Proposition: If $M$ is the disjoint union of two components $U$ and $V$, then $H^p(M) = H^p(U) \oplus H^p(V)$.

Proof. (short version) This is the 4th Axiom of De Rham cohomology. □

Proof. (longer version) Since $M$ is the union of open sets there exists a Mayer-Vietoris sequence of cohomology groups. Note that $U \cap V = \emptyset$, so there is no cohomology. Observe

$$
\ldots \rightarrow H^{p-1}(U \cap V) \rightarrow H^p(M) \rightarrow H^p(U) \oplus H^p(V) \rightarrow H^p(U \cap V) \rightarrow \ldots
$$

becomes

$$
\ldots \rightarrow 0 \rightarrow H^p(M) \rightarrow H^p(U) \oplus H^p(V) \rightarrow 0 \rightarrow \ldots
$$

and the zeroes give an isomorphism for all $p$ by exactness. □

Proof. (sans axioms) Recall from class ($\frac{4}{23}$) $\rho_\ast: H^p(M) \rightarrow H^p(U) \oplus H^p(V)$ comes from the restriction of $\ast$-forms. We will show $\rho_\ast$ is an isomorphism.

Let $[\omega]$ class $\rho_\ast$. Then $\omega_U$ and $\omega_V$ are exact, so $\omega = d\eta$ on $U$ and $d\mu$ on $V$ for some $(p-1)$-forms $\eta$, $\mu$. Extend $\eta$ and $\mu$ to all of $M$ by setting them equal to zero outside $U$ and $V$ respectively. This is smooth because $U$ and $V$ are disjoint. Then $\omega = d\eta + d\mu = d(\eta + \mu)$ and so $\omega$ is exact. Hence $[\omega] = 0$ and $\rho_\ast$ is injective.

Let $([\eta], [\mu]) \in H^p(U) \oplus H^p(V)$. Extend $\eta$ and $\mu$ as above. Then $\eta + \mu \in \Omega^p(M)$ and is closed because $d(\eta + \mu) = d\eta + d\mu = 0$.

Now, $\rho_\ast([\eta + \mu]) = ([\eta + \mu], [\eta + \mu]) = ([\eta], [\mu])$, so $\rho_\ast$ is surjective. □

Therefore $\rho_\ast$, which is known to be linear, is an isomorphism between $H^p(M)$ and $H^p(U) \oplus H^p(V)$.

Was the assumption that $U \cap V$ be components necessary? Couldn't they just be open, disjoint sets?
2) See the "zigzag lemma" on pages 395-396. We will verify the details.

a. We verified in class that \([a''']\) is indeed in \(H^p(B)\) and that the choice of \(b' \in B^p\) does not matter.

Independence of representative from \([c^p] \in H^p(B)\). Let \(c^p \in [c^p]\).

Then \(c^p = c^p + b\eta\) for some \(\eta \in C^{*\*}\). Since \(g\) is onto \(\eta = g(B)\) for some \(B \in B^{*\*}\). Then \(c^p = c^p + g(B) = c^p + g(dB) = g(B^p + dB)\). So \(B^p\) is replaced by \(B^p + dB\) in the definition of \(\delta\), but they are from the same cohomology class which we know doesn't matter. √

Independence of choice of \(a''' \in A^{*\*}\). Let \(a''' \in A^{*\*}\) be such that \(f(a''') = dB^p\). Note that \(f(a''' - a'''') = f(a''') - f(a''') = dB^p - dB^p = 0\).

So \(a''' - a'''' \in \ker f\) and by exactness is also in the image of the map into \(A^{*\*}\). But this map is zero, so \(a''' - a''' = 0\) and \(a''' = a''''\). √

b. We will show that \(\delta\) is linear.

Let \(\delta([c_1,j]) = [a_1]\) and \(\delta([c_2]) = [a_2]\) via the elements \(b_1\) and \(b_2\), resp. That is, \(g(b_1) = c_1\), \(g(b_2) = c_2\) and \(f(a_1) = db_1\), \(f(a_2) = db_2\). Since \(f, g,\) and \(\delta\) are linear,

\[ g(b_1 + b_2) = g(b_1) + g(b_2) = c_1 + c_2 \]

and

\[ f(a_1 + a_2) = f(a_1) + f(a_2) = db_1 + db_2 = d(b_1 + b_2). \]

So by definition, \(\delta([c_1] + [c_2]) = [a_1] + [a_2] = \delta([c_1]) + \delta([c_2])\) via the element \((b_1 + b_2)\)

\[
\delta \left( r \left[ c_1 \right] \right) = r \delta \left( \left[ c_1 \right] \right) \text{ for } r \in \mathbb{R}.
\]
We will show that the long exact sequence is indeed exact. Note that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact.

