) = \max \left( \frac{s - Q^2}{18q^2}, X^L \right), \quad s \in [0, \bar{s}].
\]

The value of the investors’ objective function is \( \Pi^B = X^L - 2F \), which is the maximum attainable value under any contract, state-variable distribution, or first-period profit function. For sufficiently inefficient \( L \), therefore, an optimal contract is a simple loan renewal schedule under which the limited liability constraint binds; the same contract that would be obtained if investors were to take the profit function \( \pi^L \) as exogenous (and unaffected by the contract).

For intermediate differences in relative efficiency, the optimal loan-renewal schedule may not be simple: Bondholders can benefit by flattening the schedule in one region and steepening it in another to induce the \( U \) to charge higher prices and therefore increase \( L \)'s profit function. We show that a step function contract is optimal in this intermediate case.

**Proposition 6** Suppose \( x \geq 9/4, q + c^L - c^U \geq 0 \), and that the renewal schedule is the step function

\[
\beta (\pi) = \begin{cases} 
0 & \text{if } \pi < K, \\
1 & \text{if } \pi \geq K,
\end{cases}
\]

for some \( K \geq X^L \) (in order to satisfy the limited liability constraint in (4)). Then in equilibrium \( \pi^L (s) \geq K \) for each \( s > 0 \), and therefore renewal is certain. If \( s > \frac{18qK}{Q^2} \), there is a unique equilibrium with \( \pi^L (s) > K \) given by Lemma 1 after making the substitution \( \delta (s) = 0 \); if \( s \in \)

---

\(^{15}\)Assuming that the optimal \( \beta \) is increasing in profits, as it is in all our solutions.

\(^{16}\)When \( \frac{X^U}{X^L} > \max \left( \frac{(3q + c^L - c^U)}{2q}, 1 \right) \), a flatter renewal schedule, with any kink point satisfying

\[
K \in \left[ \frac{X^L}{\max \left( \frac{(3q + c^L - c^U)}{2q}, 1 \right)}, \frac{X^U}{\max \left( \frac{(3q + c^L - c^U)}{2q}, 1 \right)} \right]
\]

results in the same equilibrium and bondholder objective.
there is an equilibrium with $\pi^L(s) = K$ given by Lemma 1 (with $\delta(s)$ solving $\pi^L(s) = K$); and if $s \in (0, K/2q]$ there is a corner equilibrium given by Lemma 2 (with $\hat{\pi} = K$).\footnote{We ignore the dominated alternative equilibrium with $\beta(\pi^L(s)) = 0$ (and both firms’ profits are lower) that may exist under some parameter ranges. See the proof for details.}

The proposition shows that if $X^U/X^L > 9/4$, an optimal contract is a step-function with $K = X^L$, because this contract insures renewal. In particular, in the parameter range $(3q + c^U - c^L) / 2q > 9/4$ and $X^U/X^L \in [9/4, (3q + c^L - c^U) / 2q)$, a step function contract is optimal, but no simple contract is. Bondholders exploit $U$’s incentive to encourage $L$’s survival by concentrating the steepness of the contract at the kink point $K$. It follows from the incentive compatibility constraint (2) that the contract payout is that of a cash-or-nothing binary option: no payment to bondholders if profits fall short of $K$ (and therefore certain nonrenewal), and the payment $X^L$ if profits exceed $K$ (and therefore certain renewal). This contract induces $U$ to raise prices enough so that $L$’s profits are at least $K$ for all each positive demand state.

### 3.2 The Case When $X^U < 0$

We obtain an explicit solution for the optimal contract when the maximum demand, $\bar{s}$, cannot be too high relative to $|X^U|$. We show that the optimal contract is simple, but that when the relative efficiency of $L$ (as measured by $X^U$) is sufficiently high, the limited liability constraint is nonbinding. That is, bondholders and equity holders share in $L$’s profits, giving bondholders an equity stake in the firm.

**Proposition 7** Suppose that

$$\bar{s} \leq \frac{24q|X^U|}{Q^2},$$

and that the optimal $\beta()$ is absolutely continuous.

a) If

$$\frac{X^U}{X^L} \leq -3,$$

then the optimal contract is

$$\beta(\pi^L(s)) = \frac{3\pi^L(s)}{|X^U|}, \quad s \in [0, \bar{s}].$$

There is a interior unique equilibrium given by Lemma 1 after making the substitution $\delta(s) = 3/|X^U|$.

b) If

$$-3 < \frac{X^U}{X^L} < 0,$$

then the optimal contract within the space of continuous functions with piecewise-continuous first derivatives is

$$\beta(\pi^L(s)) = \frac{\pi^L(s)}{X^L}, \quad s \in [0, \bar{s}].$$

There is a unique interior equilibrium given by Lemma 1 after making the substitution $\delta(s) = 1/X^L$.

\footnote{Note that $(3q + c^U - c^L) / 2q > 9/4$ together with assumption (7) are equivalent to $c^L - c^U \in (1.5q, 3q)$.}
When \(-3 < X^U / X^L < 0\), the optimal contract is a simple contract with binding limited liability constraint: \(K = X^U\). As mentioned earlier, this is the same as the optimal contract if \(L\)'s profit function were exogenous (that is, if \(L\)'s profit function were invariant to the contract terms). The limited liability constraint precludes bondholders from steepening the \(\beta\) schedule to extract more income. As \(|X^U|\) increases within this range, competition escalates and prices and \(L\)'s profits decrease; the ex-post bondholder payout for each state \(s\), given by

\[
\Pi^B(s) = -F + \frac{sQ^2}{2qX^L (3 + |X^U|/X^L)^2} \left( X^L - F \right),
\]

therefore declines.

