Problem Set 1

Samuel Otten
Michigan State University
MTH 425 · Summer 2008

1

We will consider the set

\[ S = \left\{ z : \frac{z}{z^2 + 1} < \frac{1}{\sqrt{2}} \right\}. \]

Let \( z = x + iy \). Then \( z \in S \) if and only if

\[ \frac{1}{\sqrt{2}} > \left| \frac{x + iy}{x + iy + 1} \right| = \frac{|x + iy|}{|(x + 1) + iy|} = \frac{\sqrt{x^2 + y^2}}{\sqrt{(x + 1)^2 + y^2}}. \]

Note that all quantities are nonnegative, so we can square both sides and maintain the inequality, yielding

\[ \frac{1}{2} > \frac{x^2 + y^2}{(x + 1)^2 + y^2}. \]

This easily simplifies to

\[ 2x + 1 > x^2 + y^2, \]

which defines the interior of a circle. To see this, let us move the \( 2x \) to the right-hand side and add 1 to each side thus completing the square, giving

\[ 2 > x^2 - 2x + 1 + y^2 = (x - 1)^2 + y^2. \]

Hence \( S \) is the open disc in \( \mathbb{C} \) centered at \( 1 + 0i \) with radius \( \sqrt{2} \) (see figure).
Let \( S = \{ z : |z^2 - 1| < 1 \} \). Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = z^2 - 1 \). Then \( z \in S \) if and only if \( |f(z)| < 1 \). In terms of sets, then, \( S \) is the preimage under \( f \) of the unit disc \( N(0, 1) \); that is, as sets

\[
S = f^{-1}(N(0, 1)) = (N(1, 1))^1/2.
\]

Let \( w \in N(1, 1) \) and \( w = re^{i\theta} \). Then \( w^{1/2} = \sqrt{r} e^{i(\theta/2 + k\pi)} \in S \) for \( k = 0, 1 \). So every point in \( N(1, 0) \) will have two preimages under \( f \), one for \( k = 0 \) and one for \( k = 1 \). We shall work with \( k = 0 \) and later include the other portion by rotating \( \pi \) radians about the origin.

We can easily determine particular points in \( S \) (or \( \delta S \)). From \( w = 2 \) we get \( \sqrt{2} \in \delta S \). From \( w = \sqrt{2} e^{\pm i\pi/4} \) we get \( \sqrt{2}^{1/2} e^{\pm i\pi/8} \in \delta S \). When \( |w| = 1 \) its modulus is fixed by the square root and its angle is cut in half, so the arc of the unit circle contained in \( N(1, 1) \) will remain at a distance of 1 from the origin but \( S \) will contain half the arc length. Finally, looking within \( N(1, 1) \) as \( r \) approaches 0 we see \( \theta \) bounded by \( \pi/2 \) and \(-\pi/2 \). Correspondingly within \( S \), the moduli will also approach 0 but the arguments will be bounded by \( \pi/4 \) and \(-\pi/4 \). This gives us the picture of \( S \) shown below (together with the other component where \( k = 1 \)).

This is an open set because it does not contain any of its boundary points. The set \( S \) is not a domain, however, because it is not connected. There is no way to polygonally join the two components because the origin is excluded from \( S \).
We wish to find all solutions to the equation $\cos z = \sqrt{3}$.

Let $z = x + iy$. Then by the definition of cosine we have

$$\cos z = \cos x \cosh y - i \sin x \sinh y = \sqrt{3},$$

which implies

$$\begin{cases} \cos x \cosh y = \sqrt{3} \\ \sin x \sinh y = 0 \end{cases}.$$ 

The second equation is satisfied if $x = k\pi$ for some $k \in \mathbb{Z}$ or if $y = 0$.

In the first case ($x = k\pi$) we see from the first equation that $\cos x = \pm 1$, thus forcing $\cosh y = \pm \sqrt{3}$. Of course, $\cosh y$ is never negative. This means $x = 2n\pi$, $y = \cosh^{-1}\sqrt{3}$ (which is double valued) is one set of solutions to the original equation.