Consider $\bar{f}(\alpha) = \bar{g}(\beta) = \gamma$. Since $f(a)$ is in the kernel of $g$, we get $g(f(a)) = [g(a)] = 0$. So $f \in \ker g$.

Let $b \in B^0$ with $[b] \in \ker \bar{g}$. Then $[g(b)] = \bar{g}([b]) = \gamma = 0$ so $g(b) = dc$ for some $c \in C_1$. Since $g$ is onto, there is a $b' \in B_1$ such that $g(b') = c$. Then

$$g(b - db') = g(b) - g(db') = g(b) - dc = 0.$$ 

So $b - db' \in \ker g$ and hence $b - db' = f(a)$ for some $a \in A^0$ by exactness. Now $\bar{f}(\alpha) = [f(a)] = [b - db'] = [b]$, so $[b] \in \text{Im} \bar{f}$.

Consider $\bar{g}(\beta) = \bar{h}(\gamma)$. Put $g(b) = c$ and note that by definition $\beta \in \ker \bar{g}$, so $db = 0$. By definition of $\delta$, we have $\delta([c]) = [a]$ where $f(a) = db = 0$. Since $f$ is one-one, $a = 0$ and so $\delta([c]) = 0$ implying $\bar{g}(\beta) \in \ker \delta$. So $\bar{g} \in \ker \delta$.

Let $[c] \in \ker \delta$. Then $\delta([c]) = [a]$ via $b$, we know $g(b) = c$, $f(a) = db$, and $a = da'$ for some $a' \in A^0$. Note that $d(b - f(a')) = db - dc = db - (db) = db - db = 0$ so $g(b - f(a')) = 0$. Also note $\bar{g}(b - f(a')) = [g(b) - g(f(a'))] = [g(b)] = [c]$, so $[c] \in \text{Im} \bar{g}$.

Consider $\delta([c]) = \bar{g}(\beta)$. By definition, $\delta([c]) = [a]$ via $b$ where $g(b) = c$ and $db = f(a)$. Then $\bar{f}(\delta([c]) = \bar{f}([a]) = [f(a)] = [db] = 0 \in H^1(B)$ so $\delta \in \ker \bar{f}$.

Let $[a] \in \ker \bar{f}$. This means $f(a) = db$ for some $b \in B_1$.

Consider $[g(b)] \in H^1(B)$. If $g(b) = c$ then $\delta([c]) = [a]$ because $g(b) = c$ and $f(a) = db$. So $[a] \in \text{Im} \delta$. 

\[ \text{im} \delta = \ker \bar{f} \]

\[ \text{im} \bar{f} = \ker \delta \]

\[ \text{im} \bar{g} = \ker \delta \]

\[ \text{im} \delta = \ker \bar{f} \]

\[ \text{im} \bar{f} = \ker \delta \]

\[ \text{im} \bar{g} = \ker \delta \]
3) We will compute \( H^*(T^2) \).

a. Since the interval \((0,3)\) is contractible it follows that the cylinder \( C = (0,3) \times \mathbb{S}^1 \) deformation retracts to \( \mathbb{S}^1 \). Thus by corollary 39.1, \( H^*(C) = H^*(\mathbb{S}^1) \).

b. \( T^2 = C_1 \cup C_2 \). \( C_1, C_2, U, \) and \( V \) deformation retract to \( \mathbb{S}^1 \).

C. By part (a), \( H^*(C_i) = H^*(\mathbb{S}^1) = H^0(U) = H^0(V) = H^0(\mathbb{S}^1) = \begin{cases} \mathbb{R}, & p = 0, 1 \\ 0, & \text{otherwise} \end{cases} \).

Using \( T^2 = C_1 \cup C_2 \) from part (b) we have the Mayer-Vietoris sequence

\[
0 \to H^0(T^2) \to H^0(C_1) \oplus H^0(C_2) \to H^0(U) \oplus H^0(V) \to 0.
\]

We rewrite this as

\[
0 \to H^0(T^2) \to \mathbb{R}^2 \to \mathbb{R}^2.
\]

Since the torus is connected we know \( H^0(T^2) = \mathbb{R} \). By counting dimensions (using rank-nullity) we know \( 1 - 2 + 2 - \dim H^0(T^2) + 2 - 2 + \dim H^1(T^2) = 0 \). Thus \( \dim H^0(T^2) = 1 = \dim H^1(T^2) \). Now \( T^2 \) is orientable so there exists an orientation form \( \omega \in \Omega^{2}(T^2) \). By proposition 146(e) of the text, \( \int_T \omega > 0 \) so the integral map from \( H^2(T^2) \) to \( \mathbb{R} \) is necessarily onto. This implies \( \dim H^2(T^2) \geq 1 \) and of course equality holds. So \( H^2(T^2) = \mathbb{R} \).

