Abstract

This paper suggests “soft debt” as a social convention that facilitates long-term reciprocal relationships. In each round of a two-player repeated game, one player develops a need for help from his counterpart, who may or may not be able to help. If help is rendered, a soft debt is tacitly accrued by him to his counterpart and added to the soft debt balance between them. The roles of potential beneficiary and provider of help, as well as the benefit and cost involved, are randomly drawn in each round. If the players follow soft debt strategies, i.e. they make decisions about offering and soliciting help based on the soft debt balance between them, then their expected future values of the relationship also depend on the balance. This consideration creates intertemporal incentives that promote reciprocity. Under discrete benefits, soft debt strategies are shown to constitute stationary Markov equilibria with trades provided that the cost structure meets certain conditions. The first best allocation is never achieved, but all trades that do occur in the equilibria are efficient.

KEYWORDS: Repeated games; Stochastic games; Reciprocity
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1 Introduction

When a person is asked by a friend to help him, say, move to a new home, how would he respond? He may anticipate a return of the favor in the future should he help. But would the return be expected to adequately compensate for his effort? He may be particularly

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concerned if there have been instances in the past when he needed help but the friend declined to help. The friend might have had excuses (e.g., time conflicts) but he cannot tell whether they were genuine. On the other hand, the friend may have helped him before, and by helping this time he will be returning past favors. But then how many past favors would it take to justify his effort? What if these past favors had been mostly trivial?

Of course both of these motives, namely anticipation of return and return of past favors, can be in play at the same time. Moreover, the friend would also have gone through similar thoughts before deciding to ask for help in the first place. Similar situations arise not just between friends, but also between family members, co-workers, neighbors, or even strangers who may become acquaintances in the future. How is reciprocity sustained in these relationships as the players pursue their self interests?

This paper suggests “soft debt” as a social institution that provides a solution to reciprocity. The premise is that whenever help is provided, a soft debt is tacitly accrued by the beneficiary to the provider. The soft debt is then added to the soft debt balance between them. If both of them follow “soft debt strategies”, i.e., they make decisions about offering and soliciting help based on this balance, then their expected future values of the relationship also depend on the balance. The more one is owed, the more likely that he will be helped and the more valuable the help is expected to be, and thus the higher his expected future value. This consideration creates intertemporal incentives that promote reciprocity.

Instead of trying to set up some defection detection and punishment mechanism, the players rely on the incentives created by the soft debt mechanism to maintain cooperation. When a player helps the other (if he is able to), he earns an increase in his expected future value. Similarly, if he has need and is helped, he incurs a drop in his expected future value. In other words, there are “soft prices” to be earned or paid. A player will “sell” a favor whenever the soft price he would earn exceeds the cost. Conversely, he will “buy” a favor whenever the soft price he would pay is less than the benefit. The benefit and cost can vary from round to round, but the players follow the same calculation. I call such voluntary favor trading without contracts “soft transactions.”

The soft debt mechanism simplifies bilateral relationships by summarizing the full history between two individuals into a single number. The notion of soft debt is evident from the daily use of words like “owe,” “repay,” “price,” “indebted” between acquaintances even when no contract is involved. The soft debt mechanism also implies that in contrast
to the usual assumption, an individual who needs help do not automatically accept favor when offered, because there is a soft price to be paid for the favor. A soft transaction goes through only if both players agree.

Along with hard transactions, soft transactions constitute an unified approach to analyzing a host of economic activities. When individuals have needs (e.g. to move to a new home), often they can compare the hard (hire a mover) and soft (ask a friend for help) alternatives side-by-side.\(^1\) Conversely when they have goods or services to offer (e.g. baby-sitting), again they can opt for either hard (get paid) or soft (help friends for “free”) transactions. Both types of transactions stem from similar cost and benefit analysis mediated by prices, be them hard or soft.

This paper starts with a general set-up for a two-player repeated stochastic game. In each round, one of the players develops a need for help from his counterpart. Call them the (soft) buyer and (soft) seller for this round respectively. The seller may or may not be able to help, and the ability to help is her private information (as in Möbius (2001)). Help is rendered if only if both the buyer buys and the seller sells. The roles of buyer and seller, as well as the benefit and cost involved, are drawn randomly in each round. Suppose both players follow soft debt strategies, I show that an autarky equilibrium always exists, but the first best outcome (i.e., trade occurs if and only if the benefit exceeds the cost) can never be achieved.

Under discrete benefits, I show that stationary Markov equilibria with trades could be sustained when the players follow the “debt limit strategy”, a particular form of soft debt strategy whereby players buy and sell as long as the debt balance is within a certain limit. In one version of the model, the cost-to-benefit ratio is fixed. In another version, the cost associated with each benefit is random. The cost may exceed the benefit and its realized value is observable only to the seller. The debt limit depends on the the cost structure and parameters such as the discount factor and the probability that the seller is able to help. Although the first best allocation cannot be attained, all trades that do occur are efficient.

I then discuss the advantages and disadvantages of soft transactions as compared to their hard counterparts. As a result of their differences, soft transactions are popular in long-term relationships involving trading of personalized favors, while hard transactions dominate anonymous exchanges of mass-produced goods. I finally suggests some directions

\(^1\)Of course, self production is usually another alternative.
for further research related to soft transactions.

This paper is closely related to the favor trading literature. Möbius (2001) assume that the benefit and cost are fixed and that the ability to help is private information. He shows that favor trading could be maintained between two players when they grant favors if and only if the net number of favors granted is below a certain number. Therefore the debt limit strategy in the current paper is similar to this mechanism. He also demonstrates how cooperation is sustainable in a group of players who rarely meet each other, if indirect favors are allowed. Hauser and Hopenhayn (2008) build on the bilateral model and make two relaxations: (i) the exchange rate between current and future favors is allowed to deviate from one; (ii) the balance of favors is allowed to appreciate or depreciate. They show that with these relaxations higher payoffs can be achieved. Nayyar (2009) presents a discrete time version of Hauser and Hopenhayn’s model and allows the opportunities to offer help to arise at different rates for the two players. Kalla (2010) studies the model with different discount factors. In another variation, he assumes complete information and that the players have concave utility functions instead of being risk neutral, hence favor trading is considered a form of insurance.

The model presented in this paper differ from the favor trading literature mainly in two ways. First, the benefit and cost are random (instead of fixed), so that the players need to take into consideration the specific benefit and cost drawn in each round. In reality, people often face a new and unique situation each time they meet. Even between close partners there are variations among established routines. Second, in the spirit of transactions, help is rendered only if both parties agree. In the above mentioned papers, only the provider unilaterally decides whether to help.

