Thirty Years of Problem Solving in Mathematics Education: Policy and Promise

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In a famous address that influenced the direction of mathematical research in the 20th Century, David Hilbert (1900) claimed of mathematicians, “We feel within us the perpetual call: There is a problem. Seek its solution.” Echoing this sentiment, another well-respected mathematician, Paul Halmos (1980), stated that “the mathematician’s main reason for existence is to solve problems” (p. 519). Others have made similar remarks (e.g., Polya, 1981; Schoenfeld, 1985), but the point is clear—mathematical activity has problem solving at its center. Thus it becomes important to clarify what is meant by the term “problem.” A problem is not an exercise of some mathematical skill or procedure that is already known. Polya (1957), for example, made sure to distinguish between authentic problems and “routine problems,” which he defined as a task that “can be solved either by substituting special data into a formerly solved general problem, or by following step by step, without any trace of originality, some well-worn conspicuous example” (p. 171). In contrast, a problem of the sort that involves problem solving is “a task for which the solution method is not known in advance” (NCTM, 2000, p. 52). As Schoenfeld (1992) characterized it, problems are “problematic” (p. 338).

Although there have been calls for school mathematics to represent the discipline of mathematics in “intellectually honest” ways (Bruner, 1960; Lampert, 1990), classroom practice in the United States has typically not been focused on authentic problem solving. Instead, teachers tend to present a mathematical idea or procedure, work several examples from the front of the room, and then assign exercises in which students practice whatever has just been presented (Smith, 1996; Stigler, Fernandez, & Yoshida, 1996; Stigler & Hiebert, 1999). Recently, the Common Core State Standards Initiative (2010)
has called for a shift in school mathematics instruction toward the inclusion of problem solving so that students are able to “make sense of problems and persevere in solving them” (p. 6), which they identify as a key mathematical practice that should be developed in students at all levels.

The Common Core Standards, however, are simply the most recent chapter in the story of problem solving. Reform efforts in mathematics education since the time of Hilbert’s 1900 address have included some form of call for problem solving (Stanic & Kilpatrick, 1988; Wilson, 2003). This has been especially true over the past 30 years since the publication of the National Council of Teacher’s of Mathematics’ [NCTM] Agenda for Action (1980), whose first recommendation was that “problem solving must be the focus of school mathematics” (p. 2). In this paper, I trace this call for problem solving by reviewing key policy documents of the past three decades. But first, I review some of the research and philosophical positions on problem solving in mathematics education that have led to its prominence in reform efforts. I then conclude with a brief discussion of some of the factors that may be contributing to the difficulty of achieving rich problem solving experiences in school mathematics classrooms.

**Justifications for Problem Solving**

The purpose for this section is to provide background that can inform the policy review in the next section. In particular, various studies and philosophical positions are reviewed that help to provide a rationale for why problem solving has been identified as a desirable feature in mathematics education. (Note that, due to the scope of this paper, this review does not deal with the large body of work in cognitive psychology on problem solving processes (e.g., Newell & Simon, 1972; Sternberg & Frensch, 1991).)
Although problem solving research has somewhat faded away in the early 21st Century (Schoenfeld, 2007), it received a great deal of attention in the 1970s and 1980s. Initially, the research was primarily quantitative in nature and was designed to identify the characteristics of difficult problems, the characteristics of successful problem solvers, and to investigate methods of training students to use problem solving heuristics (Lester, 1994). Kantowski (1977) observed that prior problem solving research seemed to focus on the *product* of students’ problem solving, so she set to work, as did others at that time, to better understand the *process* of problem solving. What processes can be observed when middle school students solve nonroutine geometry problems? How do those processes change as the students’ problem-solving abilities are developed? To answer these questions, Kantowski implemented a pretest-posttest design with an intermediate phase of instruction based on Polya’s (1957, 1981) heuristics. Eight subjects were included in the study and the tests included a think-aloud protocol for the purpose of uncovering the thought processes to an extent which is unachievable via paper-and-pencil tasks alone. Kantowski (1977) found, first of all, that the use of heuristics increased from the pretest to the posttest, suggesting that direct instruction of heuristics does influence the frequency of their use. Second, her results revealed a correlation between the use of heuristics and student success in solving problems, suggesting that problem solving skills are related to measurable student outcomes.