In summary,

\[
H^p(T^2) = \begin{cases} \mathbb{R}, & p = 0, 2 \\ \mathbb{R}^2, & p = 1 \\ 0, & \text{otherwise} \end{cases}
\]
4) We will compute the De Rham cohomology of $\mathbb{R}^n$ minus two points 
(To avoid silliness, let $n \geq 2$)

Let $x, y \in \mathbb{R}^n$ be distinct and put $M = \mathbb{R}^n \setminus \{x, y\}$. Now $x$ and $y$ must differ in at least one coordinate. After possibly reordering the coordinates, suppose $x' \neq y'$ and without loss assume $x' < y'$.

Since $\mathbb{R}$ is Hausdorff, there exists an $\varepsilon > 0$ such that the intervals $(x' - \varepsilon, x' + \varepsilon)$ and $(y' - \varepsilon, y' + \varepsilon)$ are disjoint. Set $z' = \frac{x' + y'}{2}$. Note that $z'$ is in neither $(x' - \varepsilon, x' + \varepsilon)$ nor $(y' - \varepsilon, y' + \varepsilon)$. Specifically, $z' > x' + \varepsilon$ and $z' < y' - \varepsilon$. This implies

$$x \in \tilde{U} := \{p \in \mathbb{R}^n | p > z' + \varepsilon\} \quad \text{and} \quad y \in \tilde{V} := \{p \in \mathbb{R}^n | p > z' - \varepsilon\}$$

whereas $y \notin \tilde{U}$ and $x \notin \tilde{V}$. Set $U = M \cap \tilde{U}$ and $V = M \cap \tilde{V}$.

Clearly, $M = U \cup V$ and we will use this to employ Mayer-Vietoris. Note that $U$ and $V$ are once-punctured $n$-planes and so deformation retract to $S^{n-1}$. Also, $U \cap V$ is precisely $(z' - \varepsilon, z' + \varepsilon) \times \mathbb{R}^{n-1}$ and so deformation retracts to $\mathbb{R}^{n-1}$.

This brings us to

$$0 \rightarrow H^0(M) \rightarrow H^0(M) \oplus H^0(V) \cong \mathbb{R}^2 \rightarrow H^0(U \cap V) \cong \mathbb{R}$$

We know $H^0(M) = \mathbb{R}$ because $M$ is connected (since $n \geq 2$).

Also, it is obvious that $H^p(M) = 0$ for $2 \leq p \leq n - 2$, and at the end $0 \rightarrow H^n(M) \rightarrow 0$ implies $H^n(M) = 0$. 

\[ \text{\footnotesize Otten} \]
Counting dimensions at the beginning of the sequence we find
\[ 1 - 2 + 1 = \dim H'(M) = 0, \] so necessarily \( H'(M) = 0. \)

Finally, we consider \( 0 \rightarrow H''(M) \rightarrow \mathbb{R}^2 \rightarrow 0. \) The map
in the middle must be an isomorphism and thus \( H''(M) = \mathbb{R}^2. \)

In summary,
\[ H^p(M) = \begin{cases} \mathbb{R}^2 & p = 0 \\ \mathbb{R} & p = n-1 \\ 0 & \text{otherwise}. \end{cases} \]

5) Let \( \Sigma_g \) be a compact, orientable 2-manifold of genus \( g. \) It
is given that \( \Sigma_g \) is diffeomorphic to \( \mathbb{S}^2 \) with \( g \) handles
attached (see Figure). We will compute the De Rham cohomology.

We will use Mayer-Vietoris' and the example of \( \mathbb{S} \) minus
disks from class. To define \( U \) and \( V \) we smoothly deform
\( \Sigma_g \) so that the handleless sphere is entirely below \( z = 0 \)
(in \( \mathbb{R}^3 \)) and the handles all reach up to \( z = 1 \) (see figure).