This paper is also related to the broader literature on reciprocity. This includes for instance the literature on informal insurance (e.g. Coate and Ravallion (1993), Kocherlakota (1996)), social norms and reciprocity (e.g. Kandori (1992)), relational contracts (e.g. Bull (1987), Levin (2003)) and games of imperfect public monitoring (e.g. Green and Porter (1984)) and private monitoring (e.g. Bhaskar and Obara (2002)).

The rest of the paper is organized as follows. Section 2 presents the general model, define the various concepts related to soft debt, and introduces some general results. Section 3 and Section 4 solves for equilibria constituted by the debt limit strategy under discrete benefits. In Section 3 the cost-to-benefit ratio is assumed to be fixed. In Section 4 the cost associated with each benefit is random and its realized value (which could exceed the
benefit) is observable only to the seller. Section 5 compares soft transactions against their hard counterparts. Finally, the conclusion suggests some directions for further research.

2 General Model

2.1 Game structure

Two players, player 1 and player 2, are randomly matched from a population to play an infinitely repeated game, starting in round 1. At the beginning of each round of stage game, one player is assigned the role of (soft) buyer, who needs help from the other player. The other player is assigned as the (soft) seller, who may or may not be able to help. She is called a capable seller for the round if she can render help, or an incapable seller otherwise. The ability to help is randomly drawn and observed only by the seller. No one else in the population is able to help. In each round, player \( i \) \((i = 1, 2)\) is assigned as the buyer with probability \( \beta_i \in (0, 1) \), which is fixed across rounds, so that \( \beta_1 + \beta_2 = 1 \). Conditional on being assigned the seller, player \( i \) is capable of helping with probability \( \pi_i \in (0, 1) \), which is also fixed across rounds. All of these probabilities are common knowledge.

In round \( t \), if help is rendered, the buyer (suppose player \( i \)) will receive a benefit of \( b_{it} > 0 \) and the (capable) seller (player \( j \)) will incur a cost of \( c_{jt} > 0 \). The values \( b_{it} \) and \( c_{jt} \) are drawn from the joint distribution of random variables \( (\tilde{b}_{it}, \tilde{c}_{jt}) \). The joint distribution is independent and identically distributed (i.i.d.) across time, and is independent of whether the seller is capable. The distribution is common knowledge. For completeness, the values \( b_{jt} \) and \( c_{it} \) are zeros, i.e., any help is unilateral in each round. The role assignment, benefit and cost are revealed to both players once drawn, while the ability to help remains the seller’s private information forever.\(^2\)

After the drawing, the buyer and the capable seller simultaneously decide whether to buy and sell respectively. This is the only decision they need to make in the round. An incapable seller cannot sell. The decisions are then revealed to both players. Help is rendered if and only if both the buyer buys and the seller sells. If help is not rendered, both players receive zero payoffs for this round. The game proceeds to the next round.

Unlike typical repeated games, the rendering of help requires the consent of not just the provider (the seller), but also the beneficiary (i.e. the buyer). The reason for this requirement is that the help, as in the case of hard transactions, may not come free, as

\(^2\)In the model in Section 4, the cost will be observed only by the seller.
explained in the next subsection. Even if the seller finds it profitable to sell, the buyer may not want to buy.

In the model the buyer and seller are treated as if they make decisions simultaneously. In practice, since the soft transaction occurs only if both players agree, it makes no difference if one of them initiate the offer, and then the other respond, as long as one’s decisions does not depend on the other’s.

The players are risk neutral with discount factor $\delta_i$. Player $i$’s objective in round $\tau$ is to maximize the expectation of discounted sum of lifetime payoffs conditional on his current information:

$$E_{i\tau} \sum_{t=1}^{\infty} \delta_i^t \left( \beta_i b_{it} - \beta_j c_{it} \right)$$

where $E_{i\tau}$ denotes the expectation operator conditional on player $i$’s information set in round $\tau$. Note that in each round $t$, one or both of the realized values $b_{it}$ and $c_{it}$ will be zero(s), depending on what role he is assigned and whether help is rendered. Both discount factors are common knowledge and reflect the chance of future meetings as well as the patience levels of individual players. This completes the specification of the stochastic game.

2.2 Strategy and Equilibrium Concept

Following the notion of soft debt, I focus on strategies that prescribe actions based on the soft debt balance.

In each round, if help is rendered, then the buyer accrues a soft debt of $d(b, c)$ to the seller, where $b$ is the benefit to the buyer and $c$ is the cost to the seller. Assume $d$ is invariant across time. Also assume $d > 0$ for all $b$ and $c$; and $d$ is increasing in both $b$ and $c$. Then the soft debt holding can be defined as follows:

**Definition 1.** Player $i$’s soft debt holding (i.e. net soft debt balance owed by player $j$ to him) at the beginning of round $\tau$ is

$$D_{i\tau} = \sum_{t=1}^{\tau-1} [d(b_{jt}, c_{it}) - d(b_{it}, c_{jt})]$$

Obviously, $D_{1\tau} = -D_{2\tau}$ and $D_{i1} = 0$. Note also that for simplicity no “interest” (positive or negative) is charged on the debt.$^3$

$^3$Unlike the nominal interest rate in hard transactions, the interest rate here can be negative, which
Let $\mathcal{A} \equiv \{\text{buyer, capable seller, incapable seller}\}$ be the set of role assignments.

**Definition 2.** A *soft debt strategy* for player $i$ is a mapping from the product set of the sets of current roles, benefit to the buyer, cost to the seller, and soft debt holding into the set of actions:

$$\sigma_i : \mathcal{A} \times \mathbb{R}^2_+ \times \mathbb{R} \rightarrow \{\text{trade, not trade}\}$$

In the definition, “trade” means to buy (in the case of the buyer) or to sell (in the case of the capable seller). Note that $\sigma_i$ is time independent (i.e. stationary). The player’s decisions depend on the entire history of past actions and information only through the current debt balance. That is, the whole history is condensed to the debt balance.

**Definition 3.** A *soft debt equilibrium* is a stationary Markov equilibrium in which both players’ follow soft debt strategies.

A *soft transaction* is a trade that occurs (i.e. help is rendered) in a soft debt equilibrium.

### 2.3 General Analysis

With soft debt equilibrium defined, we can now define soft prices.

