REPRESENTATIONS OF $\mathfrak{sl}_2(\mathbb{C})$

Representations of Lie Algebras

Throughout, all vector spaces and algebras are over $\mathbb{C}$.

**Definition.** Let $L$ be a Lie algebra. A **representation** of $L$ is a Lie algebra homomorphism
\[ \varphi : L \to \text{End}(V) \]
In $\text{End}(V)$, the Lie bracket is the commutator. So this means for $x, y \in L$, we have
\[ \varphi([x, y]) = \varphi(x) \circ \varphi(y) - \varphi(y) \circ \varphi(x) \]
The space $V$ is then called an “$L$-module”.

**Example.** For any Lie algebra $L$, we have the “adjoint representation” of $L$ on itself:
\[ \text{ad} : L \to \text{End}(L) \]
given by $\text{ad}(x)(y) = [x, y]$. We usually write $\text{ad}_x$ instead of $\text{ad}(x)$. The fact that $\text{ad}$ is a Lie algebra homomorphism is the Jacobi identity!
\[ \text{ad}_{[x, y]}(z) = \text{ad}_x \text{ad}_y(z) - \text{ad}_y \text{ad}_x(z) \]
\[ [[x, y], z] = [x, [y, z]] - [y, [x, z]] \]
In fact, the image $\text{ad}(L)$ lies in the Lie subalgebra $\text{Der}(L) \leq \text{End}(L)$. This is again the Jacobi identity!
\[ \text{ad}_x([y, z]) = [\text{ad}_x(y), z] + [y, \text{ad}_x(z)] \]
\[ [x, [y, z]] = [[x, y], z] + [y, [x, z]] \]
The kernel of $\text{ad}$ is the **center** of $L$:
\[ Z(L) = \{ x \in L \mid [x, y] = 0 \forall y \in L \} \]
Universal Enveloping Algebras

Let $L$ be a Lie algebra, with basis $x_1, \ldots, x_n$. The tensor algebra $T(L) = \bigoplus_k V^\otimes k$ is isomorphic to $\mathbb{C} \langle x_1, \ldots, x_n \rangle$, the free associative algebra on $n$ generators. The universal enveloping algebra of $L$ is the quotient

$$U(L) = T(L)/ \langle a \otimes b - b \otimes a - [a, b] \rangle$$

$$\cong \mathbb{C} \langle x_1, \ldots, x_n \rangle / \langle ab - ba - [a, b] \rangle$$

It is a quotient of the algebra of non-commutative polynomials, where the variables satisfy

$$x_j x_i = x_i x_j - \sum_k c_{ij}^k x_k \quad (i < j)$$

Let $i: L \to U(L)$ be the inclusion.

It is “universal” in the following sense. Any associative algebra $A$ can be made into a Lie algebra under the commutator bracket. For any Lie algebra $L$ and any associative algebra $A$ (thought of as a Lie algebra), and any Lie algebra homomorphism $\varphi: L \to A$, there is an associative algebra homomorphism $\tilde{\varphi}: U(L) \to A$ such that $\varphi = \tilde{\varphi} \circ i$.

\[
\begin{array}{ccc}
U(L) & \xrightarrow{\tilde{\varphi}} & A \\
\downarrow & & \downarrow \\
L & \xrightarrow{\varphi} & A
\end{array}
\]

**Theorem.** The categories of $L$-modules and $U(L)$-modules are equivalent.

**Observation.** $V$ an irreducible $U(L)$-module $\implies$ $V$ a cyclic module $\implies$ $V$ is a quotient of $U(L)$ (as a module over itself)

**Proposition.** $U(L)$ is a Hopf algebra with

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\varepsilon(x) = 0$$

$$s(x) = -x$$

Thus, if $V$ and $W$ are $L$-modules, then $V \otimes W$ is an $L$ module via

$$L \xrightarrow{i} U(L) \xrightarrow{\Delta} U(L) \otimes U(L) \xrightarrow{\rho_V \otimes \rho_W} \text{End}(V \otimes W)$$

$$x \mapsto x \mapsto x \otimes 1 + 1 \otimes x \mapsto \rho_V(x) \otimes \text{Id}_W + \text{Id}_V \otimes \rho_W(x)$$
**Semisimple Lie Algebras**

**Definition.** Let $L$ be a lie algebra. Define $L^0 = L$, $L^1 = [L, L]$, and $L^{k+1} = [L^k, L]$. Then $L$ is “solvable” if the chain $L \supseteq L^1 \supseteq \cdots \supseteq L^k \supseteq \cdots$ is eventually zero.

**Example.** Any abelian Lie algebra (i.e. $[x, y] = 0$ for all $x, y \in L$) is solvable, since $[L, L] = 0$.

**Definition.** The “radical” of the Lie algebra $L$ is the unique maximal solvable ideal $\text{Rad}(L) \leq L$.

**Definition.** The “Killing form” is a symmetric bilinear form $K$ on $L$ given by $K(x, y) = \text{tr}(\text{ad}_x \text{ad}_y)$

**Definition.** A Lie algebra $L$ is “semisimple” if any of the following equivalent conditions hold:

- Every $L$-module is “completely reducible”.
- $\text{Rad}(L) = 0$
- $L$ has no nonzero solvable ideals
- $L$ has no nonzero abelian ideals
- $K$ is nondegenerate

Let $L$ be a semisimple Lie algebra, and $\varphi: L \to \text{End}(V)$ any representation. Then for $x \in L$, $\varphi(x)$ has its “Jordan decomposition” $\varphi(x) = \delta + \varepsilon$

where $\delta$ is diagonalizable and $\varepsilon$ is nilpotent, and $[\delta, \varepsilon] = 0$.