When \(X^U / X^L \leq -3\), the slope of the optimal \(\beta\) flattens as \(|X^U|\) increases. The flatter \(\beta\) requires equity holders to keep a larger fraction of period-one income to meet the incentive compatibility constraint; therefore the limited liability constraint is nonbinding (the \(\beta \leq 1\) constraint is also nonbinding under the above parameter restrictions). Equilibrium prices, profits and firm \(L\)'s market share (substituting \(\delta(s) = -3/X^U\) into the equations of Lemma 1) are the same for all \(X^U\) in this range:

\[
\begin{align*}
    p^L(s) - c^L &= \frac{Q}{6}, & p^U(s) - c^U &= \frac{2(e^L - e^U)}{3}, \\
    \pi^L(s) &= \frac{sQ^2}{72q}, & \pi^U(s) &= \frac{s}{18q} \left( 9q - e^U + c^L \right) \left( e^L - e^U \right) + \frac{sQ^2}{24q}.
\end{align*}
\]

The flattening of the optimal contract completely counteracts \(U\)'s incentive to compete more aggressively as \(|X^U|\) increases. The flatter schedule lowers the probability of renewal for each reported profit, but the increase in profit for each given state results in a renewal probability that is higher for each state than if the limited liability constraint were left binding. Nonetheless, \(L\)'s markups and market share are only half (and resulting profits are therefore one quarter) what they would be if \(L\) were unlevered.

As \(|X^U|\) increases, the flattening of the renewal schedule and the shrinking fraction of profits extracted from equity holders \((R(\pi^L(s)) / \pi^L(s) = 3X^L / |X^U|)\) results in an ex-post payout that again declines:

\[
\Pi^B(s) = -F + \frac{sQ^2}{24q |X^U|} \left( X^L - F \right).
\]

Whenever \(X^U < 0\), therefore, expected profits to bondholders are decreasing in \(|X^U|\); that is, it is less profitable to finance more efficient firms.

### 4 Conclusion

We show that the interaction of a firm’s capital structure with product market competition can result in relatively efficient firms being preyed upon (when demand is weak) while relatively inefficient firms are propped up by their competitor. When bondholders anticipate this interaction, they can design the contract to mitigate predation against efficient levered firms, but magnify the incentive to support the inefficient levered firm. Since deterring predation is costly to bondholders, the profitability of financing falls as the relative efficiency of the levered firm increases compared to the entrant (holding fixed the levered firm’s expected second-period profits).
Our study suggests that markets may not necessarily achieve socially desirable outcomes. Given potential entry, incumbents prefer the status quo if the entrant is relatively more efficient, but try to eliminate a competitor if he is more efficient than his replacement. Since efficient entrants face endogenously increased barriers to entry, efficient entry may occur only if the potential entrant is significantly more efficient or because of regulatory intervention. The presence of an efficient entrant ironically makes the market less competitive resulting in consumers paying higher prices. Since the equilibrium is self-enforcing, such situations may persist indefinitely.
Appendix: Proofs Omitted From the Main Text

Proof of Lemma 2

A necessary condition for an equilibrium with $\lambda_1^L(s) = 1$ is that $U$ cannot increase profits by lowering its price:

$$c^U + c^L + q + [1 + X^U \beta_-^U(\pi^L(s))] (p^L(s) - c^L) \geq 2p^U(s).$$

Substituting $p^U(s) = p^L(s) + q$ gives

$$c^U - c^L - q + [X^U \beta_-^U(\pi^L(s)) - 1] (p^L(s) - c^L) \geq 0.$$

(16)

On the other hand, firm $L$ cannot increase its profit by raising its price, which from (9) is equivalent to

$$\frac{q + p^U(s) + c^L}{2} \leq p^U(s) - q$$

and therefore

$$2q \leq p^L(s) - c^L.$$

(17)

We can rule out $p^L(s) - c^L > \tilde{\pi}/s$ (which implies $\pi^L(s) > \tilde{\pi}$ and $\beta_-^U(\pi^L(s)) = 0$) because (16) and (17) cannot both be satisfied under the standing assumption. When $q + c^L - c^L \geq 0$, the dominant equilibrium is $p^L(s) - c^L = \tilde{\pi}/s$ and therefore $\pi^L(s) = \tilde{\pi}$, and conditions (16) and (17) give the result.

When $q + c^L - c^U < 0$, the dominant equilibrium under conditions (16) and (17) is

$$p^L(s) - c^L = \frac{\tilde{\pi}}{s} \text{ if } \left\{ \begin{array}{l} \frac{\pi X^U \beta_-^U(\tilde{\pi}) - 1}{q + c^L - c^U} \leq s \leq \frac{\pi}{2q} \text{ and } 1 + \frac{q + c^L - c^U}{2q} \leq X^U \beta_-^U(\tilde{\pi}) < 1; \text{ or } \\ \frac{1}{2} \leq X^U \beta_-^U(\tilde{\pi}) < 1; \end{array} \right.$$

However, in the range

$$1 + \frac{q + c^L - c^U}{2q} \leq X^U \beta_-^U(\tilde{\pi}) < 1 \text{ and } s < \frac{\pi X^U \beta_-^U(\tilde{\pi}) - 1}{q + c^L - c^U},$$

there is no equilibrium with $p^L(s) - c^L = \tilde{\pi}/s$. That is, certain renewal cannot be achieved even when $L$ is allowed to capture the entire market. Under this range, condition (16) is

$$p^L(s) - c^L \leq \frac{-q - c^L + c^U}{1 - X^U \beta_-^U(s [p^L(s) - c^L])}.$$

(18)

The dominant equilibrium is given by the maximum $p^L(s) - c^L$ that satisfies (18).