Let \( f \) be the height function, and \( \varepsilon > 0. \) Define
\[ U = \{ p \in \Sigma_g \mid f(p) > \varepsilon \} \]
and \[ V = \{ p \in \Sigma_g \mid f(p) < 1 - \varepsilon \}. \]
Clearly \( \Sigma_g = U \cup V \) and \( U \cap V \) are
open.
Then $U$ is the disjoint union of $g$ cylinders, $U \cup V$ is the disjoint union of $2g$ cylinders, and $V$ is diffeomorphic to $S^1 \setminus \{2g\}$ disks? We know all of these cohomologies so the Mayer-Vietoris sequence is

$$
0 \rightarrow H^0(U) \rightarrow R^g \rightarrow R^{2g} \rightarrow H^1(U) \rightarrow R^g \rightarrow R^{2g+1} \rightarrow H^2(U) \rightarrow 0 \ldots
$$

Since $U$ is connected, $H^0(U) = R$. Moreover, since it is orientable there exists an orientation form and integrating this gives $H^1(U) = R^g$ 

(the reasoning here is the same as in problem 3c). Now we count dimensions using rank-nullity and find

$$
1 - (g+1) + 2g - \dim H^0(U) + (g+2g-1) - 2g + 1 = 0
$$

or equivalently

$$
\dim H^1(U) = 2g.
$$

Thus

$$
H^p(U) = \begin{cases} R & p = 0, 2 \\ R^g & p = 1 \\ 0 & \text{otherwise} \end{cases}
$$

3A3 (b) Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold without boundary and let $\omega^n$ be as in problem 5 in homework 9.

a. We will show that $\omega^n$ is not exact.

Suppose to the contrary that $\omega^n = d\eta$ for some $\eta \in \Omega^{2n-1}(M)$. Then

$$
\int_M \omega^n = \int_M d\eta = \int_M \delta^\omega \eta = 0
$$

because $\delta^\omega = 0$.

But $\omega^n$ is an orientation form so by proposition 14.6 (c), $\int_M \omega^n > 0$.

This is a contradiction. So no $\eta$ exists and $\omega^n$ is not exact.
b. Proposition \([\omega^k] \neq 0\) for \(k=1, \ldots, n\) and therefore \(H^{2k}(M) \neq 0\) for the same \(k\).

Proof. We must show that \(\omega^k\) is closed but not exact.

(closed) We induct on \(k\). Base Step: By \#5 HW9, \(d\omega = 0\) so \(\omega\) is closed. Inductive Step: Assume \(d(\omega^k) = 0\). Then
\[
d(\omega^{k+1}) = d(\omega^k \wedge \omega) = d\omega^k \wedge \omega + \omega^k \wedge d\omega = 0
\]
so \(\omega^{k+1}\) is closed. By induction, \(\omega^k\) is always closed.

(not exact) We induct on \(k = n-k\). Base Step: When \(k=0\), \(\omega^{n-k} = \omega^n\) which is not exact by part (a). Inductive Step: Assume \(\omega^{n-k}\) is not exact. Suppose to the contrary that \(\omega^{n-(k+1)}\) is exact. Then there exists \(\eta \in H^{-2}^1(M)\) such that \(d\eta = \omega^{n-(k+1)}\). This means \(\omega^{n-k} = \omega^{n-(k+1)} \wedge \omega = d\eta \wedge \omega\). But then\[
d(\eta \wedge \omega) = d\eta \wedge \omega + \eta \wedge d\omega = d\eta \wedge \omega = \omega^{n-k}
\]
which contradicts our induction hypothesis. Thus \(\omega^{n-(k+1)}\) is not exact. Therefore, \(\omega^k\) is never exact.

Since \(\omega^k\) is closed but not exact for \(k=1, \ldots, n\), \([\omega^k]\) is a nontrivial element in \(H^{2k}(M)\).

\[\square\]

C. A symplectic manifold is necessarily even dimensional, so \(S^n\) does not admit a symplectic structure for \(n\) odd.

For \(S^n\), \(n \leq 4\) even, we have \(H^2(S^n) = 0\) and hence part (b) implies \(S^n\) does not admit a symplectic structure in these cases either.

Consider \(S^2\). We showed in \#1 HW9 that \(\Omega \wedge d\omega = r dx dy + d\omega \wedge d\omega dy - d\omega x dy d\omega d\omega dx dy\) is non-vanishing on \(S^2\). It is also true that \(\Omega\) is closed because there are no 3-forms on \(S^2\). Thus by \#5 HW9, \(\Omega\) is a \(\sqrt{}\) symplectic form on \(S^2\).

\[\text{Would agree: This is a stupid case: a symplectic structure? This would violate the statement of (c).}\]

\[\text{What about } S^0? \text{ Does it not admit everyone?} \]