**Definition 4.** Let $V_1 (D_1)$ and $V_2 (D_2)$ be the players’ expected future value in a soft debt equilibrium where $D_1$ and $D_2$ ($= -D_1$) are their respective current debt holdings. Suppose player $i$ is assigned as the buyer and the benefit $b_i$ and cost $c_j$ are drawn, then his *soft buying price* is

$$p_i^{\text{buy}} \equiv \delta_i [V_i (D_i) - V_i (D_i - d(b_i, c_j))]$$

and as the seller player $j$’s *soft selling price* is

$$p_j^{\text{sell}} \equiv \delta_j [V_j (D_j + d(b_i, c_j)) - V_j (D_j)]$$

The functions $V_1$ and $V_2$ are stationary because the probability distributions for role assignment, benefits and costs of helping are all i.i.d. across time. The soft price is the increase (for the seller) or decrease (for the buyer) in the expected future value of the relationship resulting from the change in the debt balance. Unlike hard prices, the soft

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*can reflect fading of memories.*
price paid by the buyer and that received by the seller can be different. The divergence exists because in soft transactions, there is no money or other exchange media that defines a single price. This means even if the benefit exceeds the cost, mutually beneficial trade may or may not occur. This limitation may offer an explanation for the development of hard transactions to fill in the gap.

Player \( i \) will buy iff the incentive compatibility (IC) condition is met:  

\[
\text{IC (buyer): } b_i > p_i^{\text{buy}}
\]

Similarly, player \( j \), if capable, will sell iff:

\[
\text{IC (seller): } c_j < p_j^{\text{sell}}
\]

Denote by \( H_i (D) \) the event that player \( i \) is helped given that he is the buyer with debt holding \( D \) and the seller is capable, then:

\[
\Pr (H_i (D)) = \Pr \left( b_i > p_i^{\text{buy}} (D) \text{ and } c_j < p_j^{\text{sell}} (-D) \right)
\]

where

\[
p_i^{\text{buy}} (D) \equiv \delta_i \left[ V_i (D) - V_i (D - d (b_i, c_j)) \right]
\]

and

\[
p_j^{\text{sell}} (-D) \equiv \delta_j \left[ V_j (-D + d (b_i, c_j)) - V_j (-D) \right]
\]

are the random buying and selling prices respectively.

In general, the value function is composed of four components that correspond to the following scenarios: player \( i \) is the buyer and he is helped; he is the buyer but is not helped; player \( j \) is the seller and she is helped; she is the seller but is not helped.

\[\text{There is no need to check the individual rationality (IR) conditions because the "worst" expected future value cannot fall below zero (autarky). Also assume that the players do not buy or sell if they are indifferent.}\]
The value function of player $i$ at the beginning of a round can thus be formulated recursively:

$$V_i(D) = \beta_i \left\{ \begin{array}{l}
\pi_j \Pr(H_i(D)) E \left[ \tilde{b}_i + \delta_i V_i \left( D - d \left( \tilde{b}_i, \bar{c}_j \right) \right) \mid H_i(D) \right] \\
+ (1 - \pi_j \Pr(H_i(D))) \delta_i V_i(D)
\end{array} \right\} + \beta_j \left\{ \begin{array}{l}
\pi_i \Pr(H_j(-D)) E \left[ -\tilde{c}_i + \delta_i V_i \left( D + d \left( \tilde{b}_j, \bar{c}_i \right) \right) \mid H_j(-D) \right] \\
+ (1 - \pi_i \Pr(H_j(-D))) \delta_i V_i(D)
\end{array} \right\}$$

I further simplify the notations by defining the following conditional probability and conditional mean benefit and cost:

$$P_i(D) \equiv \Pr(H_i(D)), \quad \tilde{b}_i(D) \equiv E \left( \tilde{b}_i \mid H_i(D) \right), \quad \bar{c}_i(-D) \equiv E \left( \tilde{c}_i \mid H_j(-D) \right).$$

Then

$$V_i(D) = \beta_i \left\{ \begin{array}{l}
\pi_j P_i(D) \left[ \tilde{b}_i(D) + \delta_i E \left[ V_i \left( D - d \left( \tilde{b}_i, \bar{c}_j \right) \right) \mid H_i(D) \right] \right] \\
+ (1 - \pi_j P_i(D)) \delta_i V_i(D)
\end{array} \right\} + \beta_j \left\{ \begin{array}{l}
\pi_i P_j(-D) \left[ -\bar{c}_i(-D) + \delta_i E \left[ V_i \left( D + d \left( \tilde{b}_j, \bar{c}_i \right) \right) \mid H_j(-D) \right] \right] \\
+ (1 - \pi_i P_j(-D)) \delta_i V_i(D)
\end{array} \right\} \quad (1)$$

Upon rearranging terms,

$$V_i(D) = \frac{1}{1 - \delta_i} \left\{ \beta_i \left\{ \pi_j P_i(D) \left[ \tilde{b}_i(D) - \tilde{p}_i^{\text{buy}}(D) \right] \right\} \\
+ \beta_j \left\{ \pi_i P_j(-D) \left[ -\bar{c}_i(-D) + \tilde{p}_i^{\text{sell}}(D) \right] \right\} \right\} \quad (2)$$

where $\tilde{p}_i^{\text{buy}}(D) \equiv E \left[ \tilde{p}_i^{\text{buy}}(D) \mid H_i(D) \right]$ and $\tilde{p}_i^{\text{sell}}(D) \equiv E \left[ \tilde{p}_i^{\text{sell}}(D) \mid H_j(-D) \right]$, for $i = 1, 2$.

Some key terms are interpreted as follows:

- $\tilde{p}_i^{\text{buy}}(D)$: player $i$’s conditional mean buying price with debt holding $D$
- $\tilde{p}_i^{\text{sell}}(D)$: player $i$’s conditional mean selling price with debt holding $D$
- $\tilde{b}_i(D) - \tilde{p}_i^{\text{buy}}(D)$: player $i$’s conditional mean buying surplus with debt holding $D$
- $-\bar{c}_i(-D) + \tilde{p}_i^{\text{sell}}(D)$: player $i$’s conditional mean selling surplus with debt holding $D$
The two conditional mean surpluses must be positive because the player will trade only if it is profitable. Equation (2) shows that a player’s expected future value equals the discounted sum of surpluses weighted by the chances of buying and selling.

**Definition 5.** The *first best allocation* is achieved when help is rendered iff the benefit exceeds the cost and the seller is able to help.

For the general model, the following observations can be made (all proofs are relegated to the Appendix).

**Proposition 1.** (i) An autarky soft debt equilibrium always exists. (ii) The first best allocation can never be achieved in a soft debt equilibrium.

While part (i) of the Proposition confirms that a soft debt equilibrium always exists, part (ii) proclaims that inefficiency is inevitable in soft debt equilibria. The question is by how much trades would be limited under soft debt equilibria. The next two sections demonstrate that one possibility is to restrict trades so that the debt balance does not exceed certain debt limits. Solving for equilibria with trade would require some simplifying assumptions to the general model.

