We would like to use residues to find \[ \text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^2 + 2x + 2}. \]

Let \( f(z) = \frac{1}{z^2 + 2z + 2} \) and note that on the real axis \( f \) corresponds to the integrand above. From the equation \( x^2 + 2x + 2 = 0 \) and the quadratic formula, we find that the singularities of \( f \) are \(-1 \pm i\) and both are clearly simple poles.

Let \( R > \sqrt{2}, \ c_1 : z = x \ (-R \leq x \leq R) \), and \( c_2 : z = Re^{i\theta} \ (0 \leq \theta \leq \pi) \). Then \( c = c_1 + c_2 \) is a simple closed contour that contains \(-1 + i\) in its interior (see figure). By the Residue theorem,

\[
\oint_c f(z)dz = 2\pi i \text{ Res}_{-1+i} f(z).
\]

Let us compute the right-hand side of this equation.

Since \( f(z) = \frac{1}{(z-(-1+i))(z-(-1-i))} \), we can see that \( \phi(z) = \frac{1}{z-(-1-i)} = \frac{1}{z+1+i} \). Thus \( \phi(-1+i) = \frac{1}{2i} \) and so \( \text{Res}_{-1+i} f(z) = \frac{1}{2i} \). This means

\[
\oint_c f(z)dz = 2\pi i \left( \frac{1}{2i} \right) = \pi.
\]

It follows that

\[
\oint_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz = \int_{-R}^{R} \frac{dx}{x^2 + 2x + 2} + \int_{c_2} f(z)dz = \pi.
\]

In particular, as \( R \) tends to \( \infty \) we get

\[
\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \pi - \int_{c_2} f(z)dz.
\]  \hspace{1cm} (1)

All that remains is to show that the integral over \( c_2 \) tends to 0 as \( R \) tends to \( \infty \).
If \( z \) is on \( c_2 \) then \( |z| = R > \sqrt{2} \) and
\[
|z^2 + 2z + 2| = |z - (-1 + i)||z - (-1 - i)| \geq |z| - |1 + i|| |z| - |1 - i|
\]
\[=(R - \sqrt{2})(R - \sqrt{2}) = R^2 - 2\sqrt{2}R + 2.\]
So with regard to the integral we have
\[
\left| \int_{c_2} f(z)\,dz \right| \leq \int_{c_2} \frac{|dz|}{|z^2 + 2z + 2|} \leq \frac{1}{R^2 - 2\sqrt{2}R + 2} \int_0^\pi |iR e^{i\theta}| \,d\theta
\]
\[= \frac{R}{R^2 - 2\sqrt{2}R + 2} \int_0^\pi |d\theta| = \frac{\pi R}{R^2 - 2\sqrt{2}R + 2}.\]
It is clear that this tends to 0 as \( R \) tends to \( \infty \) because the denominator dominates the numerator of the bound. Therefore, from equation 1 we have
\[
\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \pi.
\]

2

We shall use the Residue theorem to show that, for \(-1 < a < 1\),
\[
\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2\pi}{\sqrt{1 - a^2}}.
\]

The left-hand side suggests integration about a circle, say \(|z| = 1\), which can be parametrized by \( z = e^{i\theta}, \, 0 \leq \theta < 2\pi \). Then \( dz = i e^{i\theta} \,d\theta \) and consequently \( d\theta = dz/iz \). Within the integrand we can substitute for \( \cos \theta \) by using the identity
\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}.
\]
All together we have
\[
\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \oint_{|z|=1} \frac{dz}{z + \frac{a z^{-1}}{2} + \frac{a}{2}} = \frac{2}{i} \oint_{|z|=1} \frac{dz}{az^2 + 2z + a}
\]
\[= \frac{2}{i} \oint_{|z|=1} \frac{dz}{az^2 + 2z + a} = \frac{2}{i} \left( 2\pi i \text{Res}_{z=z_0} f(z) \right).\]

Let \( f(z) = \frac{1}{az^2 + 2z + a} \). The singularities of \( f \) are the roots of the equation \( az^2 + 2z + a = 0 \) which, by the quadratic formula, are
\[z_0 = \frac{-1 + \sqrt{1 - a^2}}{a} \quad \text{and} \quad z_1 = \frac{-1 - \sqrt{1 - a^2}}{a}.
\]
Since \(-1 < a < 1\), it can be shown using L’hopital’s rule that \(|z_0| < 1\) and \(|z_1| > 1\). This means that \( z_0 \) is in the interior of our contour of integration, whereas \( z_1 \) is in the exterior. So by the Residue theorem,
\[
\frac{2}{i} \oint_{|z|=1} \frac{dz}{az^2 + 2z + a} = \frac{2}{i} \left( 2\pi i \text{Res}_{z=z_0} f(z) \right).
\]
Now, $f(z)$ is a quotient of polynomials, so we can use the formula below (where $q(z) = az^2 + 2z + a$),

$$\text{Res}_{z=z_0} f(z) = \frac{1}{q'(z_0)} = \frac{1}{2az_0 + 2} = \frac{1}{2a \left( \frac{-1+\sqrt{1-a^2}}{a} \right) + 2} = \frac{1}{2\sqrt{1-a^2}}.$$ 

Putting everything together yields

$$\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2}{i} \oint_{|z|=1} f(z) dz = \frac{2}{i} \left( 2\pi i \text{Res}_{z=z_0} f(z) \right) = \frac{2}{i} \left( \frac{2\pi i}{2\sqrt{1-a^2}} \right) = \frac{2\pi}{\sqrt{1-a^2}}.$$

3

We wish to find a linear fractional transformation $T(z)$ that maps the open unit disk, $\{z : |z| < 1\}$, onto the right half-plane, $\{w = u + iv : u > 0\}$.