However, the frameworks used by problem solving researchers at that time, such as Kantowski, would now be considered quite narrow (Santos-Trigo, 2007). Scholars, especially Alan Schoenfeld, worked to remedy this throughout the 1980s. Schoenfeld (1985) developed a theoretical framework which could be used for investigating problem
solving and, more broadly, mathematical thinking. This framework comprises four domains which he claims must necessarily be addressed by any work intending to investigate mathematical problem-solving performance. These are as follows:

- **Resources.** (The relevant mathematical knowledge—intuition, facts, algorithms, understanding—possessed by the individual.)
- **Heuristics.** (Strategies and techniques—drawing figures, introducing suitable notation, exploiting similar problems, reformulating—for making progress on unfamiliar problems.)
- **Control.** (Global decisions—planning, monitoring, metacognitive acts—with regard to selecting and using resources and heuristics.)
- **Belief systems.** (The mathematical worldview—conscious and unconscious—of an individual which may determine his or her behavior.)

Schoenfeld (1985) pointed out that it is often the case that resources are assumed to be the primary determinant of success in problem solving; that is, if the requisite mathematical content for a particular problem is known, then the problem should be solvable. Schoenfeld uncovered the inappropriateness of this assumption. For instance, mathematicians with powerful heuristics and control are likely to be able to solve problems even when their resources are severely lacking, and students who possess the necessary resources may be unable to solve problems because their belief systems do not allow the connections to be made (e.g., deductive results are not called upon in empirical settings). This new perspective, reiterated by Lester (1988), can be construed as an indictment of traditional instruction that does not incorporate problem solving because merely supplying students with mathematical resources in the form of pieces of content
inadequately equips them to face new situations and think mathematically. Indeed, national assessments have shown this to be the case (e.g., Silver & Kenney, 1997).

If metacognition and student beliefs are key players in problem solving, then what role is left for the knowledge of the basic mathematical facts and procedures? Resnick (1988) supplied a partial answer to this question. She studied students in the fifth grade during word problem sessions and found that insecure mathematical knowledge blocked successful problem solving from occurring. Thus, mathematical knowledge resources may be a necessary, though not sufficient, condition for successful problem solving. Furthermore, she found that there is more subtlety to the teaching of heuristics than suggested by Polya’s actual writing. For example, Resnick’s research implied that Polya-like prompting questions were often too general to provide real help. The teacher prompt “Would it help to draw a diagram?” was not helpful when the student did not know what diagram to draw.

Though the teaching of heuristics is a subtle art, there is evidence to suggest that it can be successful. Charles and Lester (1984) conducted a statewide evaluation of a problem-solving focused instructional program known as Mathematical Problem Solving (MPS). This evaluation was based on standardized tests and classroom observations, and encompassed an entire school year. They found that students in the MPS experimental classrooms were better able to understand problems, plan solution strategies, and obtain correct results than students in control classrooms. Furthermore, both students and teachers in the MPS program exhibited improved attitudes toward mathematics as measured by a survey administered at the conclusion of the study.
More recently, several school mathematics curricula were developed that emphasized problems solving as a key feature of doing (and learning) mathematics (e.g., Coxford et al., 1997; Lappan, Fey, Friel, Fitzgerald, & Phillips, 1995). Having been used since the 1990s, evaluations of their effectiveness and outcomes were collected by Senk and Thompson (2003). Overall, the findings support the notion that curricula marked by a focus on problem solving, when compared to traditional materials, are correlated with improvements in students’ success with non-trivial tasks, interpretation of mathematical representations, and conceptual understanding, while simultaneously not harming their performance on basic skills. (Although problem solving was not the only shared feature of these curricula—they also tended to support explicit reasoning, real-world contexts, and student-centered instruction, for example—it is fair to say that problem solving opportunities and development was a unifying theme.)

The above paragraphs show that engaging in problem solving can be beneficial to students’ learning of mathematics and that equipping students with facts and procedures is not sufficient to produce competent problem solvers. Such bodies of research, however, are not the only basis of justification for the inclusion of problem solving in the policy documents reviewed in the next section. There are also important and influential philosophical arguments that have been made in favor of problem solving in mathematics education.

One such argument is based on a conception of mathematics as a “dynamic, problem-driven” discipline wherein “patterns are generated and then distilled into knowledge” (Ernest, 1988, quoted in Thompson, 1992, p. 132). From this perspective, which Ernest termed the “problem-solving view,” mathematics is a process of posing,
refining, and solving problems, rather than a collection of finished products. Thompson
(1992) noted that mathematics educators often adhere to this view of mathematics and
parlay it into calls for instruction that aligns with it.