### 3 Discrete Benefit Model with Proportional Cost

Assume $\delta_1 = \delta_2 = \delta$, $\beta_1 = \beta_2 = 1/2$, $\pi_1 = \pi_2 = \pi$, $(\bar{b}, \bar{c})$ are identically distributed for the two players, and focus on symmetric equilibria where $V_1$ and $V_2$ coincide. Consider the case where $\bar{b}$ follows a uniform discrete distribution, realizing each outcome of $1, \ldots, B$ with probability $1/B$. The cost corresponding to each benefit of $b$ is fixed at $\alpha b$ where $\alpha \in (0, 1)$ is constant, so the first best outcome is for the players to trade in all rounds as long as the seller is capable. Although $b$ is not continuous, in practice the benefit can be measured in units as small as one wants.

Suppose $d(\cdot, \cdot)$ is homogenous of degree 1, then $d(b, \alpha b) = b \cdot d(1, \alpha)$, and $D_t^k = \sum_{i=1}^{\tau-1} (b_t^i - b_{t-1}^i) \cdot d(1, \alpha)$. Since $d(1, \alpha)$ is constant, we can use the net cumulative benefit (call it the debt holding level) as an index for the debt balance. Denote by $V_k$ the expected future value when the debt holding level is $k$:

$$V_k \equiv V(k \cdot d(1, \alpha))$$
Define \( p_{k+b,k} \equiv \delta (V_{k+b} - V_k) \), \( b = 1, ..., L - k \) as the soft selling price received by the seller holding debt level \( k \) for providing a benefit of \( b \) (or equivalently the soft buying price paid by the buyer for a benefit of \( b \) when his debt holding is \( k + b \)). Note that \( p_{k+b,k} = \sum_{r=0}^{b} p_{k+r+1,k+r} \). That is, the soft price for \( b \) units of benefits is the sum of \( b \) one-unit soft prices.

**Definition 6.** A *debt limit strategy* with limit \( L > 0 \) is a strategy whereby the player buys (if he is drawn as the buyer) and sells (if capable seller) iff debt levels in absolute terms do not exceed \( L \) both before and after the transaction.

A *debt limit equilibrium* with limit \( L \) is a soft debt equilibrium in which both players follow the debt limit strategy with limit \( L \).

In other words, under the strategy a player will trade as long as (i) he is capable and (ii) after the trade, neither he will owe nor he will be owed more than \( L \). Once the debt level falls outside the limit (which is off equilibrium path), autarky prevails forever. The limit \( L \) defines the range of soft debt level that the players engage in. A higher \( L \) means more opportunity for soft transactions. Therefore \( L \) can be regarded as depth of relationship.

For a strategy profile to constitute an equilibrium, the following IC and IR conditions have to be satisfied:

\[
\text{IC (buyer): } p_{k,k-b} < b \text{ if } b = 1, ..., L + k
\]

\[
\text{IC (seller): } p_{k+b,k} > \alpha b \text{ if } b = 1, ..., L - k
\]

\[
\text{IR: } V_k > 0
\]

for \( k = 0, \pm 1, ..., \pm L \).

The IC conditions ensure that it is profitable for the buyer to buy and the seller (if capable) to sell when the resulting debt level falls within the limit. The IR condition guarantees that the player will always stay in the relationship. Note that the IR conditions are interim IR conditions, i.e., they need to be satisfied not only ex ante (before the drawing in each round), but also ex post.

There are a total of \( L(L + 1) \) inequalities in the IC condition for buyer, the same number in the IC condition for seller, and \( 2L \) in the IR condition.

From Proposition 1 we already know that the first best allocation cannot be attained. The following proposition shows that when the costs are low enough, simple “debt limit
strategies” would constitute equilibria with trade.

**Proposition 2.** Consider any positive integer \( L \leq \frac{B}{2} \). In the discrete benefit model with proportional cost, if

\[
\alpha < \frac{1}{2B \left( \frac{1}{\delta} - \frac{1}{\pi} \right) + 1}
\]

then there exists a debt limit equilibrium with limit \( L \).

The lower \( B \) is, the higher the cost ratio \( \alpha \) that can be sustained in an equilibrium. Conversely, a high \( B \) means an equilibrium is possible only if the \( \alpha \) is low. This is because a high \( B \) implies a high potential limit \( L \), which means a player may find herself owed a lot from the other. The cost ratio then needs to be low enough to entice the players to engage in the relationship.

The higher \( \delta \) and \( \pi \) are, the higher \( \alpha \) that can be supported. If \( \delta \) is high, which means the players are patient and meet frequently, then an equilibrium can be sustained even with a high \( \alpha \). On the other hand, a high \( \pi \) means the players are able to help each other with high probability, which again make a high cost ratio affordable. The affordable cost ratio approaches 1 as \( \delta \) and \( \pi \) both tend to 1.

For deep relationships like those between family members, \( \delta \) and \( \pi \) are high because they meet frequently and are often able to help each other. Also, \( \alpha \) is low because they are familiar with each other’s taste, habit, information, etc., which means it takes relatively little effort to help each other. Therefore a high \( B \), and thus \( L \) can be supported. Conversely, between strangers \( \delta \) and \( \pi \) are low and \( \alpha \) is high, so only a low \( L \) is possible.

It is obvious from Proposition 2 that multiple debt limit equilibria can coexist. As long as the condition on \( \alpha \) is satisfied, all debt limit equilibria with limits smaller or equal to \( \frac{B}{2} \) exist. In addition there is always the autarky equilibrium. At first glance the equilibrium seems indeterminate. However, a closer examination would reveal that players will always incline to reach the highest equilibrium limit (\( \frac{B}{2} \) for even \( B \) or \( \frac{B}{2} - 1 \) for odd \( B \)) supported by the cost structure. First note the following proposition.

**Proposition 3.** Denote by \( V_k^L \) the expected future value in the debt limit equilibrium with limit \( L \) when the player’s debt holding level is \( k \). Then \( V_{k + 1}^{L'} > V_k^L \) iff \( L' > L \) for all \( k = 0, \pm 1, \ldots, \pm L \).

The intuition for the proposition is as follows. A higher debt limit would allow more trading opportunities that are not available under a lower limit. For instance, a player
with debt holding level \( k \) could potentially buy a benefit up to \( L' + k \) or sell a benefit up to \( L' - k \) under limit \( L' \). The range would be narrower if the limit is \( L \) instead. The difference in the scope of trade holds true in every round. Since the trades are mutually beneficial, the expected future value is higher with a higher debt limit.

In any round, the remaining candidates of equilibrium limits are bounded between the highest debt level attained thus far at the lower end, and \( B/2 \) at the upper end. Now suppose in the current round if the transaction goes through the debt level will exceed the highest attained level but not \( B/2 \). The buyer in this round would have no reservation in buying. If the seller sells, not only that the buyer will profit from the transaction, the transaction will also signify a higher limit, which means a higher expected future value (as shown in Proposition 3). If the seller does not sell, trade will not occur but there is no harm in trying to buy anyway. Conversely, the same logic applies to the (capable) seller. Being aware of each other’s calculation, the players will always agree to trade within the maximum limit \( B/2 \).