Proof of Proposition 4.

The case of $X^U > 0$

Concavity of $U$’s objective function in this case ($L$’s is always sufficient) implies that the necessary conditions of Lemmas 1 and 2 are also sufficient. We first consider interior ($\lambda^L(s) \in (0, 1)$) equilibria. An interior equilibrium to the left of the kink in $\beta$ occurs when $\pi^L(s) < X^L$, and is
obtained by substituting \( \partial \beta (\pi^L(s)) = \beta' (\pi^L(s)) = 1/K \) into Lemma (1). The conditions are 
\[
x < \frac{(3q + c^L - c^U)}{2q}
\]
and
\[
\pi^L(s) = \frac{s}{2q} \left( \frac{Q}{3-x} \right)^2 < K.
\]
An interior equilibrium to the right of the kink in \( \beta \) occurs when \( \pi^L(s) > K \), in which case 
\( \partial \beta (\pi^L(s)) = \beta' (\pi^L(s)) = 0 \), and Lemma 1 implies the condition 
\[
\pi^L(s) = \frac{s}{2q} \left( \frac{Q}{3} \right)^2 > K.
\]
Finally, an interior solution at the kink (renewal is just assured) corresponds to 
\[
K = \frac{s}{2q} \left( \frac{Q}{3 - X^U \delta(s)} \right)^2, \quad \delta(s) \in [0, 1/K], \quad X^U \delta(s) < \frac{3q + c^L - c^U}{2q}.
\]
Therefore 
\[
X^U \delta(s) = 3 - Q \sqrt{\frac{s}{2qK}},
\]
and the restrictions on \( \delta(s) \) are equivalent to 
\[
\frac{2qK \max (0, 3-x)^2}{Q^2} \leq s \leq \frac{18qK}{Q^2}, \quad K < s.
\]
Note that 
\[
\frac{2qK \max (0, 3-x)^2}{Q^2} \leq \frac{K}{2q}
\]
if and only if 
\[
\frac{3q - c^U + c^L}{2q} \leq x.
\]

We complete the proof by considering corner \((\lambda^L(s) = 1)\) equilibria, which is obtained by 
substituting \( \tilde{\pi} = K \) and \( \beta^L(\pi^L) = 1/K \) into Lemma 2.

**Proof of Proposition 5**

**The case of** \( X^U < 0 \)

In this case, \( U \)'s objective function is not necessarily concave, and we must therefore confirm 
that the necessary conditions of Lemma 1 are sufficient (in particular, the \( U \) would have no incentive 
to deviate). However, we can rule out the corner solution of Lemma 2.

From Lemma 1, an interior equilibrium to the left of the kink in \( \beta \) requires 
\[
x < \frac{(3q + c^L - c^U)}{2q}
\]
(which holds automatically by our standing assumption) and
\[
\pi^L(s) = \frac{s}{2q} \left( \frac{Q}{3-x} \right)^2 < K;
\]
or, equivalently,
\[
s < 2qK \left( \frac{3-x}{Q} \right)^2.
\]

(19)
Firm $U$’s total profit at the candidate equilibrium is

$$\Pi^U(s) = \frac{s}{2q} \left( 2q \left( 2q - \frac{Q}{3-x} \right) \left( 2q - \frac{(1-x)Q}{3-x} \right) + X^U \frac{\pi^L(s)}{K}. \right)$$

We need to check whether $U$ may want to deviate from the candidate equilibrium when we hold fixed $p^L(s) - c^L = Q/(3-x)$. We obtain $U$’s optimal deviating price in the $\pi^L(s) > X^L$ region (where $\beta(\pi^L(s)) = 1$) from (11) by substituting $\beta^\prime(\pi^L(s)) = 0$ and $p^L(s)$ to get

$$\hat{p}_1^U(s) = \frac{(e^U + c^L + q) (3-x) + Q}{2(3-x)}.$$

From equation (8), $L$’s profit under $\hat{p}_1^U(s)$ is

$$\pi^L(s) = s \frac{Q^2(2-x)}{4q(3-x)^2}.$$

Therefore within the demand range

$$s \geq 2qK \frac{(3-x)^2}{Q^2(1-x/2)},$$

$\hat{\pi}^L(s)$ exceeds $K$ (implying $\beta(\hat{\pi}^L(s)) = 1$), implying that $\hat{p}_1^U(s)$ is, in fact, a feasible deviating strategy in the region (19). $U$’s profit under the deviating strategy is

$$\hat{\Pi}^U(s) = \frac{s}{2q} \left( 2q \left( \frac{4q(3-x) - Q(2-x)}{2(3-x)} \right)^2 + X^U \right)$$