John Dewey provides another philosophical impetus behind the push for problem
solving. Although Dewey did not often refer explicitly to problem solving, his notion of
reflective thinking has been viewed as reasonably synonymous (Stanic & Kilpatrick,
1988). Dewey (1933) felt that it was the ability to think reflectively that made one human
and, to him, the attitudes of open-mindedness, whole-heartedness, and responsibility were
more important than procedural skills or knowledge of particular facts. Moreover,
techniques and skill are only truly owned by students when they are learned with
understanding. Dewey maintained problem solving as a means to learning important
subject matter and simultaneously as an end in itself because of its contribution to human
reflective thought.

A final rationale for problem solving in school mathematics is related to the
connection between school curricula and students’ lives after school. As Lesh and
Zawojewski (2007) have noted recently, “there is a growing recognition that a serious
mismatch exists (and is growing) between the low-level skills emphasized in test-driven
curriculum materials and the kind of understanding and abilities that are needed for
success beyond school” (p. 764). Problem solving, on the other hand, provides the
creativity, flexibility and metacognitive control of thought that do align with professional
and post-secondary demands. In other words, by studying problem solving in
mathematics, students can become better prepared for many aspects of their lives after
school (e.g., trades, professional careers, knowledgeable citizenship).
Problem Solving in Landmark Documents

In this section, I highlight the role of problem solving in several key documents since 1980. The section is divided according to the source of the documents: the first subsection focuses on documents written from within the mathematics education community, whereas the second focuses on government-based reports. I recognize that these categories are not mutually exclusive since the mathematics education community is often writing for the purpose of influencing governmental policy and the government reports are often prepared with input and contributions from mathematics educators. The distinction is useful, nonetheless, as an organizational tool.

From the Mathematics Education Community

As mentioned above, the first item in NCTM’s Agenda for Action (1980) was a call for problem solving to be the focus of school mathematics. This meant that curricula should be organized around problem solving, that teachers should work to create environments in which problem solving can flourish (later explicated in the Professional Standards for Teaching Mathematics (NCTM, 1991)), and also that researchers should examine effective ways in which to develop mathematical problem solvers.

The Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989), known ubiquitously as the Standards and a cornerstone of the current reform movement, continued this emphasis on problem solving as they identified it as the first standard and demarcated it (along with communicating and reasoning) as a characteristic process of mathematics. A common justification for this emphasis is the preparation of students as productive citizens, harkening to the professional and post-secondary rationale outlined above. In the Standards, problem solving is not just the ability to solve
problems but also entails the representation of problems and the understanding of the language and reasoning of mathematics that are involved in formulating problems and verifying solutions. Furthermore, these Standards identify the importance of assessing problem solving; if problem solving is a key focus of instruction, then it does not suffice to assess only recall of facts and execution of procedures. This assessment, according to NCTM, should take place in the classroom (e.g., observing students as they work to solve problems, listening to students as they discuss their solution strategies) as well as on written examinations.

The next landmark document from NCTM was the Principles and Standards for School Mathematics (2000) which identified ten standards that “mathematics instruction should enable students to know and do” (p. 7, emphasis added). This reference to doing mathematics played out in the five process standards, the first of which is problem solving (the others being reasoning and proof, communication, connections, and representation). Again, there is an appeal to the benefits of problem solving beyond the classroom:

By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom. In everyday life and in the workplace, being a good problem solver can lead to great advantages. (p. 52)

There are also, however, justifications for problem solving that do not appeal to contexts outside the domain of mathematics. For example, the Principles and Standards point out that, by engaging in problem solving, students can explore and make sense of new
mathematical ideas and can also “solidify and extend what they already know” (p. 52). There is also a discussion of how problem posing and problem solving can lead students to a mathematical disposition, which is related to the alignment between school mathematics and the discipline of mathematics described above. (Similar rationales for the connection between problem solving, sense making, and mathematics learning are also present in (NCTM, 2009).)