In the particular case of the adjoint representation, $\varphi(x) = \text{ad}_x \in \text{End}(L)$. Let $x_\delta, x_\varepsilon \in L$ be such that $\text{ad}_{x_\delta} = \delta$ and $\text{ad}_{x_\varepsilon} = \varepsilon$. Then necessarily $x = x_\delta + x_\varepsilon$. The fact that $L$ is semisimple guarantees that for any representation, the Jordan decomposition of $\varphi(x)$ is exactly $\varphi(x) = \varphi(x_\delta) + \varphi(x_\varepsilon)$

**Corollary.** If $L$ is semisimple and $x \in L$ such that $\text{ad}_x$ is diagonalizable, then $\varphi(x)$ is diagonalizable for any representation $\varphi$. 
Definition. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is a 3-dimensional vector space with basis $h, e, f$ and bracket relations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h$$

Representations

Example. The “standard” representation of $\mathfrak{sl}_2(\mathbb{C})$ is the map $\mathfrak{sl}_2(\mathbb{C}) \to \text{End}(\mathbb{C}^2)$ given by

$$h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

The image is the space of matrices $A$ with $\text{tr}(A) = 0$.

Example. The adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$ in the basis $h, e, f$ looks like $\mathfrak{sl}_2(\mathbb{C}) \to \text{End}(\mathbb{C}^3)$:

$$h \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad e \mapsto \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

The Killing form $K$ in the basis $h, e, f$ looks like

$$K = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$$

The determinant is $\det(K) = -128$, so $K$ is nondegenerate. Therefore $\mathfrak{sl}_2(\mathbb{C})$ is semisimple.

Since $\mathfrak{sl}_2$ is semisimple, the Corollary tells us that $\varphi(h)$ is diagonalizable for any representation.

So let $V$ be any (finite-dim) $\mathfrak{sl}_2$-module, and write it as sum of $h$-eigenspaces

$$V = \bigoplus_{\lambda} V_{\lambda}$$

where $V_{\lambda} = \{v \in V \mid h \cdot v = \lambda v\}$. 
**Question**: How do $e$ and $f$ act on $V$, relative to $V_\lambda$?

Since $[h, e] = 2e$, we have for any $v \in V$:

$$h \cdot (e \cdot v) = e \cdot (h \cdot v) + 2e \cdot v$$

In particular, if $v \in V_\lambda$, then

$$h \cdot (e \cdot v) = (\lambda + 2)e \cdot v$$

This means that $e \cdot v \in V_{\lambda + 2}$. Similarly, $f \cdot v \in V_{\lambda - 2}$.

So for some $\lambda$ for which $V_\lambda \neq 0$, we have

$$V = \bigoplus_{\mu \in \lambda + 2\mathbb{Z}} V_\mu$$

Since $V$ is finite-dimensional, this sum is finite, so there is some “maximal” $\lambda$ so that

$$V = \bigoplus_{k=0}^{\ell} V_{\lambda - 2k}$$

**Proposition.** Let $v \in V_\lambda$, where $\lambda$ is “highest”. If $V$ is irreducible, then the set $v, fv, f^2 v, \ldots$ is a basis for $V$.

**Proof.** A calculation shows that

$$e f^m v = m(\lambda - m + 1) f^{m-1} v$$

□

**Corollary.** All the $V_\lambda$ are 1-dimensional.

**Corollary.** $\lambda$ is a positive integer.

**Proof.** Since $\lambda - 2\ell$ is the “lowest” eigenvalue, we have

$$0 = e f^{\ell+1} v = (\ell + 1)(\lambda - \ell) f^{\ell}$$

In particular, $\lambda = \ell$.

□

**Theorem.** For every positive integer $n$, there is a unique irreducible $\mathfrak{sl}_2$-module of dimension $n$, with eigenvalues $n-1, n-3, \ldots, -n + 3, -n + 1$. It is denoted $V^{(n-1)}$.  

Example. The standard representation (on $\mathbb{C}^2$ with basis $x, y$) is exactly $V^{(1)}$, where $\langle x \rangle = V_1$ and $\langle y \rangle = V_{-1}$. The action looks like

\[
\begin{align*}
   hx &= x, & hy &= -y \\
   ex &= 0, & ey &= x \\
   fx &= y, & fy &= 0
\end{align*}
\]

The symmetric algebra $\text{Sym}(V)$ can be naturally identified with $\mathbb{C}[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ is a basis for $V$. So the symmetric powers $\text{Sym}^k(V)$ can be naturally identified with the space of monomials of degree $k$.

There is an $\mathfrak{sl}_2$-action on $\mathbb{C}[x, y]$ given by

\[
\begin{align*}
   h &\mapsto x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, & e &\mapsto x \frac{\partial}{\partial y}, & f &\mapsto y \frac{\partial}{\partial x}
\end{align*}
\]

Each homogeneous piece of this module is invariant, and isomorphic to $V^{(k)}$. 