which exceeds the profit from the candidate equilibrium if

$$s > 2qK \frac{(3-x)^2}{Q^2(1-x/4)}.$$

Therefore the candidate equilibrium with $\pi^L(s) < K$ is in fact an equilibrium only when

$$s < 2qK \frac{(3-x)^2}{Q^2(1-x/4)}.$$

An equilibrium to the right of the kink in $\beta$ occurs when $\pi^L(s) > K$, in which case loan renewal is certain. Prices and profits for this candidate equilibrium are given by Lemma 1 with $\delta(s) = 0$. As in the case when $X^U > 0$, a necessary condition is $s > 18qK/Q^2$. Now fix $p^L(s)$ and consider the possibility of $U$ reducing $\hat{\pi}^U_1(s)$ to within the region where $\pi^L(s) < K$. Consider first the case when $x \in (-2,0)$. Fixing $p^L(s) = c^L(s) + Q/3$, $L$’s optimal deviating price, from (11), is

$$\hat{p}^U(s) = \frac{e^U + c^L + q + (1+x)(p^L(s) - c^L)}{2}.$$

\(^{20}\)P’s profit function is the maximum of two concave functions which intersect at the the value of $p^L_1(s)$ such that $\pi^L_1(s) = K$ (the kink point in $\beta$). It can be shown that if $\pi^L_1(s) < K$, then $P$ increasing $p^U_1(s)$ to the kink point never results in higher profits that the candidate equilibrium.
which results, from (8), in first-period profit for $L$ of
\[ \hat{\pi}^L(s) = \frac{s}{18q} (1 + x/2) Q^2. \]

If
\[ s < \frac{18qK}{(1 + x/2) Q^2} \]
then $\hat{\pi}^L(s) < K$; therefore we need only consider whether deviating from the certain-renewal candidate equilibrium may be optimal when
\[ \frac{18qK}{Q^2} \leq s < \frac{18qK}{(1 + x/2) Q^2}, \quad x \in (-2, 0). \]

$U$’s total profit at the candidate equilibrium (from Lemma 1 with $\delta(s) = 0$) is
\[ \Pi'(s) = \frac{s}{18q} (3q + c^L - c^U)^2 + X^U \]
which compares to the deviating-strategy total profit (from (10))
\[ \hat{\Pi}^p(s) = \frac{s}{2q} \left( 2q - \frac{(2 + x) Q}{6} \right) \left( \frac{q + c^L - c^U + (1 + x) Q/3}{2} \right) + x \frac{s}{2q} \frac{(2 + x) Q^2}{18} \]
Deviating results in higher profits when
\[ \frac{s}{2q} \left( 2q - \frac{(2 + x) Q}{6} \right) \left( \frac{q + c^L - c^U + (1 + x) Q/3}{2} \right) + x \frac{s}{18q} \frac{(2 + x) Q^2}{2} \]
\[ > \frac{s}{18q} (3q + c^L - c^U)^2 + X^U, \]
or, equivalently,
\[ s < \frac{18qK}{(1 + x/4) Q^2}. \]

So we have a certain-renewal equilibrium when
\[ \frac{18qK}{(1 + x/4) Q^2} \leq s, \quad x \in (-2, 0). \]

For the case when $x \leq -2$, again fixing $p^L(s) = c^L(s) + Q/3$, then $U$’s optimal deviating strategy is to capture 100% of the market by charging
\[ \bar{p}^U_1(s) = p^L(s) - q = \frac{c^U + 2c^L}{3}, \]
which results in the profit
\[ \hat{\Pi}^p(s) = \frac{2}{3} s \left( c^L - c^U \right). \]

Deviating from the certain-renewal equilibrium is optimal if
\[ \frac{2}{3} s \left( c^L - c^U \right) > \frac{s}{18q} (3q + c^L - c^U)^2 + X^U, \]
or, equivalently,

\[ s < \frac{18qK}{Q^2}. \]

Therefore we have a certain-renewal equilibrium when

\[ \frac{18qK}{Q^2} \leq s, \quad x \leq -2. \]

The proof is completed by deriving the parameter ranges under which multiple or no pure-strategy equilibria exist. Define as \( \hat{\alpha} \) the (unique) negative solution to

\[ 9 (4 - \alpha) - (3 - \alpha)^2 (4 + \alpha) = 0 \]

(it can be solved numerically to get \( \hat{\alpha} \approx -1.646 \)).

When \( x \in (\hat{\alpha}, 0) \), then \((1 + x/4)^{-1} < (1 - x/3)^2 (1 - x/4)^{-1} \), and therefore both a predation and nonpredation equilibrium are supported for the parameter range

\[ s \in \left( \frac{18qK}{Q^2 (1 + x/4)}, \frac{18qK}{Q^2 (1 - x/4)} \right), \quad x \in (\hat{\alpha}, 0). \]

When \( x \in (\hat{\alpha}, 0) \), the inequality reverses, and therefore there is no pure strategy equilibrium when

\[ s \in \left( \frac{18qK (1 - x/3)^2}{Q^2 (1 - x/4)}, \frac{18qK}{1 + x/4) Q^2} \right), \quad x \in (-2, \hat{\alpha}), \]

and also when

\[ s \in \left( \frac{18qK (1 - x/3)^2}{Q^2 (1 - x/4)}, \frac{18qK |x|}{Q^2} \right), \quad x \leq -2. \]

**Proof of Proposition 6.**

The equilibrium is similar to parts a) and b) of Proposition 4 (for the simple renewal schedule); however \( U \)'s objective function is not concave here (\( L \)'s always is) and we must therefore confirm that \( U \) has no incentive to deviate. We can rule out some deviations because \( U \)'s objective function is the sum of a strictly concave (first-period profit) function and a step function (the second period payout); for example, \( U \) would never deviate from an interior certain-renewal equilibrium by increasing its price, because this would reduce both the first period profit. Therefore the proof of the (unique) equilibrium for the range \( s > \frac{18qK}{Q^2} \) is essentially the same as in the proof of Proposition 4.