Although NCTM has been a leader in the past few decades with respect to policy recommendations for school mathematics, they are not alone in their positions. Kilpatrick, Swafford, and Findell (2001) edited a volume entitled *Adding It Up: Helping Children Learn Mathematics* that has also played an important role in and had significant resonance with the mathematics education community (prompting me to include it in this section, even though it was produced by the National Resource Council). *Adding It Up* focuses particularly on mathematics at the elementary level but their articulation of the strands of mathematical proficiency—conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition—applies to all grade levels. The report argues that all five strands are interwoven and that mathematical proficiency depends on all five working together. Of particular relevance here is the notion of *strategic competence*, defined as the ability of students to “formulate, represent, and solve mathematical problems” (Kilpatrick et al., 2001, p. 116). This strand maps strongly to NCTM’s identification of the problem solving process standard. Furthermore, the interweaving of the strands described in *Adding It Up* relates to ideas described above such as problem solving experiences helping to build conceptual understanding and engender a productive mathematical disposition.
From Various Government Bodies

In 1983, only a few years after NCTM’s original *Agenda for Action* was released, the National Commission on Excellence in Education raised an alarm concerning the state of education in the U.S.: “Our once unchallenged preeminence in commerce, industry, science, and technological innovation is being overtaken by competitors throughout the world” (p. 1). Clearly citing the need for competitive and competent citizens (and soldiers), *A Nation at Risk* called for vast improvements in educational outcomes, with a particular focus on mathematics and science. One of the pieces of evidence used by this report to paint the discouraging portrait was the fact that “only one-third [of 17-year-olds] can solve a mathematics problem requiring several steps” (p. 3). An inference that can be drawn, therefore, is that the Department of Education viewed the traditional modes of mathematics teaching up to that point, which (as noted above) tended to have a paucity of problem solving opportunities, as inadequately meeting the needs of the nation.

Near the end of that same decade, the National Research Council (1989) reiterated the inadequacy of such problem-free approaches to mathematics instruction: “Evidence from many sources shows that the least effective mode for mathematics learning is the one that prevails in most of America’s classrooms” (p. 57). As an alternative, a different vision of classroom practice is offered that resonates with the recommendations found in the previous section as well as the descriptions of doing mathematics from mathematicians at the outset of this paper:

Just as children need the opportunity to learn from mistakes, so students need an environment for learning mathematics that provides generous room for trial and
error. In the long run, it is not the memorization of mathematical skills that is particularly important… but the confidence that one knows how to find and use mathematical tools whenever they become necessary. There is no way to build this confidence except through the process of creating, constructing, and discovering mathematics. (NRC, 1989, p. 60)

This quote exemplifies an appeal to a constructivist philosophy of learning which leads to recommendations for instruction that seem to align with a problem-solving rich approach.

Closing in on the present day, the National Mathematics Advisory Panel (2008) used problem solving success as a desirable outcome variable throughout their final report—that is, students in our mathematics education system should be able to solve both routine and non-routine problems. However, this report also treated problem solving in mathematics classrooms as a means to a wide variety of desirable ends. In particular, the report echoed the proficiency strands of Adding It Up when it made the point that “the curriculum must simultaneously develop conceptual understanding, computational fluency, and problem-solving skills” (p. 19) because all three are mutually reinforcing. Moreover, the National Mathematics Advisory Panel report cites “high-quality” research in stating that, “[e]ven during the preschool period, children have considerably greater reasoning and problem-solving ability than was suspected until recently” (p. 30), and that these abilities should be built upon throughout the school curriculum. This desired pervasiveness of problem solving resonates with NCTM’s call for the problem solving process standard to be enacted at all grade levels.

This brings us to the recent release of the Common Core State Standards for Mathematics (2010). Whereas the other government reports in this section made allusions
(i.e., *A Nation at Risk, Everybody Counts*) or repeated in-line references (i.e., *Final Report of the National Mathematics Advisory Panel*) to problem solving in mathematics education, the *Common Core Standards* elevated problem solving into a position of prominence by naming it as the first of their “standards for mathematical practice” (p. 6, which are explicitly linked by the authors of the *Common Core Standards* to NCTM’s process standards and the strands of proficiency from *Adding It Up*). As such, problem solving is one of the key ways in which “developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years” (p. 8). Therefore, we see an appeal to the discipline of mathematics as another form of rationale for focusing on problem solving in mathematics education.