Therefore although multiple equilibria exist, the one with the maximum limit will prevail ultimately.

4 Discrete Benefit Model with Random Private Cost

In this section, the discrete benefit model is modified by assuming instead of a fixed cost-to-benefit ratio, that the cost is random and only observable to the seller. In particular, for each benefit \( b \in \{1, 2, ..., B\} \), the corresponding cost \( \tilde{c}_b \) follows some distribution over \((\underline{c}_b, \overline{c}_b)\) where \( 0 < \underline{c}_b < \overline{c}_b \). The upper limit \( \overline{c}_b \) can exceed \( b \), therefore rendering help can be inefficient. The distribution of \( \tilde{c}_b \) is common knowledge. Let \( \hat{c}_b \equiv E(\tilde{c}_b) \).

Since the actual cost is unobservable to the buyer, costs do not enter the soft debt formula. The cumulative net benefit is taken as the soft debt balance, i.e., \( D_\tau = \sum_{t=1}^{\tau-1} (b^t_i - b^t_j) \). Like before, \( V_k \) denotes the expected future value when the soft debt holding is \( k \), and \( p_{k+b,k} \) denotes the soft price \( p_{k+b,k} = \delta (V_{k+b} - V_k) \).

The IC condition for buyer and the IR condition are the same as in Section 3. The IC condition for (capable) seller is:

\[
\text{IC (seller): } p_{k+b,k} > \overline{c}_b \text{ if } b = 1, ..., L - k; \ k = 0, \pm 1, ..., \pm L
\]

This IC condition ensures that the capable seller is willing to sell even if the highest cost
for delivering the benefit is drawn.

**Proposition 4.** Consider any positive integer $L \leq B/2$. In the discrete benefit model with random private cost, if

$$
\tilde{c}_b < \frac{b \left( L + k + \frac{b+1}{2} \right) + \sum_{r=0}^{b-1} \tilde{c}_{L-k-r}}{2B \left( \frac{1}{\delta-1} - \frac{1}{\pi} \right) + 2L + 1}
$$

for $b = 1, ..., 2L$, $k = -L, -L + 1, ..., L - b$, then there exists a debt limit equilibrium with limit $L$.

All transactions that occur under the equilibrium are efficient, i.e. $\tilde{c}_b < b$ for all $b \leq 2L$. Moreover, there always exists some distributions of $\{\tilde{c}_b, b = 1, ..., 2L\}$ that satisfy (4).

Again $L$ measures the depth of the relationship. The more frequently the players meet (higher $\delta$), or the more likely the seller is able to help (higher $\pi$), or the smaller the maximum benefit $B$ (which means a lower potential $L$), the easier it is for the cost structure to support the equilibrium. On the other hand, the more efficient they are in helping each other (lower $\tilde{c}_b$’s in general), the higher $L$ can be sustained. Therefore the debt limit is highest in closely knit groups where members understand each other’s needs well, such as the family. At the other extreme, the debt limit would be very low between strangers.

As shown in the proof, the right-hand side of (4) in the proposition is just $p_{k+b,k}$. Therefore (4) is equivalent to IC (seller). To understand the formula for the soft price, it would be easier to start with the soft price for one unit of benefit:

$$
p_{k+1,k} = \frac{L + k + 1 + \tilde{c}_{L-k}}{2B \left( \frac{1}{\delta-1} \right) + 2L + 1}
$$

This is the increase in future value when the debt holding increases from $k$ to $k+1$. The value increase for two reasons. First, the higher debt holding opens up the opportunity to receive a maximum benefit of $L + k + 1$ (which will bring his debt position to the lowest limit). Second, at the same time he owes his counterpart one less (or she owes him one more), so he will not help if the benefit drawn is $L - k$ or more (the $\tilde{c}_{L-k}$ term).

These factors are adjusted by the denominator for realizing these additional values under different possibilities in different future time. The higher $B$ is, the lower the chance
that the future benefits and costs fall within the debt limit and hence the less likely that
the values can be realized soon. On the other hand, the higher $\delta$ is, either because the
agents meet more frequently or they are more patient, the higher the values will be. The
soft price for $b$ units of benefit $p_{k+b,k}$ is just the summation of $b$ number of one-unit soft
price.

As in Section 3, although multiple equilibrium may exist, the players will gravitate
toward the highest limit $L = \frac{B}{2}$. But note one difference between the two models. In for
the constant cost ratio model, if the equilibrium with limit $L/2$ exists, then all equilibria
with lower limits exist too. In the current model, the existence of an equilibrium does
not guarantee that all equilibria with lower limits exist too. Whether the equilibria exist
depends on whether (4) is satisfied by the corresponding costs.

5 Soft vs Hard Transactions

The availability of both hard and soft transactions begs the question of which one is chosen
under what situations. The key advantage of soft transactions over the hard alternatives
is their saving in transaction costs. By their very nature, soft transactions involve no
(or minimal) negotiation of price and conditions. (The absence of formal contracts also
means no formal record keeping and no taxes.) On the other hand, the lack of formal
enforcement mechanism means that they are harder to start between strangers.

It is therefore unsurprising that soft transactions are preferred when the needs are
personal and specific, for which customization would be valuable but contracting would
be costly. They are well suited for interactions between acquaintances, especially when
repeated interactions would further lower the costs as players learn more about each others’
tastes, habits and costs.

This observation is consistent with the conditions for existence of soft debt equilibria
in the previous two sections. Also, since the favors are often non-standardized, they tend
to be personal services rather than tangible goods. In contrast, when the need is general
and standardized, hard transactions gain the advantage by exploiting the economies of
scale through mass production and routine transactions.\(^5\)

A second factor that distinguish between the two alternatives concerns the medium

\(^5\)Economists have long recognized the differences between personal interactions between acquaintances
and anonymous market transactions. Adam Smith examined the two types of social interactions in The
Theory of Moral Sentiments and The Wealth of Nations respectively.
of exchange. While hard transactions enjoy the benefits of having money as the medium of exchange, soft transactions are essentially tacit barters that occur over time, which require both parties to have something that the other wants. Even if multilateral favor trading is allowed in a group through indirect favors, as in Möbius (2001), trading in soft debts still cannot match the flexibility and convenience of trading in money. Therefore soft transactions are more viable when there are common interests between the parties.

As a result of these two factors, soft transactions are pervasive everywhere from the family to the neighborhood to the workplace.\(^6\) Spouses share household duties; neighbors trade favors such as baby-sitting and house-sitting; research collaborators take turns in contributing to their projects, all probably without formal contracts. The reliance on personal contacts also suggests that soft transactions play a particularly strong role in the social fabric of developing countries. Their lack of formal records makes it difficult to compare their size with the hard sector, but a moment of casual observation would reveal that they are pervasive in essentially everyone’s daily economic activities.