In the second case ($y = 0$) we see from the first equation that $\cosh y = \cosh 0 = 1$. Then necessarily $\cos x = \sqrt{3}$, which is impossible since $\sqrt{3} > 1$.

So this case yields no solutions to the original equation.

Therefore the solution set to the equation $\cos z = \sqrt{3}$ is

$$\{ z \mid x = 2n\pi, y = \cosh^{-1}\sqrt{3} \}$$

for $n \in \mathbb{Z}$ and with the understanding that $\cosh^{-1}\sqrt{3}$ is double-valued.

4

Let $f(z) = \text{PV}(z^{3/2})$ and $g(z) = z^2 + 1$, and define $F(z) = f(g(z))$. We will determine the natural domain $D$ of analyticity of $F$.

Because of $f$’s relationship to the square root function, we know that the domain of analyticity of $f$ is the plane with the nonpositive real axis removed. The function $g$, on the other hand, is entire. To find $D$ we must make sure that points are not mapped by $g$ into the branch cut of $f$.

What points would be mapped by $g$ into the branch cut of $f$? This would occur if and only if $\text{Re}(g(z)) \leq 0$ and $\text{Im}(g(z)) = 0$; that is, if and only if

$$x^2 - y^2 + 1 \leq 0,$$

and

$$2xy = 0.$$

Suppose $x = 0$. Then the equation is satisfied and the inequality becomes $1 - y^2 \leq 0$, or equivalently, $y \geq 1$. Thus we must exclude $\{ z \mid x = 0, y \geq 1 \}$ from $D$. Now suppose $y = 0$. Then the equation is satisfied and the inequality becomes $x^2 + 1 \leq 0$. Since $x$ is real this inequality has no solutions.

Therefore $D = \mathbb{C} \setminus \{ z \mid x = 0, y \geq 1 \}$ (see figure).
By the chain rule, $F'(z) = f'(g(z)) \cdot g'(z)$. We can use the power rule to find $g'(z) = 2z$ and $f'(z) = 3/2 \cdot PV\sqrt{z^2}$. If we take $\sqrt{z}$ to mean this principal value, then we have

$$F'(z) = \frac{3\sqrt{z^2 + 1}}{2} \cdot 2z = 3z\sqrt{z^2 + 1}.$$ 

5

Let $u(x, y) = 2y - 2xy$. We will verify that $u$ is harmonic. Recall that this requires $u \in C^2(D)$ and $u_{xx} + u_{yy} = 0$ in $D$, where $D$ is the domain of analyticity of $u$. Since $u$ is a polynomial in two variables, we know that it is an entire function (i.e., $D = \mathbb{C}$) and $u \in C^2(D)$. Thus all that remains is to verify Laplace’s equation.

Observe

$$u_x = -2y \quad u_y = 2 - 2x,$$

$$u_{xx} = 0 \quad u_{yy} = 0.$$ 

Thus we have $u_{xx} + u_{yy} = 0 + 0 = 0$. So $u$ is harmonic. What is its harmonic conjugate?

Recall that $v(x, y)$ is the harmonic conjugate of $u$ if $v$ itself is harmonic and together they satisfy the Cauchy-Riemann equations. Thus

$$v_x = -u_y = 2x - 2,$$

$$v_y = u_x = -2y.$$ 

The first equation implies $v(x, y) = x^2 - 2x + \phi(y)$ for some function $\phi$. From the second equation we know $\phi'(y) = v_y = -2y$, so $\phi(y) = -y^2 + c$ for some constant $c$. So the harmonic conjugate of $u(x, y)$ is

$$v(x, y) = x^2 - 2x - y^2 + c.$$ 

Note that $v$ is harmonic because it is a polynomial and hence smooth, and also $v_{xx} + v_{yy} = 2 - 2 = 0$.

Define $f(z) = u(x, y) + i v(x, y) = (2y - 2xy) + i(x^2 - 2x - y^2)$. It follows from a theorem in class (May 21st) that $f$ is analytic.