1

CLAIM: A bijective map between manifolds is not necessarily a diffeomorphism.

Consider the example \( f : \mathbb{R} \to \mathbb{R}, x \to x^3 \). We know that \( \mathbb{R} \) is a manifold by example (1) from class (8/29). Furthermore, \( f \) is one-one because \( x^3 = y^3 \) implies \( x = y \), and \( f \) is onto because for any \( x \in \mathbb{R} \) we have \( \sqrt[3]{x} \in \mathbb{R} \) with \( f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x \). So \( f \) fulfills the hypothesis of the claim.

However, we will show that \( f \) is not a diffeomorphism.

By definition, the inverse of a diffeomorphism is smooth. In our example, \( f^{-1} : \mathbb{R} \to \mathbb{R}, x \to \sqrt[3]{x} \).

Using the concept of total derivative, we know that \( f^{-1} \) is differentiable at \( a \in \mathbb{R} \) if and only if there exists a linear map \( L : \mathbb{R} \to \mathbb{R} \) such that
\[
\lim_{v \to 0} \frac{|f^{-1}(a + v) - f^{-1}(a) - L(v)|}{|v|} = 0.
\]

Note that at \( a = 0 \) this becomes
\[
\lim_{v \to 0} \frac{|f^{-1}(v) - f^{-1}(0) - L(v)|}{|v|} = \lim_{v \to 0} \frac{\sqrt[3]{v} - L(v)}{|v|}.
\]

The right-hand side of the equation is unbounded and hence \( L \) does not exist. This means \( f^{-1} \) is not differentiable at \( x = 0 \) and so is not smooth. Therefore, \( f \) is not a diffeomorphism.

2

Proposition. Let \( M \) be a manifold and \( f : M \to \mathbb{R} \) a map. The graph of \( f \) is defined to be \( G_f = \{(x, f(x)) \mid x \in M\} \). Put
\[
\phi : M \to G_f, \quad x \mapsto (x, f(x)).
\]

If \( f \) is smooth, then \( \phi \) is a diffeomorphism. This implies that \( G_f \) is a manifold because it is diffeomorphic to \( M \).

Proof. Suppose \( \phi(x) = \phi(y) \) for some \( x, y \in M \). Then by definition of \( \phi \) we have \( (x, f(x)) = (y, f(y)) \) and in the first component we see \( x = y \). Thus \( \phi \) is one-one. Now, let \( (x, f(x)) \in G_f \).

Then by definition of \( G_f \) we have \( x \in M \). Applying \( \phi \) to \( x \) yields \( \phi(x) = (x, f(x)) \) and so \( \phi \) is onto.

We know that \( \phi \) is smooth because we can view it component-wise as \( \phi(x) = (\phi^1(x), \phi^2(x)) \) where \( \phi^1 \) is the identity function on \( M \) and \( \phi^2 = f \), both of which are smooth. We also know that \( \phi^{-1} : G_f \to M, (x, f(x)) \to x \) is smooth because it is a linear projection from \( G_f \) to \( M \). Therefore, \( \phi \) is a diffeomorphism.
3

Proposition. Let \( C = \{(2 \cos t, 2 \sin t, 0) \mid 0 \leq t < 2\pi\} \subset \mathbb{R}^3 \) and let \( T = \{p \in \mathbb{R}^3 \mid d(p, C) = 1\} \). Then the torus \( T \) is diffeomorphic to \( S^1 \times S^1 \).

Proof. We will view \( S^1 \) as the set \( \{(\cos \mu, \sin \mu) \mid 0 \leq \mu < 2\pi\} \) in \( \mathbb{R}^2 \), and so \( S^1 \times S^1 = \{(\cos \mu, \sin \mu), (\cos \lambda, \sin \lambda) \mid 0 \leq \mu, \lambda < 2\pi\} \). For convenience we will omit the extra parentheses. Let

\[
\phi : S^1 \times S^1 \rightarrow T, \quad (\cos \mu, \sin \mu, \cos \lambda, \sin \lambda) \rightarrow ((2 + \cos \mu) \cos \lambda, (2 + \cos \mu) \sin \lambda, \sin \mu).
\]