Soft transactions may also be advantageous in situations where silent mutual understanding is preferred to explicit agreement (e.g. tacit collusion between businesses). In some other cases, hard transactions are simply unavailable for legal and ethical reasons. For instance, researchers can contract out tasks only to a certain extent.

Soft transactions could also play a role in property rights allocation problems. For example, although auctioning would be an efficient way for a family to allocating the right to choose TV programs, it is seldom adopted in reality. Instead family members make compromises without negotiating explicit terms of compensation to the conceder, who nevertheless expects to be compensated (or have some of his soft debt offset). The same kind of tacit give-and-take reciprocity is as well practiced between friends, neighbors and coworkers, etc.

6 Conclusion

This paper highlights soft debt as an incentive device that facilitates reciprocity under incomplete information. Players engaged in soft transactions perform profit calculations using soft prices just like in hard transactions. The first best allocation can never be

\(^6\)Although a marriage is often accompanied by a contract, the “contract” is probably far too vague to make the marriage a hard transaction.
achieved. Nevertheless, trading within certain debt limits are possible depending on the
discount factor, chance of the seller being able to help, and the distribution of benefit and
cost.

There are several directions in which the notion of soft transactions can be extended.
First, the paper focuses on isolated bilateral relationship and avoids mentioning soft mar-
kets. The model assumes the players are randomly matched and only the seller can help
the buyer. But often the buyer would be able to choose from different sellers, and vice
versa. Just like in the hard market, the soft prices will be the driving force behind the
player’s choices. In this setting, whether a relationship is exclusive (e.g. marriage) or non-
exclusive (e.g. friends) would affect the player’s decisions since breaking up an exclusive
relationship can be costly. Another possibility is to consider multilateral relationship in a
group, where indirect favors could be granted (as in Möbius (2001)).

Second, the scope of soft transactions is inherently limited by the lack of money as a
medium of exchange. Unlike hard prices, the soft price paid by the buyer and that received
by the seller can be different. The divergence means that mutually beneficial trades may
not occur. This limitation may offer an explanation for the rise of hard transactions to fill
the gap. This approach may complement the existing literature on the interface between
personal and market exchanges (e.g. Kranton (1996), Araujo (2004)).

Third, this paper presumes a simple dichotomy of hard and soft transactions, while
in reality there are many “hybrid” transactions. For example, an employment contract
contains both the hard (employment contract) and soft (vague duties within “reasonable”
bounds). The relational contract literature (e.g. Bull (1987), Levin (2003)) is pertinent to
the subject since it is concerned with combining explicit and implicit incentives in repeated
long-term relations. In fact, to the extent that a contract is incomplete, there is always
some “softness” in it and hence there is potential for long-term relationship. In purely
hard transactions which allow no room for ambiguity, if they ever exist, the parties would
simply trade and part.

The presence of hybrid transactions may also help explain the puzzle that providing
rewards and punishments sometimes has perverse effects (see Bénabou and Tirole (2006)
for a collection of examples). These phenomena suggest that analysis based purely on the
hard elements are incomplete. For example, Gneezy and Rustichini (2000) reports that
fining late parents in an Israeli day-care center actually resulted in more late arrivals. Soft
debt may shed some light on the puzzle. Without the fine system, if the parents are late
they may expect to pay a soft price in one form or another in the future. They would avoid being late if there are limited means to repay the soft debt. But with the fines in place, they may take the hard prices as substitutes for the soft prices, and therefore worry less about being late.

Lastly, as in standard neo-classical economics, this paper takes self interest maximization as individuals’ only objective. It actually even portraits what normally perceived as the most intimate relationships as long series of self-interest maximizing behaviors. But undeniably altruism plays an important role in human relationship. Evolutionary biology (e.g. Trivers (1971)) provides the theoretical basis of altruism. However, the two perspectives need not be mutually exclusive. People may sacrifice for no return, especially for loved ones and sometimes even for strangers. But even between the closest relationships, there are often give-and-take interactions that are better analyzed by soft transactions.

References


Appendix

Proof of Proposition 1

Proof. (i) If one player’s strategy is to never buy or sell regardless of the debt level, there will be no trade regardless of the other player’s response. Therefore a best response of the other player is to follow the same strategy. An autarky equilibrium is thus sustained. In terms of future values, this means setting $V_1$ and $V_2$ to zero invariably.

(ii) Assume on the contrary that the first best allocation is attained in a soft debt equilibrium. Suppose player 1 deviates from the existing strategy by not selling although he is able to help at a cost less than the benefit to player 2. Since player 2 cannot observe whether he could help, she would continue the existing strategy which prescribe her to help whenever she can at a cost below the benefit. Player 1 therefore can deviate and
profit. The same is true for player 2. This means the soft debt equilibrium that supports the first best allocation does not exist.

Proof of Proposition 2

Proof. I first compute the expected future values and soft prices assuming that both players follow the debt limit strategy. Next by using these results I verify that the incentive compatibility (IC) and individual rationality (IR) conditions hold. Finally, I argue that the debt limit equilibria exist using the one-shot deviation principle.

1. Computation of expected future value and soft prices

Suppose the debt limit has never been exceeded. Starting with a debt level between $-L$ and $L$ in any arbitrary round, since $B \geq 2L$, the debt level after a transaction (if it occurs) can be any integer within the same range. Given that all transactions go through if the debt level after transaction remains within $[-L, L]$ and the seller is capable, then for $-L \leq k \leq L$,

\[
V_k = \frac{1}{2} \left[ \frac{\pi}{B} \sum_{b=1}^{L+k} (b + \delta V_{k-b}) + \left( 1 - \frac{\pi}{B} (L + k) \right) \delta V_k \right] + \frac{1}{2} \left[ \frac{\pi}{B} \sum_{b=1}^{L-k} (-\alpha b + \delta V_{k+b}) + \left( 1 - \frac{\pi}{B} (L - k) \right) \delta V_k \right]
\]  

(5)

In the first bracket the player is assigned as the buyer. The summation term in the bracket refers to the benefits and resulting future values if transaction occurs (with his resulting debt holding level falling to $k-1, k-2, ..., -L$), while the next term captures the no transaction case. Similarly, in the second bracket the player is drawn as the seller, with his debt holding level rising to $k+1, k+2, ..., L$ if trade goes through.

Group all $V_k$ terms to the left hand side,

\[
2 \left[ \frac{B (1 - \delta)}{\pi} + L \delta \right] V_k = \sum_{b=1}^{L+k} (b + \delta V_{k-b}) + \sum_{b=1}^{L-k} (-\alpha b + \delta V_{k+b})
\]

(6)

where $\sum_{t=1}^{0} \equiv 0$.  