In summary, a variety of policy documents have identified school mathematics as being in a unique position to develop students’ abilities related to problem solving and have recommended that mathematics educators cultivate these abilities whole-heartedly. Ideally, this emphasis on problem solving would transfer into society in positive ways by promoting a knowledgeable citizenry, a capable workforce, a strong national defense, and by creating pathways of advancement for students that would not exist otherwise (Hiebert et al., 1996; Schoenfeld, 2007). There are also potential benefits that are relatively self-contained to the domain of mathematics. For example, problem solving experiences more accurately represent the discipline of mathematics than drill-and-kill exercises and can cultivate in students an aptitude for mathematical thinking (Schoenfeld, 1992)—an ability to face unfamiliar situations, assess and analyze the contributing factors, generate a plan for resolution, and evaluate the success of that plan. But calls for such a problem-solving
approach have been made from both mathematics education and governmental communities for at least the last 30 years, as evidenced by the review above. Why is it that we have not yet been able to achieve this goal? This question, which is both deeper and broader than the scope of this paper, is addressed briefly in the concluding section.

**Issues of Implementing Problem Solving**

There are several possible factors that might contribute to the difficulty of moving mathematics instruction away from more exercise-oriented activities toward more problem-oriented activities. First of all, this shift toward problem solving might be difficult for the same reason that any move away from traditional forms of instruction are difficult—that is, students and teachers are not accustomed to it. Thinking about students first, it is now widely accepted that students do not come to mathematics classes as “blank slates” but carry with them past experiences and notions of what is expected of them in mathematics classes and what it means to learn mathematics (Muis, 2004). Even children coming to formal educational settings for the first time have preconceived notions about what these will entail from parents, peers, siblings, and the media. Since the broader cultural image of mathematics education in the United States is not rich in problem solving, and since it is likely that students’ past experiences in mathematics classes were of the traditional variety, any changes will require a great deal of work and effort to reverse that momentum.

With respect to teachers, the story is much the same. Prior to taking over their own classrooms, teachers have experienced thousands of hours of instruction in schools (Lortie, 1975) and, as with students, these are likely to have been in a traditional style. Thus, these teachers have a great deal of momentum in the direction of teaching the way
they have been taught and are likely to view mathematics in a corresponding light—as a collection of static facts and procedures rather than a process of investigation (Felbrich, Muller, & Blomeke, 2008). As Thompson (1992) has pointed out, “the task of modifying long-held, deeply rooted conceptions of mathematics and its teaching in the short period of a course in methods of teaching remains a major problem in mathematics teacher education” (p. 135).

Related to teacher beliefs about mathematics and mathematics teaching is teacher knowledge of mathematics. It is possible that even teachers with strong content knowledge of mathematics may have weak process knowledge of mathematics. (The situation is likely worse for teachers who have neither a strong content nor a strong process knowledge foundation.) For those teachers who are not comfortable engaging in the processes of mathematics such as problem solving, or who have not had adequate experiences engaging in them in the past, can reasonably be expected to give pushback when asked to teach in this way.

A final potential contributor to the difficulty of enacting a problem-solving rich form of instruction is one that might seem circular on the surface—problem solving is hard. I mean this in at least two ways. First, it is harder to engage in mathematical problem solving than it is to enact a given mathematical procedure. For students, it is harder to understand what a problem is asking, develop a means for representing the ideas at play, grapple with various solution approaches, and article your reasoning to others than it is to proceed from step one to step two to step three in an imitative fashion. For teachers, it is harder to pose good problems, guide students through arduous mathematical terrain, and not tell them the key that will unlock their difficulty than it is to
model a procedure and assign a set of exercises. In a culture that seems to prize convenience and the avoidance of strain (e.g., there is an entire advertising campaign built on the idea of an “easy” button), the deck is certainly not stacked in favor of problem solving flourishing in schools—though perhaps this is an argument for the necessity of problem solving in mathematics education as a balancing agent in modern society, but exploring this thought would take us too far afield.

A second reason that it is hard to make problem solving a central goal of mathematics instruction is because problem solving is inherently general and difficult to characterize and measure. For example, Polya’s (1957) heuristics are often taken as a model of problem solving, but it has been suggested that they are so general that they may be unhelpful in teaching those who are not already accomplished problem solvers (Resnick, 1988). Moreover, when compared to something like the distributive property, it is more difficult to determine when problem solving has been taught and when it has been learned. Until appropriate assessments and a collective body of experiences around problem solving are developed, it will be easier for teachers, districts, and states to avoid this uncertainty and unfamiliarity by continuing with a content-focused approach.

However, there is much evidence to suggest that a focus on problem solving has benefits in many areas: problem solving supports students’ mathematical learning of both concepts and procedures, accurately reflects what it means to do mathematics, and plays a vital role in developing a strong and mathematically powerful citizenry. Because of this promise that it holds, problem solving has been a constant feature of mathematics education policy documents for the past three decades and is likely to remain so for the decades to come.
References


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