First, we must verify that \( \phi(S^1 \times S^1) \subseteq T \). Let \( s = (\cos \mu, \sin \mu, \cos \lambda, \sin \lambda) \in S^1 \times S^1 \). It is our task to show \( \phi(s) \in T \); that is, \( d(\phi(s), C) = \min_{c \in C} |\phi(s) - c| = 1 \). Using the definition of \( C \) and the identity \( \sin^2 \theta + \cos^2 \theta = 1 \), we see

\[
d(\phi(s), C) = \min_{0 \leq t < 2\pi} \sqrt{((2 + \cos \mu) \cos \lambda - 2 \cos t)^2 + ((2 + \cos \mu) \sin \lambda - 2 \sin t)^2 + (\sin \mu - 0)^2}
= \min_{0 \leq t < 2\pi} \sqrt{(2 + \cos \mu)^2 + 4 - 4(2 + \cos \mu)(\cos \lambda \cos t + \sin \lambda \sin t) + \sin^2 \mu}
= \min_{0 \leq t < 2\pi} \sqrt{4 + 4 \cos \mu + \cos^2 \mu + 4 - 4(2 + \cos \mu)(\cos \lambda \cos t + \sin \lambda \sin t) + \sin^2 \mu}
= \min_{0 \leq t < 2\pi} \sqrt{1 + (8 + 4 \cos \mu) - (8 + 4 \cos \mu)(\cos \lambda \cos t + \sin \lambda \sin t)}
= \min_{0 \leq t < 2\pi} \sqrt{1 + (8 + 4 \cos \mu)(1 - \cos \lambda \cos t + \sin \lambda \sin t)}.
\]

Note that \( d(\phi(s), C) \) is bounded above by 1 because for \( t = \lambda \) we have within \((*)\)

\[
\sqrt{1 + (8 + 4 \cos \mu)(1 - \cos \lambda \cos t + \sin \lambda \sin t)} = \sqrt{1 + (8 + 4 \cos \mu)(1 - \cos^2 \lambda + \sin^2 \lambda)}
= \sqrt{1 + (8 + 4 \cos \mu)}(1 - 1) = \sqrt{1} = 1.
\]

Moreover, \( d(\phi(s), C) \) is bounded below by 1 because within \((*)\) the expression \( \cos \lambda \cos t + \sin \lambda \sin t \) is at most 1, and thus \( 1 - \cos \lambda \cos t + \sin \lambda \sin t \) is at least 0. It follows from \((*)\) that

\[
d(\phi(s), C) \geq \sqrt{1 + (8 + 4 \cos \mu)(0)} = \sqrt{1} = 1.
\]

So indeed \( d(\phi(s), C) = 1 \) and consequently \( \phi(s) \in T \).

Now, to show injectivity suppose \( \phi(\cos \mu, \sin \mu, \cos \lambda, \sin \lambda) = \phi(\cos m, \sin m, \cos \ell, \sin \ell) \) in the image of \( \phi \). It follows that

\[
(i) \quad (2 + \cos \mu) \cos \lambda = (2 + \cos m) \cos \ell
(ii) \quad (2 + \cos \mu) \sin \lambda = (2 + \cos m) \sin \ell
(iii) \quad \sin \mu = \sin \ell.
\]

By solving \((i)\) and \((ii)\) for \( \cos \mu \) and setting them equal, we find that

\[
\frac{(2 + \cos m) \cos \ell}{\cos \lambda} - 2 = \frac{(2 + \cos m) \sin \ell}{\sin \lambda} - 2,
\]

which implies \( \frac{\cos \ell}{\cos \lambda} = \frac{\sin \ell}{\sin \lambda} \) or equivalently \( \cot \ell = \cot \lambda \). By the properties of the cotangent function and the restriction of \( \lambda, \ell \) to \( [0, 2\pi) \), we know that \( \lambda = \ell + \pi \cdot a \) for some \( a \in \{0, 1\} \). Suppose \( a = 1 \) and so \( \lambda = \ell + \pi \). This means \( \cos \lambda = -\cos \ell \) and by \((i)\)

\[
2 + \cos \mu = -(2 + \cos m).
\]
But this is absurd because the left-hand side is at least 1 while the right-hand side is at most -1. Hence \( a = 0 \) and \( \ell = \lambda \). Using this with (ii) we see \( 2 + \cos \mu = 2 + \cos m \) and thus \( \cos \mu = \cos m \). By (iii) we also have \( \sin \mu = \sin m \). It follows that \( \mu = m \). Therefore, \( (\cos \mu, \sin \mu, \cos \lambda, \sin \lambda) = (\cos m, \sin m, \cos \ell, \sin \ell) \) in \( S^1 \times S^1 \) and \( \phi \) is one-one.