20
Iterate \( k \) forward to \( k + 1 \), then for \(-L - 1 \leq k \leq L - 1\),

\[
2 \left[ \frac{B (1 - \delta)}{\pi} + L \delta \right] V_{k+1} = \sum_{b=1}^{L+k+1} (b + \delta V_{k-b+1}) + \sum_{b=1}^{L-k-1} (-ab + \delta V_{k+b+1}) \tag{7}
\]

Subtract (6) from (7), for \(-L \leq k \leq L - 1\),

\[
2 \left[ \frac{B (1 - \delta)}{\pi} + L \delta \right] (V_{k+1} - V_k) = (L + k + 1) + \delta V_k + \alpha (L - k) - \delta V_{k+1}
\]

\[
\left[ \frac{2B (1/\delta - 1)}{\pi} + 2L + 1 \right] \delta (V_{k+1} - V_k) = L + k + 1 + \alpha (L - k)
\]

Recall the definition \( p_{k+1,k} \equiv \delta (V_{k+1} - V_k) \),

\[
p_{k+1,k} = \frac{L + k + 1 + \alpha (L - k)}{2B \left( \frac{1/\delta - 1}{\pi} \right) + 2L + 1} \tag{8}
\]

Since \( p_{k+b,k} = \sum_{r=0}^{b-1} p_{k+r+1,k+r} \) for \( b = 1, \ldots, L - k \),

\[
p_{k+b,k} = \frac{L + k + \frac{b+1}{2} + \alpha \left( L - k - \frac{b-1}{2} \right)}{2B \left( \frac{1/\delta - 1}{\pi} \right) + 2L + 1} \tag{9}
\]

2. Verification of IC and IR

Now I show that under (3) the players indeed find it profitable to trade whenever the debt level will remain within \( L \). The IC and IR conditions are restated below:

IC (buyer): \( p_{k,k-b} < b \) if \( b = 1, \ldots, L + k \)

IC (seller): \( p_{k+b,k} > ab \) if \( b = 1, \ldots, L - k \)

IR: \( V_k > 0 \)

By (9), IC (seller) can be rewritten as:

\[
\alpha < \frac{L + k + \frac{b+1}{2}}{2B \left( \frac{1/\delta - 1}{\pi} \right) + L + k + \frac{b+1}{2}}
\]
Since $k$ is lowest at $k = -L$ and $b$ is lowest at $b = 1$, therefore $L + k + \frac{b+1}{2} \geq 1$. By substituting this lowest value in IC (seller), we can obtain a sufficient condition for IC (seller):

$$\alpha < \frac{1}{2B \left( \frac{1}{\delta} - \frac{1}{\pi} \right) + 1}$$

which is just (3) in the proposition.

To show that IC (buyer) is met, first note that IC (buyer) is equivalent to:

$$p_{k+1,k} < 1, \ k = -L, -L + 1, \ldots, L - 1$$

(This restated IC (buyer) is obviously a sufficient condition for the original one. For the necessity part, note that if $p_{r+1,r} \geq 1$ for any $r = -L, -L + 1, \ldots, L - 1$, then the corresponding inequality for $b = 1$ and $k = r + 1$ in the original IC (buyer) will be violated.)

Use (8) and rearrange terms, IC (buyer) becomes:

$$\alpha < \frac{2B \left( \frac{1}{\delta} - \frac{1}{\pi} \right) + L - k}{L - k}$$

which must hold because $\alpha < 1$.

To verify IR, since $V_k$ is increasing in $k$ (as $p_{k+1,k} > 0$), it is sufficient to show that $V_{-L} > 0$.

By (6), when $k = -L$,

$$2 \left[ \frac{B (1 - \delta)}{\pi} + L\delta \right] V_{-L} = \sum_{b=1}^{2L} (-\alpha b + \delta V_{-L+b})$$

$$= \sum_{b=1}^{2L} (-\alpha b + \delta V_{-L} + p_{-L+b,-L})$$

$$2B \left( \frac{1}{\delta} - \frac{1}{\pi} \right) V_{-L} = \sum_{b=1}^{2L} (-\alpha b + p_{-L+b,-L})$$

which is positive according to IC (seller) for $k = -L$. Therefore IR also holds given (3).

3. Existence of equilibrium

The one-shot deviation principle states that a strategy profile constitutes a subgame perfect equilibrium iff there is no profitable one-shot deviation for any player at any
history. Consider three cases. (i) If the debt level has ever exceeded the limit, then the other player will never buy or sell. So there can be no profitable deviations. (ii) If the debt level has never exceeded the limit but will after the transaction, then again the other player will not buy or sell, and there can be no profitable deviations. (iii) If the debt level has never exceeded the limit and will remain so after the transaction, the IC conditions above guarantee that any deviation will be unprofitable.

In conclusion, by the one-shot deviation principle, the debt limit strategies with limit \( L \leq B/2 \) do constitute debt limit equilibria provided that (3) holds.

Proof of Proposition 3

Proof. Restate (5) for \( V^L_k \):

\[
V^L_k = \frac{1}{2} \left[ \frac{\pi}{B} \sum_{b=1}^{L+k} \left( b + \delta V^L_{k-b} \right) + \left( 1 - \frac{\pi}{B} (L + k) \right) \delta V^L_k \right]
+ \frac{1}{2} \left[ \frac{\pi}{B} \sum_{b=1}^{L-k} \left( -c_b + \delta V^L_{k+b} \right) + \left( 1 - \frac{\pi}{B} (L - k) \right) \delta V^L_k \right]
\]

where \( c_b = \alpha b \)

Compare it to \( V^L'_k \), written as:

\[
V^L'_k = \frac{1}{2} \left\{ \left[ \frac{\pi}{B} \sum_{b=1}^{L+k} \left( b + \delta V^L'_{k-b} \right) + \left( 1 - \frac{\pi}{B} (L + k) \right) \delta V^L'_k \right] \right\}
+ \frac{1}{2} \left\{ \left[ \frac{\pi}{B} \sum_{b=1}^{L-k} \left( -c_b + \delta V^L'_{k+b} \right) + \left( 1 - \frac{\pi}{B} (L - k) \right) \delta V^L'_k \right] \right\}
\]

There are two differences between the formula for \( V^L_k \) and that for \( V^L'_k \). First, for \( V^L_k \), there are two extra summation terms outside the brackets. These are summations of surpluses from trades and thus are both positive. Second, the \( V^L \) terms in the first formula are replaced by \( V^L' \) terms in the second. But we can expand the \( V^L \) and \( V^L' \) terms again in the same fashion as above. \( V^L' \) again has two extra positive terms over \( V^L \). Continue the process repeatedly, the difference due to the unexpanded \( V^L \) and \( V^L' \) terms tend to zero because of discounting. However, \( V^L'_k \) accumulates two extra positive terms in each iteration, therefore \( V^L'_k > V^L_k \).