We next consider the range \( s \in (K/ (2q), \frac{18qK}{Q^2}] \), in which we show that a unique equilibrium exists with \( \lambda^r_t \in (0, 1) \). As in Proposition 4, equilibrium prices and profits are found from Lemma 1 by solving for \( \delta(s) \) that solves \( \pi^L(s) = K \) (we will confirm the result, despite the discontinuity in the step renewal function):

\[ p^L(s) - c^L = \sqrt{\frac{2qK}{s}}, \quad p^U(s) - c^U = c^L - c^U - q + 2 \sqrt{\frac{2qK}{s}} \]
and
\[ \Pi^U(s) = \frac{s}{2q} \left( 2q - \sqrt{\frac{2qK}{s}} \right) \left( c^L - c^U - q + 2\sqrt{\frac{2qK}{s}} \right) + X^U. \] (20)

Holding fixing \( p^L(s) \), we check whether \( U \) has an incentive to reduce its price so that \( \pi^L(s) < K \) (and therefore \( \beta(\pi^L(s)) = 0 \)). Firm \( U \)'s optimal deviating price (from (11) with \( \delta(s) = 0 \)) is\(^{21}\)
\[ \hat{p}^U(s) = \frac{q + c^U + p^L(s)}{2} = \frac{q + c^U + c^L}{2} + \frac{1}{2} \sqrt{\frac{2qK}{s}} \]
(note that \( s < 18qK/Q^2 \) implies that \( \hat{p}^U(s) < p^U(s) \) and therefore \( \hat{\pi}^L(s) < K \)), which results in the total profits of
\[ \hat{\Pi}^U(s) = \frac{s}{2q} \left( \frac{q + c^U - c^L}{2} + \frac{1}{2} \sqrt{\frac{2qK}{s}} \right)^2. \] (21)

Defining
\[ y = \sqrt{\frac{2qK}{s}}, \] (22)
the candidate equilibrium at \( \pi^L(s) = K \) is in fact an equilibrium if \( \Pi^U(s) \geq \hat{\Pi}^U(s) \), or, equivalently,
\[ (2q - y) \left( c^L - c^U - q + 2y \right) + xy^2 \geq \left( \frac{q + c^U - c^L}{2} + \frac{1}{2} y \right)^2, \]
which simplifies to
\[ (9 - 4x)y^2 - 6Qy + Q^2 \leq 0. \] (23)

If \( 9 - 4x \leq 0 \) then it is easy to show that condition (23) always holds when \( s \leq 18qK/Q^2 \) (which is equivalent to \( y > Q/3 \)). Therefore the step function contract (14) is optimal when \( x \geq 9/4 \) and \( s \in (K/(2q), 18qK/Q^2] \).\(^{22}\)

Finally, we consider the corner equilibrium case when \( s \leq K/(2q) \). An equilibrium at \( \pi^L(s) = K \) requires the prices
\[ p^L(s) - c^L = \frac{K}{s}, \quad p^U(s) = p^L(s) + q, \]
(and therefore \( \lambda^L(s) = 1 \)) and total profits
\[ \Pi^U(s) = X^U. \]

\(^{21}\)The nonnegativity constraint on \( \lambda^L \) may be violated, but imposing the constraint will only reduce the profitability of the deviating strategy, which we show is suboptimal without this constraint.

\(^{22}\)Since we need only consider the range
\[ 2q \geq y \geq Q/3, \]
we can generalize the result slightly. It is sufficient to confirm the condition (23) only at the points \( \{2q, Q/3\} \). The condition holds at \( Q/3 \) if \( x \geq 0 \) and at \( 2q \) if
\[ x \geq \frac{9}{4} - \frac{Q(12q - Q)}{16q^2}. \]

T, with the latter

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Now fix this $p^L (s)$ and consider price-reducing strategies by $U$ that necessarily reduce $L$’s profits below $K$, resulting in certain nonrenewal. $U$’s optimal deviating price, $\hat{p}^U (s)$, is one of two cases: a) $U$ leaves some market share for $L$: $\lambda^L (s) > 0$; or b) $U$ captures the entire market: $\lambda^L (s) = 0$. Omitting, for now, the constraint $\lambda^L (s) \in [0,1]$, the optimal deviating price is (from (11) with $\delta (s) = 0$)

$$\hat{p}^U (s) = \frac{q + c^U + p^L (s)}{2} = \frac{q + c^U + c^L}{2} + \frac{K}{2s},$$

This price satisfies $\hat{p}^U (s) \geq p^L (s) - q$ (resulting in $\lambda^L (s) \geq 0$) only if $s \geq K/Q$. Consider first case (a), which requires $2q < Q$ (implying a nonempty intersection of the ranges $s \leq K/(2q)$ and $s \geq K/Q$). Substituting $\hat{p}^U (s)$ and $p^L (s) = c^L + \frac{K}{s}$ into (10), the total profit under the deviating strategy is

$$\hat{\Pi}^U (s) = \frac{s}{2q} \left( \frac{q - c^U + c^L}{2} + \frac{K}{2s} \right)^2.$$