To show surjectivity, let \( p \in T \). Then \( p \) is determined by three parameters \((x, z, \theta)\) where the \( 1 \leq x \leq 3 \) and \(-1 \leq z \leq 1\) establish a position on the circle \((x - 2)^2 + z^2 = 1\) and \( \theta \) rotates this about the \( z \)-axis in \( \mathbb{R}^3 \). Consider \((x - 2, z, \cos \theta, \sin \theta) \in S^1 \times S^1 \). When we apply \( \phi \) we get

\[
\phi(x - 2, z, \cos \theta, \sin \theta) = ((2 + x - 2) \cos \theta, (2 + x - 2) \sin \theta, z) = (x \cos \theta, x \sin \theta, z).
\]

This is precisely the point \( p \in T \), therefore \( \phi \) is onto.

The smoothness of \( \phi \) follows from the fact that it comprises only the trigonometric functions sine and cosine, which are smooth. For

\[
\phi^{-1} : T \to S^1 \times S^1, (x, y, z) \to (\cos(\sin^{-1} z), z, x/(2 + \cos(\sin^{-1} z)), y/(2 + \cos(\sin^{-1} z)))
\]

we have smoothness because the definition of \( T \) guarantees \(-1 \leq z \leq 1\) and we can take \( \sin^{-1} z \) to be well-defined as the value between 0 and \( 2\pi \). Moreover, the denominators in the third- and fourth-coordinate are well-behaved because \( 2 \neq \cos(\sin^{-1} z) \) for any \( z \).

Therefore, \( \phi \) is a diffeomorphism and \( T \) is diffeomorphic to \( S^1 \times S^1 \).

\[\square\]

4

CLAIM: The sphere \( S^n \) cannot be parametrized by a single chart \( \varphi : S^n \to V \subset \mathbb{R}^n \).

Suppose to the contrary that such a function \( \varphi : S^n \to V \) exists. By the definition of a chart, \( V \) is an open subset of \( \mathbb{R}^n \) and \( S^n \) is homeomorphic to \( V \). Now, we know from point-set topology that \( S^n \) is compact, and compactness is a homeomorphic invariant, so also \( V \) is compact in \( \mathbb{R}^n \).

Let \( C \) be the open cover of \( V \) defined by

\[
C = \left\{ B \left( v, \frac{1}{2} \cdot d(v, \mathbb{R}^n \setminus V) \right) \right\}_{v \in V}.
\]

Note that since \( V \) is open, \( V^o = V \) and thus the distances appearing in the definition of \( C \) are nonzero. Let

\[
S = \left\{ B \left( v_i, \frac{1}{2} \cdot d(v_i, \mathbb{R}^n \setminus V) \right) \right\}_{i=1}^m
\]

be any finite subcover of \( C \). The finiteness grants the existence of \( D := \min_{1 \leq i \leq m} d(v_i, \mathbb{R}^n \setminus V) \). Since \( d(V, \mathbb{R}^n \setminus V) = 0 \), there exists some \( v_* \in V \) such that \( d(v_*, \mathbb{R}^n \setminus V) < \frac{1}{2} \cdot D \).

Since \( S \) covers \( V \), it must be the case that \( v_* \in B(v_k, \frac{1}{2} \cdot d(v_k, \mathbb{R}^n \setminus V)) \) for some \( 1 \leq k \leq m \). But then

\[
d(v_k, \mathbb{R}^n \setminus V) \leq d(v_k, v_*) + d(v_*, \mathbb{R}^n \setminus V) < \frac{1}{2} \cdot D
\]

which implies

\[
\frac{1}{2} \cdot d(v_k, \mathbb{R}^n \setminus V) < \frac{1}{2} \cdot D,
\]