When there arise a round where the drawing reveals that a transaction would push the
debt level over \( k \) for the first time but remain below \( k' \). Each player will be better-off by trading given the other player will trade too. Their future value for the same debt level will be higher, and they will gain from the transaction. The equilibrium with the highest limit dominates all the rest.

**Proof of Proposition 4**

*Proof.* Like in the proof for Proposition 2, I first compute the expected future values and soft prices assuming that both players follows the debt limit strategy, and then verify the IC and IR conditions. In verifying the IC conditions, I also show that all transactions that occur are efficient. Lastly I confirm that (4) is always met by some distributions of costs.

1. *Computation of expected future value and soft prices*

Suppose the debt limit has never been exceeded. Given that all transactions go through if the debt after transaction falls within \([-L, L]\) and the seller is able to help, then for \(-L \leq k \leq L\),

\[
V_k = \frac{1}{2} \left[ \frac{\pi}{B} \sum_{b=1}^{L+k} (b + \delta V_{k-b}) + \left(1 - \frac{\pi}{B} (L+k)\right) \delta V_k \right] \\
+ \frac{1}{2} \left[ \frac{\pi}{B} \sum_{b=1}^{L-k} (-\hat{c}_b + \delta V_{k+b}) + \left(1 - \frac{\pi}{B} (L-k)\right) \delta V_k \right]
\]

Compared to the \( V_k \) in (5) of the proof of Proposition 1, the expected cost \( \alpha b \) is replaced by \( \hat{c}_b \).

Following similar steps as in Proposition 1, we get:

\[
p_{k+1,k} = \frac{L + k + 1 + \hat{c}_{L-k}}{2B \left( \frac{1}{2} \delta - \frac{1}{\pi} \right) + 2L + 1} \tag{10}
\]

Again, since \( p_{k+b,k} = \sum_{r=0}^{b-1} p_{k+r+1,k+r} \) for \( b = 1, ..., L - k \),

\[
p_{k+b,k} = \frac{b \left( L + k + \frac{b+1}{2} \right) + \sum_{r=0}^{b-1} \hat{c}_{L-k-r}}{2B \left( \frac{1}{2} \delta - \frac{1}{\pi} \right) + 2L + 1} \tag{11}
\]

2. *Verification of IC and IR*

Now I show that under (4) the players indeed find it profitable to trade whenever
the debt level will remain within \( L \). For \( k = 0, \pm 1, \ldots, \pm L \), the IC and IR conditions are restated as follows:

IC (buyer): \( p_{k,k-b} < \bar{b} \) if \( b = 1, \ldots, L + k \)

IC (seller): \( p_{k+b,k} > \bar{c}_b \) if \( b = 1, \ldots, L - k \)

IR: \( V_k > 0 \)

Note that the right-hand side of (4) in the proposition is just \( p_{k+b,k} \). Therefore (4) is equivalent to IC (seller).

I now show that IC (buyer) is guaranteed by IC (seller). First note again that IC (buyer) is equivalent to:

\[
p_{k+1,k} < 1, \quad k = -L, -L + 1, \ldots, L - 1
\]

Use (10) and rearrange terms, IC (buyer) becomes:

\[
\hat{c}_{L-k} < 2B \left( \frac{1/\delta - 1}{\pi} \right) + L - k
\]

Substitute \( b \) for \( L - k \), IC (buyer) can be rewritten as:

\[
\hat{c}_b < 2B \left( \frac{1/\delta - 1}{\pi} \right) + b, \quad b = 1, \ldots, 2L
\]

Therefore showing that all transactions that occur are efficient (i.e. \( \bar{c}_b < \bar{b} \)) is more than sufficient to prove that IC (buyer) holds.

I will prove the efficiency by induction. From (4) and (10), pick \( k = L + b \) and \( b = 1 \),

\[
\bar{c}_1 < p_{L,L-1} = \frac{2L + \hat{c}_1}{2B \left( \frac{1/\delta - 1}{\pi} \right) + 2L + 1} \implies \bar{c}_1 < \frac{2L}{2B \left( \frac{1/\delta - 1}{\pi} \right) + 2L} < 1
\]

So all transactions of single-unit benefits are expected to be efficient.

Next, assume efficiency \( \bar{c}_r < r \) holds for \( r = 1, \ldots, b - 1 \), where \( b \leq 2L \), then from IC (seller) and (11),

\[
\bar{c}_b < p_{L,L-b} = \frac{b \left( 2L + \frac{1-b}{2} \right) + \sum_{r=1}^{b} \hat{c}_r}{2B \left( \frac{1/\delta - 1}{\pi} \right) + 2L + 1}
\]
But

\[
\sum_{r=1}^{b} \hat{c}_r \leq \sum_{r=1}^{b} c_r < \sum_{r=1}^{b-1} b + c_b = \frac{b(b - 1)}{2} + c_b
\]

(The second step holds as a result of the above assumption.) Combine the two inequalities and rearrange terms,

\[
\hat{c}_b < \frac{2Lb}{2B\left(\frac{1/δ-1}{π}\right) + 2L} < b
\]

Therefore efficiency holds for transactions involving benefit \(b\) if it holds for transactions of all lower benefits. By induction, all transactions that occur are efficient. IC (buyer) is satisfied.

IR can be verified in similar manner as in Proposition 2.

Applying the one-shot deviation principle as in the proof of Proposition 2, the debt limit equilibrium is shown to exist.

3. Existence of distribution of costs that support the equilibrium

To show that (4) can always be satisfied by some set of \(\{\bar{c}_b, \tilde{c}_b, \ b = 1, .., 2L\}\) so that the equilibrium exists, consider the case where \(\bar{c}_b = \bar{c}\) and \(\tilde{c}_b = \bar{c}/2\) for all \(b\) (for example, \(\tilde{c}_b\) is uniformly distributed over \((0, \bar{c})\) for all \(b\)). Since \(p_{k+b,k}\) is increasing in \(b\), if (4) is satisfied for \(b = 1\), then it is for all \(b\). This condition simply requires

\[
\bar{c} < \frac{L + k + 1 + \bar{c}/2}{2B\left(\frac{1/δ-1}{π}\right) + 2L + 1} \iff \bar{c} < \frac{L + k + 1}{2B\left(\frac{1/δ-1}{π}\right) + 2L + 1/2}
\]

Since \(k \geq -L\), picking a \(\bar{c}\) smaller than \(\frac{1}{2B\left(\frac{1/δ-1}{π}\right) + 2L + 1/2}\) guarantees (4) is met. \(\Box\)