Within the demand range $s \in [K/Q, K/(2q)]$ we get

$$\hat{\Pi}^U (s) \leq \frac{K}{4q^2} \left( \frac{q - c^U + c^L}{2} + \frac{Q}{2} \right)^2 = \frac{K}{4q^2} \left( \frac{4q}{2} \right)^2 = \frac{K}{4},$$

which is less than $\Pi^U (s) = X^U$ if $x > 1/4$ (we are assuming $x \geq 9/4$). In case (b), The optimal feasible deviating price is $\hat{p}^U_1 (s) = p^L (s) - q = K/s + c^L - q$. Firm $U$’s total profit from this price is

$$\hat{\Pi}^U (s) = s (\hat{p}^U (s) - c^L) = s \left( \frac{K}{s} + c^L - q - c^U \right).$$

The deviating strategy results in higher profits if

$$X^U < K + s (c^L - q - c^U).$$

But this is never the case when demand satisfies $s \leq K/(2q)$ because

$$X^U > K + \frac{K}{2q} (c^L - q - c^U).$$

(This is easily shown by dividing by $K$, and recalling that $c^L - c^U < 3q$ by the standing assumption, and $x > 9/4$ for the parameter range under consideration.)

Finally, we show that in the demand range

$$\left[0, \frac{3K}{Q}\right] \cup \left\{s \in \left[\frac{3K}{Q}, \frac{18qK}{Q^2}\right] : 0 < \frac{1}{9} Q^2 s^2 - 2q (2K + X^U) s + \left(\frac{6qK}{Q}\right)^2 \right\},$$

an alternative equilibrium with $\beta (\pi^L (s)) = 0$ may exist, but it is Pareto inferior (both $L$ and $U$ are worse off in this equilibrium). For any $s < 18qK/Q^2$, the candidate alternative equilibrium has prices and profits given by Lemma 1 with $\delta (s) = 0$; in particular, $p^L (c) - c^L = Q/3$. We fix $L$’s price and determine whether $U$ has an incentive to lower its price to $\hat{p}^U (s)$ that equals $L$’s profit to $K$ (resulting in certain loan renewal):

$$\hat{p}^U (s) = \frac{c^U + 2c^L}{3} + \frac{6q}{sQ} K.$$
However this price is feasible only if \( p^U(s) \leq p^L(s) + q \), (otherwise \( U \) would have to give \( L \) more than 100% market share to achieve \( \pi^L(s) = K \): which is equivalent to \( s \geq 3K/Q \) (the assumption \( c^U - c^L < 3q \) implies \( 3K/Q < 18qK/Q^2 \)). Therefore the alternative candidate is in fact an equilibrium when \( s < 3K/Q \). Considering now just the demand range \([3K/Q < 18qK/Q^2]\), the deviating strategy results in total profits (from (10))

\[
\hat{\Pi}^U(s) = \frac{s}{2q} \left( 2q - \frac{6q}{sQ} K \right) \left( \frac{2(c^L - c^U)}{3} + \frac{6q}{sQ} K \right) + X^U,
\]

which satisfies \( \hat{\Pi}^U(s) \leq \Pi^U(s) \) (that is, the candidate alternative equilibrium is in fact an equilibrium) if

\[
\left( 2q - \frac{6q}{sQ} K \right) \left( \frac{2(c^L - c^U)}{3} + \frac{6q}{sQ} K \right) + \frac{2q}{s} X^U \leq \frac{1}{9} \left( 3q + c^L - c^U \right)^2;
\]

or, simplifying,

\[
\frac{1}{9} Q^2 s^2 - 2q \left( 2K + X^U \right) s + \left( \frac{6q}{Q} K \right)^2 > 0.
\]

This condition may be satisfied for \( s \) in a neighborhood of \( 3K/Q \) when \( Q \) is small.

We conclude by showing that the alternative equilibrium with \( \hat{\Pi}^U(s) \leq 0 \) results in lower profits for both firms. Firm \( L \) is obviously worse off since its profit falls short of \( K \) in the alternative equilibrium. In the demand range \( s \in (K/(2q), 18qK/Q^2] \), firm \( U \) is worse off if

\[
\frac{s}{18q} \left( 3q + c^L - c^U \right)^2 < \frac{s}{2q} \left( 2q - \sqrt{2qK/s} \right) \left( c^L - c^U - q + 2\sqrt{2qK/s} \right) + X^U,
\]

or, equivalently,

\[
(6q - Q)^2 < 9 \left( 2q - y \right) \left( 2q - Q + 2y \right) + 9y^2 x,
\]

where \( y \) is again defined by (22). Rearranging we get

\[
(6q + Q) Q < 9 \left( 2q + Q \right) y + 9 \left( x - 2 \right) y^2.
\]

This inequality is always true because \( s < 18qK/Q^2 \) is equivalent to \( y > Q/3 \), and because of our assumption that \( x > 9/4 \). In the demand range \( s \leq K/(2q) \), firm \( U \) is worse off if

\[
\frac{s}{18q} \left( 3q + c^L - c^U \right)^2 < X^U,
\]

which follows from \( 3q + c^L - c^U < 6q \) (by our standing assumption) and \( K < X^U \) (which follows from \( x \geq 9/4 \)).