or more specifically, \( d(v_k, \mathbb{R}^n \setminus V) < D \). This contradicts the minimality of \( D \), hence \( v_* \) is not covered by \( S \) and \( C \) has no finite subcover. This means that \( V \) is not compact. However, we stated above that \( V \) is compact. This contradiction allows us to conclude that \( \varphi \) does not exist.
Let $N = (0, \ldots, 0, 1)$ be the “north pole” in $S^n \subset \mathbb{R}^{n+1}$, and let $S = -N$ be the “south pole.” Define stereographic projection by

$$
\sigma : S^n \setminus \{N\} \longrightarrow \mathbb{R}^n, \quad (x^1, \ldots, x^{n+1}) \longmapsto \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.
$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$; that is,

$$
\tilde{\sigma} : S^n \setminus \{S\} \longrightarrow \mathbb{R}^n, \quad (x^1, \ldots, x^{n+1}) \longmapsto \frac{(x^1, \ldots, x^n)}{1 + x^{n+1}}.
$$

(a) We will show that $\sigma(x)$ is the point where the line through $N$ and $x$ intersects the $n$-dimensional subspace where $x^{n+1} = 0$. We will then show that, similarly, $\tilde{\sigma}(x)$ is the point where the line through $S$ and $x$ intersects the same subspace.

The general form of a line in $\mathbb{R}^{n+1}$ is $k_0 + k_1 x^1 + \ldots + k_{n+1} x^{n+1} = 0$. Let $a \in S^n \setminus \{N\}$. We will find an equation for the line $L$ passing through $N$ and $a = (a^1, \ldots, a^{n+1})$. Since $N = (0, \ldots, 0, 1)$ satisfies the equation of $L$, we know that $k_0 + k_{n+1} = 0$ or $k_{n+1} = -k_0$. Put $k_i = 1$ for $2 \leq i \leq n+1$. Then necessarily $k_0 = -1$. The coefficient $k_1$ will be determined by the point $a$. Since $a$ satisfies the equation of $L$, we know

$$
-1 + k_1 a^1 + \ldots + a^{n+1} = 0.
$$

Solving this for $k_1$ gives $k_1 = \frac{1-a^2-a^{n+1}}{a^1}$. Hence an equation for $L$ is

$$
-1 + \left(\frac{1-a^2-\ldots-a^{n+1}}{a^1}\right) x^1 + x^2 + \ldots + x^{n+1} = 0. \quad (*)
$$

Now, the point $\sigma(a)$ is clearly contained in the subspace $x^{n+1} = 0$ because its $(n+1)$-coordinate is 0. Moreover, $\sigma(a)$ is contained in $L$ because plugging $\sigma(a)$ into $(*)$ yields

$$
-1 + \left(\frac{1-a^2-\ldots-a^{n+1}}{a^1}\right) \left(\frac{a^1}{1-a^{n+1}}\right) + \left(\frac{a^2}{1-a^{n+1}}\right) + \ldots + \left(\frac{a^n}{1-a^{n+1}}\right) + (0)
$$

$$
= -1 + \left(\frac{1-a^2-\ldots-a^{n+1}}{1-a^{n+1}}\right) + \left(\frac{a^2}{1-a^{n+1}}\right) + \ldots + \left(\frac{a^n}{1-a^{n+1}}\right)
$$

$$
= -1 + \frac{1-a^2-a^2+\ldots+a^n-a^n-a^{n+1}}{1-a^{n+1}}
$$

$$
= -1 + \frac{1-a^{n+1}}{1-a^{n+1}}
$$

$$
= -1 + 1
$$

$$
= 0.
$$

Therefore, $\sigma(a)$ is the point of intersection between $L$ and the subspace $x^{n+1} = 0$.

For the stereographic projection from the south pole, let $b \in S^n \setminus \{S\}$. We will use the same general form for the line $L$ except now the point $S = (0, \ldots, 0, -1)$ implies $k_{n+1} = k_0$. Again we put $k_i = 1$ for $2 \leq i \leq n+1$. Then necessarily $k_0 = 1$ also. The coefficient $k_1$ will be determined by the point $b = (b^1, \ldots, b^{n+1})$. Since $b$ satisfies the equation of $L$, we know

$$
1 + k_1 b^1 + \ldots + b^{n+1} = 0.
$$
Solving this for $k_1$ gives $k_1 = \frac{-1 - b^2 - \ldots - b^{n+1}}{b^1}$. Hence an equation for $\tilde{L}$ is

$$1 + \left( \frac{-1 - b^2 - \ldots - b^{n+1}}{b^1} \right) x^1 + x^2 + \ldots + x^{n+1} = 0. \quad (\dagger)$$