By a theorem from class, a linear fractional transformation is uniquely determined by the specification of three distinct images of three distinct points. For this problem, we need to send three points from the boundary circle of the unit disk to the boundary line of the right half-plane in such a way that the interior of the disk is mapped onto the right half-plane. There are many ways to do this. One approach is to define a map $f$ that sends the points $(1, i, -1) \mapsto (0, 1, \infty)$ and a map $g$ that sends $(0, 1, \infty) \mapsto (i, 0, -i)$, and then put $T = g \circ f$.

Using cross ratios, we have

$$f(z) = \frac{(z - 1) (i + 1)}{(z + 1) (i - 1)} = \frac{(1 + i)z + (-1 - i)}{(-1 + i)z + (-1 + i)},$$

and

$$g^{-1}(w) = \frac{(w - i) (0 + i)}{(w + i) (0 - i)} = \frac{iw + 1}{-iw + 1}.$$
If we put \( g^{-1}(w) = Z \) then we can solve for \( w \) as follows,

\[
Z = \frac{iw + 1}{-iw + 1}
\]

\[(1 - iw)Z = iw + 1 \]

\[Z - iZw = iw + 1 \]

\[Z - 1 = w(iZ + i) \]

\[iZ + i = w. \]

Thus we define

\[g(Z) = \frac{Z - 1}{iZ + i}\]

and consequently

\[T(z) = g(f(z)).\]

Using the formulas for \( f \) and \( g \) (but sparing some of the algebraic details) we have

\[T(z) = \frac{f(z) - 1}{if(z) + i} \]

\[= \frac{i(z + z - 1 - i) - 1}{i \left( \frac{i(z + z - 1 - i)}{i(z - z + 1 - i)} \right) + i} \]

\[= \frac{i(z + z - 1 - i) - iz + 1 - i}{iz - z - 1 + i} \]

\[= \frac{2z - 2i}{-2z - 2i}. \]

It can be verified that \( T \) maps \((1, i, -1)\) to \((i, 0, -i)\). Moreover, the unit disk is taken to the right half-plane as desired. Observe, as an example,

\[T(0) = \frac{2(0) - 2i}{-2(0) - 2i} = \frac{-2i}{-2i} = 1 = \text{Re} \ 1 > 0. \]

4

We are considering a uniform semi-infinite thin metal plate in the shape of the upper half-plane, and it is insulated on both surfaces. What is the bounded steady temperature \( T(x, y) \) in the plate if the boundary conditions are:

1. \( T(x, 0) = 0 \) for \( x < -1 \),
2. \( T(x, 0) = 1 \) for \( x > 1 \), and
3. the boundary \(-1 < x < 1, y = 0\) is insulated?
We know that the sine function maps the semi-infinite strip above the interval \([-\pi/2, \pi/2]\) injectively onto the upper half-plane and is conformal except at the corners. Thus we can use the function

$$w = \arcsin z$$

to map our original domain onto a more manageable one (see figure).

Observe that our domain is now \(\{u + iv \mid -\pi/2 < u < \pi/2, v > 0\}\) and our boundary conditions are

1. \(T(-\pi/2, v) = 0\) for \(v > 0\),
2. \(T(\pi/2, v) = 1\) for \(v > 0\), and
3. the boundary along \(-\pi/2 < u < \pi/2, v = 0\) insulated.

The temperature function for this boundary value problem is

$$T(u, v) = \frac{1}{\pi} u + \frac{1}{2}.$$

We must pull this back so it is in terms of \(x\) and \(y\).

Since we mapped forward by \(\arcsin z\), we pull back along \(\sin w\). Thus \(x = \sin u \cosh v\) and \(y = \cos u \sinh v\). Except at \(u = 0\) (where \(\sin u = 0\)) and \(u = \pm \pi/2\) (where \(\cos u = 0\)), we have

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1.$$

In this case, by properties of hyperbolas (see figure), we arrive at the equation

$$\sqrt{(x + 1)^2 + y^2} - \sqrt{(x - 1)^2 + y^2} = 2 \sin u. \quad (2)$$
The equation 2 also holds when \( u = 0 \) and \( u = \pm \pi/2 \). Pulling back the imaginary axis gives \( x = 0 \) and so
\[
\sqrt{(0 + 1)^2 + y^2} - \sqrt{(0 - 1)^2 + y^2} = \sqrt{1 + y^2} - \sqrt{1 + y^2} = 0 = 2 \sin(0).
\]
Pulling back \( u = \pi/2 \) gives \( y = 0 \) and so
\[
\sqrt{(\cosh u + 1)^2} - \sqrt{(\cosh u - 1)^2} = \cosh u + 1 - \cosh u + 1 = 2 = 2 \sin(\pi/2),
\]
and pulling back \( u = -\pi/2 \) also gives \( y = 0 \) but now \( \cosh u < 0 \). So
\[
\sqrt{(\cosh u + 1)^2} - \sqrt{(\cosh u - 1)^2} = -\cosh u - 1 + \cosh u - 1 = -2 = 2 \sin(-\pi/2).
\]
Therefore we can solve equation 2 for \( u \)
\[
u = \arcsin \left( \frac{\sqrt{(x + 1)^2 + y^2} - \sqrt{(x - 1)^2 + y^2}}{2} \right)
\]
and substitute this into the formula for \( T \), which yields
\[
T(x, y) = \frac{1}{\pi} \arcsin \left( \frac{\sqrt{(x + 1)^2 + y^2} - \sqrt{(x - 1)^2 + y^2}}{2} \right) + \frac{1}{2},
\]
where arcsine has the range \(-\pi/2\) to \( \pi/2 \) because \(-\pi/2 < u < \pi/2\).