Proof of Proposition 7

Part (a).

The bondholders’ objective function and constraints are given in (3) and (4), respectively. We will solve the problem without the limited liability constraint or the nonnegativity constraint on
\( \beta \), and show that these constraints are nonbinding. Assuming \( \beta \) is a.e. differentiable, denoting derivatives by primes, and , the bondholder’s problem can, using integration by parts, be written

\[
\Pi_B = \max_{\beta} - (1 + \beta'(0)) F + (X^L - F) \int_{u=0}^{\bar{s}} \left( \int_{s=u}^{\bar{s}} \theta(s) ds \right) \beta'(\pi^L(u)) \pi^L(u) du.
\]

Using \( \pi^L(u) = \pi^L(u)/u \), and assuming that the necessary equilibrium conditions of from Lemma (1) are sufficient (we will confirm below that it is), we can substitute for \( \pi^L(u) \) to get

\[
\Pi_B = \max_{\beta} - (1 + \beta(0)) F + \frac{(X^L - F) Q^2}{2q} \int_{u=0}^{\bar{s}} \left( \int_{s=u}^{\bar{s}} \theta(s) ds \right) \frac{\beta'(\pi^L(u))}{(3 - X^U \beta'(\pi^L(u)))^2} du.
\]

The unconstrained maximum is achieved at \( \beta(0) = 0 \) and \( \beta'(\pi^L(u)) = -3/X^U \) for all \( u \in [0, \bar{s}] \). When \( X^U \leq -3X^L \), it is easy to show that the constraints (4) are satisfied (and nonbinding when \( X^U < -3X^L \)).

**Part (b)**

We first derive the optimum when \( \beta' \) is restricted to be a step function, and then prove the general case by approximating \( \beta' \) with a step function. We partition the interval \([0, \bar{s}]\) into the subintervals \( 0 = u_0 < u_1 < \cdots < u_n = \bar{s} \) and let \( \beta'(\pi^L(s)) = b_i \) for \( s \in (u_{i-1}, u_i), i = 1, \ldots, n \). Omitting the constraints \( 0 \leq \beta \leq 1 \) (which are satisfied by the solution nonetheless), the problem is

\[
\max_{(\beta(0), b_1, \ldots, b_n)} - \beta(0) F + \sum_{i=1}^{n} \Theta_i \frac{b_i}{(3 - X^U b_i)^2} (u_i - u_{i-1}),
\]

subject to the limited liability constraints

\[
\sum_{i=1}^{j} \frac{b_i}{(3 - X^U b_i)^2} (u_i - u_{i-1}) \leq \frac{u_j}{X^L (3 - X^U b_j)^2} \quad j = 1, \ldots, n, \tag{24}
\]

where we have defined

\[
\Theta_i = \frac{(X^L - F) Q^2}{2q} \frac{1}{u_i - u_{i-1}} \int_{u=u_{i-1}}^{u_i} \left( \int_{s=u}^{\bar{s}} \theta(s) ds \right) du,
\]

which is proportional to the average over \([u_{i-1}, u_i]\) of the complementary distribution function. First note that the function \( f(y) = y/(3 - X^U y)^2 \) is monotonically increasing for \( y < -3/X^U \) and monotonically decreasing towards zero for \( y > -3/X^U \). Since the right-hand side of the constraint is decreasing in \( b_i \) for each \( j \), it follows that the solution must satisfy \( b_j \leq -3/X^U \). Let \( \kappa_1, \ldots, \kappa_n \) denote the Lagrangian multipliers for the \( n \) constraints. The Kuhn-Tucker necessary conditions for \( \{b_1, \ldots, b_n\} \) are (note that the gradients of the \( n \) constraints are linearly independent)

\[
\frac{3 + X^U b_i}{(3 - X^U b_i)^3} \Theta_i (u_i - u_{i-1}) + \kappa_i \frac{2X^U u_i}{X^L (3 - X^U b_i)^3} = (\kappa_i + \cdots + \kappa_n) \frac{3 + X^U b_i}{(3 - X^U b_i)^3} (u_i - u_{i-1}).
\]

\(23\) This constraint is slightly weaker than that in (4) because we haven’t also considered the left limits at jumps, but we will show that the optimal contract will satisfy the original contraint anyway. Also, we omit the contraint \( b_i \in [0, 1] \), which we show is nonbinding.

\(24\) For any \( b_j > -3/C \), we could obtain the same value \( b_j/(3 - Cb_j)^2 \) with a larger \( 1/(3 - Cb_j)^2 \) by choosing instead the appropriate \( b_j < -3/C \).
\( \kappa_i \geq 0, i = 1, \ldots, n, \) together with the \( n \) complementary slackness conditions. The above condition simplifies to

\[
(3 + X^U b_i) \{ \Theta_i - \kappa_i - \cdots - \kappa_n \} + \kappa_i 2 \frac{X^U}{X^L} \frac{u_i}{u_i - u_{i-1}} = 0, \quad i = 1, \ldots, n. \tag{25}
\]

Starting with \( i = n \) and working backwards, the only possibilities for each condition \( i \) are that \( \kappa_i = 0 \) and \( 3 + X^U b_i = 0 \), or \( \kappa_i > 0 \) (which implies the \( i \)th constraint is binding) and \( 3 + X^U b_i > 0 \). In addition, each constraint \( i \) satisfies \( \Theta_i > \kappa_i + \cdots + \kappa_n \). For the first condition, however, we can rule out \( \kappa_1 = 0 \) and \( 3 + X^U b_1 = 0 \) because this violates the first constraint in (24) (which can be written \( b_i \leq 1/X^L \)). Therefore \( \kappa_1 > 0 \) and \( b_1 = 1/X^L \).