As before, $\tilde{\sigma}(b)$ is clearly contained in the appropriate subspace because its $(n+1)$-coordinate is 0. Furthermore, $\tilde{\sigma}(b)$ is contained in $\tilde{L}$ because plugging $\tilde{\sigma}(b)$ into $(\dagger)$ yields

$$1 + \left( \frac{-1 - b^2 - \ldots - b^{n+1}}{b^1} \right) \left( \frac{b^1}{1 + b^{n+1}} \right) + \ldots + \left( \frac{b^n}{1 + b^{n+1}} \right) + (0)$$

$$= 1 + \left( \frac{-1 - b^2 - \ldots - b^{n+1}}{1 + b^{n+1}} \right) + \ldots + \left( \frac{b^n}{1 + b^{n+1}} \right)$$

$$= 1 + \frac{-1 - b^{n+1}}{1 + b^{n+1}}$$

$$= 1 - 1$$

$$= 0.$$

Therefore, $\tilde{\sigma}(b)$ is the point of intersection between $\tilde{L}$ and the subspace $x^{n+1} = 0$.

(b) We will show that $\sigma$ is bijective with inverse

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

To demonstrate injectivity, let $a$ and $b$ be points from $S^n \setminus \{N\}$ such that $\sigma(a) = \sigma(b)$. Suppose to the contrary that $a \neq b$. Then the line from $N$ to $\sigma(a) = \sigma(b)$ intersects $S^n$ at three distinct points—$N, a, b$. But this is absurd because $S^n$ is convex. Thus $a = b$ and $\sigma$ is one-one.

To demonstrate surjectivity, let $p$ be any point in the subspace of $\mathbb{R}^{n+1}$ where $x^{n+1} = 0$. Let $L$ be the line through $p$ and $N$. Since $L$ has no curvature and $S^n$ has positive curvature, it is necessarily the case that $L$ intersects $S^n \setminus \{N\}$ at some point $a$. Then by definition of $\sigma$ we have $\sigma(a) = p$. So $\sigma$ is onto.

To demonstrate that $\sigma$ and $\sigma^{-1}$ are indeed inverses, observe

$$\sigma(\sigma^{-1}(u^1, \ldots, u^n)) = \sigma \left( \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1} \right)$$

$$= \frac{(2u^1, \ldots, 2u^n)}{|u|^2 + 1} \cdot \frac{1}{\frac{|u|^2 - 1}{|u|^2 + 1}}$$

$$= \frac{(2u^1, \ldots, 2u^n)}{|u|^2 + 1} \cdot \frac{1}{\frac{|u|^2 + 1 - |u|^2}{|u|^2 + 1}}$$

$$= \frac{(2u^1, \ldots, 2u^n)}{|u|^2 + 1} \cdot \frac{|u|^2 + 1}{2}$$

$$= (u^1, \ldots, u^n).$$

Since their composition is the identity function on $\mathbb{R}^n$, $\sigma$ and $\sigma^{-1}$ are inverses of each other.

(c) We will compute the transition map $\tilde{\sigma} \circ \sigma^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and then verify that the atlas $\mathcal{A} = \{(S^n \setminus \{N\}, \sigma), (S^n \setminus \{S\}, \tilde{\sigma})\}$ defines a smooth structure on $S^n$. 

5
Using parts (a) and (b), we have

\[
(\tilde{\sigma} \circ \sigma^{-1})(u^1, \ldots, u^n) = \tilde{\sigma} \left( \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1} \right)
\]

\[
= \frac{(2u^1, \ldots, 2u^n)}{|u|^2 + 1} \cdot \frac{1}{1 + \frac{|u|^2 - 1}{|u|^2 + 1}}
\]

\[
= \frac{(2u^1, \ldots, 2u^n)}{|u|^2 + 1} \cdot \frac{1}{\frac{|u|^2 + 1 + |u|^2 - 1}{|u|^2 + 1}}
\]

\[
= \frac{(2u^1, \ldots, 2u^n)}{|u|^2 + 1} \cdot \frac{|u|^2 + 1}{2|u|^2}
\]

\[
= \frac{(u^1, \ldots, u^n)}{|u|^2}.
\]

To verify that \( \mathcal{A} \) defines a smooth structure on \( S^n \) we must check that \( \mathcal{A} \mathcal{A} \) is a smooth atlas (i.e, \( \tilde{\sigma} \circ \sigma^{-1} \) is a diffeomorphism). Once we know \( \mathcal{A} \) is smooth, lemma 1.10(a) gives that \( \mathcal{A} \) is contained in a maximal smooth atlas, which is by definition a smooth structure on \( S^n \).

By part (b), \( \sigma^{-1} \) is bijective and by the symmetry of \( \tilde{\sigma} \) to \( \sigma \), it follows that \( \tilde{\sigma} \) is also bijective. Thus \( \tilde{\sigma} \circ \sigma^{-1} \) is bijective. We saw above that \((\tilde{\sigma} \circ \sigma^{-1})(u) = u/|u|^2 \) which is clearly smooth. Also, \((\tilde{\sigma} \circ \sigma^{-1})^{-1} \) can be shown to be smooth. Therefore \( \tilde{\sigma} \circ \sigma^{-1} \) is a diffeomorphism and \( \mathcal{A} \) is a smooth atlas.

(d) By 1.10(b), two smooth atlases determine the same smooth structure if and only if their union is itself a smooth atlas. We will show that \( \mathcal{A} \) and the standard atlas \( \{(U_i^\pm, \phi_i^\pm)\} \) for \( 1 \leq i \leq 2 \) (see problem 7) determine the same smooth structure.

Incomplete.

6

Let \( f : \mathbb{R} \to \mathbb{R} \) be the bump function

\[
f(x) = \begin{cases} 
  e^{-1/x^2} & x > 0 \\
  0 & x \leq 0.
\end{cases}
\]

(a) We will prove that \( f \) is smooth.

Clearly \( f \) is smooth to the left of \( x = 0 \). Note that the functions \( x \to e^{-x} \) and \( x \to 1/x^2 \) are smooth for \( x > 0 \). In this domain, \( f \) is simply the composition of these two functions and so \( f \) is smooth to the right of \( x = 0 \). It remains to be checked that \( f \) is smooth at \( x = 0 \).
We know $f$ is continuous at $x = 0$ because

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{-1/x^2} = \lim_{x \to 0^+} \frac{1}{e^{1/x^2}} = 0.$$ 

To show that $f^{(k)}(x)$ exists and is continuous we will induct on $k$. For $x < 0$ this is obvious, so we will restrict our attention to $x \geq 0$ with special attention to $x = 0$ at the conclusion. We will also include in this induction the form of the derivative, namely

$$f^{(k)}(x) = \frac{p_k(x)}{x^{(3)2^k-1}} e^{-1/x^2}$$

for some polynomial $p_k(x)$.

**Base Step:** When $k = 1$ we have

$$f'(x) = \frac{2}{x^3} e^{-1/x^2} = \frac{2}{x^{(3)2^0}} e^{-1/x^2}$$

which is of the proper form and is continuous because three applications of l’Hopital’s to $\lim_{x \to 0^+} 2x^{-3}/e^{1/x^2}$ shows that this is 0.

**Inductive Step:** Suppose

$$f^{(k)}(x) = \frac{p_k(x)}{x^{(3)2^k-1}} e^{-1/x^2}$$

exists, is continuous, and is of the form shown. By the product and quotient rules,

$$f^{(k+1)}(x) = \frac{p'_{k}(x) \cdot x^{(3)2^k-1} - p_k(x) (3)2^{k-1}x^{(3)2^k-2} \cdot 2^{k-1}}{(x^{(3)2^k-1})^2} e^{-1/x^2} + \frac{2p_k(x)}{x^{3(3)2^{k-1}}} e^{-1/x^2}$$

$$= \left( \frac{p'_{k}(x) \cdot x^{(3)2^k-1} - p_k(x) (3)2^{k-1}x^{(3)2^k-2} \cdot 2^{k-1}}{(x^{(3)2^k-1})^2} + \frac{2p_k(x)}{x^{3(3)2^{k-1}}} \right) e^{-1/x^2}$$

$$= \frac{p'_{k}(x)(x^{(3)2^k-1}) - (3)2^{k-1}p_k(x) x^{(3)2^k-2} + p_k(x) (x^{(3)2^k-2} - 1)}{x^{(3)2^k}} e^{-1/x^2}.$$ 