Substituting \( b_1 = 1/X^L \), the second constraint in (24) can be rewritten

\[
\frac{b_2}{(3 - X^U b_2)^2} (u_2 - u_1) - \frac{u_2}{X^L (3 - X^U b_2)} \leq - \frac{u_1}{X^L (3 - X^U/X^L)^2}. \tag{26}
\]

The left-hand side is strictly increasing in \( b_2 \) when \( b_2 \leq -3/X^U \) (where the solution lies) with equality occurring at \( b_2 = 1/X^L \). It follows that we can rule out the possibility \( \kappa_2 = 0 \) and \( 3 + X^U b_2 = 0 \), because if \( b_2 = -3/X^U > 1/X^L \) (since \( X^U > -3X^L \) the constraint (26) would be violated. We therefore have \( \kappa_2 > 0 \) and \( 3 + X^U b_2 \geq 0 \); because (26) must hold with equality, we get \( b_2 = 1/X^L \). Repeating the same arguments for constraints \( i = 3, \ldots, n \) we obtain \( b_i = 1/X^L \) for all \( i \). Obviously \( \beta(0) = 0 \). It is easy to show that this unique solution to the necessary conditions must be the maximum.

The last step of the proof is to use an approximation argument to extend the proof to \( \beta \) schedules with piecewise continuous first derivatives. Suppose some \( \tilde{\beta} \) with piecewise continuous derivatives results in an objective-function value \( V \) which exceeds that in (??) (which represents the objective function corresponding to \( \beta'(0) = 1/X^L \)). With a sufficiently fine mesh, we can approximate \( \tilde{\beta} \) arbitrarily closely yielding an objective function value within any \( \epsilon > 0 \) of the \( \tilde{\beta} \) value and also satisfying the relaxed constraints (again using the fact that \( \tilde{\beta} \) must satisfy \( \tilde{\beta}'(\pi^L(s)) \leq -3/X^U \) for all \( s \))

\[
\int_{u=0}^{s} \frac{\beta'(\pi^L(u))}{(3 - X^U \beta'(\pi^L(u)))} du \leq \frac{s(1+\epsilon)}{X^L (3 - X^U \beta'(\pi^L(s)))} \text{ for all } s \in [0,s]
\]

(which are just the original constraints with \( X^L \) replaced by \( \tilde{X}^L = X^L/(1+\epsilon) \), and again has the optimal step function solution \( \beta'(\pi^L(u)) = \pi^L(u)/\tilde{X}^L \) and \( \tilde{\pi}^V \) given by (??) with \( X^L \) replaced by \( \tilde{X}^L \)). As \( \epsilon \downarrow 0 \) we get a contradiction, however, because the objective function for the approximating step function \( \beta' \) approaches \( V \), but \( V \) exceeds the maximum objective function value (for step function \( \beta' \) in (??)).

Since the loan-renewal function is linear in \( \pi^L(s) \), firm \( U \)'s maximization problem for any given \( p^L(s) \) is concave \( p^U(s) \); therefore the above necessary equilibrium conditions in Lemma 1 are in fact sufficient.

The rest of the proof shows that firm \( U \)'s profits are higher under the endogenous contract. Define \( y = X^U/X^L \). Firm \( U \)'s profit functions for the endogenous and exogenous cases are the same for \( y = -3 \) (since the contracts are the same) and diverge as \( y \) falls. It is sufficient to show that the profit function with the exogenous contract falls as \( y \) falls. In the case of first-period profits, writing

\[
\pi^U(s) = \frac{s}{q} \left( 2q - \frac{Q}{3-y} \right) \left( q - Q/2 + \frac{Q}{3-y} \right),
\]

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it is sufficient to show that
\[
\frac{d\pi^U(s)}{dy} = Q \frac{s}{q} \left\{ \left( 2q - \frac{Q}{3-y} \right) - \left( q - Q/2 + \frac{Q}{3-y} \right) \right\} \frac{d}{dx} \left( \frac{1}{3-y} \right) > 0,
\]
which is true iff
\[2q (3-y) + (-1-y) Q > 0.\]
But this is implied by our standing assumption whenever \( y < 0 \). In the case of total profits, writing
\[
\pi^U(s) = \frac{s}{q} \left( 2q - \frac{Q}{3-y} \right) \left( q - \frac{Q}{3-y} \right) + \frac{s}{2q} y \left( \frac{Q}{3-y} \right)^2,
\]
\[
\frac{d\pi^U(s)}{dy} = Q \frac{s}{q} \left\{ \left( 2q - \frac{Q}{3-y} \right) - \left( q - Q/2 + \frac{Q}{3-y} \right) + y \frac{Q}{3-y} \right\} \left( \frac{1}{3-y} \right)^2 + \frac{s}{2q} \left( \frac{Q}{3-y} \right)^2 > 0,
\]
which is true if and only if \( q (3-y) + Q > 0 \), which always holds when \( y < 0 \).
References