So the form of $f^{(k+1)}$ is correct. This derivative tends to 0 because, again, we can view it as a polynomial tending to infinity over an exponential tending to infinity, and repeating l’Hopital’s rule at most $(3)2^k$-times will show that the numerator will become 0 while the denominator still tends to infinity, making $\lim_{x \to 0^+} f^{(k)}(x) = 0$. Moreover, the derivative has a value of 0 at $x = 0$ as shown inductively below using the definition of derivative:

$$f^{(k+1)}(0) = \lim_{h \to 0^+} \frac{f^{(k)}(h) - f^{(k)}(0)}{h}$$

$$= \lim_{x \to 0^+} \frac{p_k(h)}{h^{(3)2^{k-1}}} e^{-1/h^2} - 0$$

$$= \lim_{x \to 0^+} \frac{p_k(h)}{h^{(3)2^{k-1}+1}} e^{-1/h^2}$$

$$= 0.$$ 

So the derivatives of $f$ exist and are continuous, therefore $f$ is smooth.
(b) Fix $0 < a < b$. Define $g(x) = f(x - a)f(b - x)$.

We can write $g$ explicitly as

$$g(x) = \begin{cases} 
0 & x \leq a \\
\frac{-1}{e^{(x-a)^2}} & a < x < b \\
0 & x \geq b.
\end{cases}$$

From this it is clear that $g$ is positive between $a$ and $b$ (because a power of $e$ is never negative) and 0 elsewhere. The smoothness of $g$ follows from the fact that it is a product of smooth functions.

(c) Sketch

$$h(x) = \frac{\int_{-\infty}^x g \, dx}{\int_{-\infty}^\infty g \, dx}.$$ 

(d) Incomplete.

7

(a) Let $p = (a, b)$ be a point in $S^1 \subset \mathbb{R}^2$. We will show that the tangent space to $S^1$ at the point $p$, denoted $T_p S^1$, is precisely span\{(-b, a)\}.

We will employ the standard smooth structure on $S^1$. So $U_1^-$ is the left semicircle, $U_1^+$ is the right semicircle, $U_2^-$ is the lower semicircle, and $U_2^+$ is the upper semicircle. Furthermore, $\varphi_1^\pm$ and $\varphi_2^\pm$ are the corresponding graph coordinates from Example 1.2 in the text. The inverse maps from $\mathbb{R}$ to $S^1$ are

$$\psi_1^- (y) = (-\sqrt{1-y^2}, y) \quad \psi_1^+ (y) = (\sqrt{1-y^2}, y)$$
$$\psi_2^- (x) = (x, -\sqrt{1-x^2}) \quad \psi_2^+ (x) = (x, \sqrt{1-x^2}).$$
Note that since $\psi_i^{\pm}$ is the inverse of $\varphi_i^{\pm}$ for $1 \leq i \leq 2$, we know $\psi_i^{\pm}(\varphi_i^{\pm}(p)) = (a, b)$. Now, suppose $p \in U_1^-$. By the definition of tangent space from class (8/31), $T_p S^1$ is the span of $(D\psi_1^-)_p$ in $\mathbb{R}^2$. In this case, we compute

$$(D\psi_1^-)_p = \frac{d}{dy}(-\sqrt{1 - y^2}, y)$$

$$= \left(-\frac{1}{2}(1 - y^2)^{-1/2}(-2y), 1\right)$$

$$= \left(\frac{y}{\sqrt{1 - y^2}}, 1\right)$$

$$= \left(\frac{b}{-a}, 1\right).$$

But $T_p S^1 = \text{span}\{(-b/a, 1)\} = \text{span}\{(-b, a)\}$, so this case is verified. The result is consistent across any location of $p$. This is verified in the APPENDIX.

(b) Consider $S^2 \subset \mathbb{R}^3$ with the standard smooth structure. As above, let $\psi_i^{\pm}: \mathbb{R}^2 \to S^2$ be the inverse maps for $1 \leq i \leq 3$. So

$$\psi_1^+(y, z) = (\pm \sqrt{1 - y^2 - z^2}, y, z)$$

$$\psi_2^+(x, z) = (x, \pm \sqrt{1 - x^2 - z^2}, z)$$

$$\psi_3^+(x, y) = (x, y, \pm \sqrt{1 - x^2 - y^2}).$$

Let $p = (a, b, c)$ be a point in $S^2$. We will determine $T_p S^2$ in terms of $a, b, c$.

Note that $\psi_i^+(\varphi_i^{\pm}(p)) = (a, b, c)$. Now, suppose $p \in U_3^+$. Then we compute

$$\frac{\partial}{\partial x} \psi_3^+ = (1, 0, \frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2x))$$

$$= (1, 0, \frac{-x}{\sqrt{1 - x^2 - y^2}})$$

$$= (1, 0, \frac{-a}{c})$$

$$\frac{\partial}{\partial y} \psi_3^+ = (0, 1, \frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2y))$$

$$= (0, 1, \frac{-y}{\sqrt{1 - x^2 - y^2}})$$

$$= (0, 1, \frac{-b}{c}).$$

From this we can see that $T_p S^2 = \text{span}\{(-c, 0, a), (0, -c, b)\}$. Just as before, this result is consistent across all charts. We will not compute all of them here, but as an example we consider $p \in U_2^-:

$$\frac{\partial}{\partial x} \psi_2^- = (1, -\frac{1}{2}(1 - x^2 - z^2)^{-1/2}(-2x), 0)$$

$$= (1, -\frac{x}{\sqrt{1 - x^2 - z^2}}, 0)$$

$$= (1, -\frac{a}{\sqrt{\sqrt{1 - x^2 - z^2}}}, 0)$$

$$\frac{\partial}{\partial z} \psi_2^- = (0, -\frac{1}{2}(1 - x^2 - z^2)^{-1/2}(-2z), 1)$$

$$= (0, -\frac{z}{\sqrt{1 - x^2 - z^2}}, 1)$$

$$= (0, -\frac{z}{\sqrt{\sqrt{1 - x^2 - z^2}}}, 1).$$
Observe that the resulting vectors \((-b, a, 0)\) and \((0, -c, b)\) are different than those above, but we have the linear combinations
\[
\frac{b}{c}(-c, 0, a) + \frac{-a}{c}(0, -c, b) = \left(-b, 0, \frac{ab}{c}\right) + \left(0, a, -\frac{ab}{c}\right) = (-b, a, 0)
\]
\[0(-c, 0, a) + 1(0, -c, b) = (0, -c, b).
\]
So \((-b, a, 0), (0, -c, b) \in \text{span}\{(-c, 0, a), (0, -c, b)\}. \) It follows that \(\text{span}\{(-b, a, 0), (0, -c, b)\} = \text{span}\{(-c, 0, a), (0, -c, b)\}.

**APPENDIX**

We have already seen \(p \in U_1^-\). If \(p \in U_1^+\), then
\[
(D\psi_1^+)_p = \frac{d}{dy}(\sqrt{1-y^2}, y)
= \left(\frac{1}{2}(-y^2)^{-1/2}(-2y), 1\right)
= \left(\frac{-y}{\sqrt{1-y^2}}, 1\right)
= \left(\frac{-b}{a}, 1\right).
\]

If \(p \in U_2^-\), then
\[
(D\psi_2^-)_p = \frac{d}{dx}(x, -\sqrt{1-x^2})
= \left(1, -\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right)
= \left(1, \frac{x}{\sqrt{1-x^2}}\right)
= \left(1, \frac{a}{b}\right).
\]

Finally, if \(p \in U_2^+\), then
\[
(D\psi_2^+)_p = \frac{d}{dx}(x, \sqrt{1-x^2})
= \left(1, \frac{1}{2}(1-x^2)^{-1/2}(-2x), 1\right)
= \left(1, \frac{-x}{\sqrt{1-x^2}}\right)
= \left(1, \frac{-a}{b}\right).
\]

And of course they all agree, because
\[
\text{span}\{\left(\frac{b}{-a}, 1\right)\} = \text{span}\{\left(-\frac{b}{a}, 1\right)\} = \text{span}\{\left(1, \frac{a}{-b}\right)\} = \text{span}\{\left(1, \frac{-a}{b}\right)\} = \text{span}\{(-b, a)\} = T_p S^1.
\]