1

Proposition. Let \((M, g)\) and \((\tilde{M}, \tilde{g})\) be Riemannian manifolds. If \(f : M \to \tilde{M}\) is a smooth map such that \(f^* \tilde{g} = g\) (i.e., \(f\) is an isometry), then \(f\) is an immersion.

Proof. We must show that \(f_*\) is injective at each point in \(M\). Let \(p \in M\) and \(X_p \in T_p M\) such that \(X_p \in \ker f_*\). Then

\[
g_p(X_p, X_p) = (f^* \tilde{g})_p(X_p, X_p) = \tilde{g}_{f(p)}(f_*(X_p), f_*(X_p)) = \tilde{g}_{f(p)}(0, 0) = 0.
\]

By (ii) from the definition of a Riemannian metric we have \(X_p = 0\), so \(f_*\) has trivial kernel. Since this is true for all \(p \in M\), \(f\) is an immersion. \(\square\)

2

Proposition. In Euclidean space, the shortest distance between two points is a straight line.

Proof. We write \(L\) for \(L_{\tilde{g}}\). For \(p, q \in \mathbb{R}^n\), we will show that the path \(\gamma : [0, 1] \to \mathbb{R}^n, \ t \mapsto (1-t)p + tq\) is not longer than any other path from \(p\) to \(q\). In other words, we will show that \(L(\tilde{\gamma}) \geq L(\gamma)\) for any path \(\tilde{\gamma}\) from \(p\) to \(q\).

Since any two points can be translated isometrically to the \(x^1\)-axis, it suffices to show this for \(p = (0, \ldots, 0)\) and \(q = (c, 0, \ldots, 0)\), \(c > 0\). Note that in this case \(\gamma(t) = (tc, 0, \ldots, 0)\) and \(\dot{\gamma}(t) = (c, 0, \ldots, 0)\). Thus \(|\dot{\gamma}(t)| = c\) and

\[
L(\gamma) = \int_0^1 |\dot{\gamma}(t)| \, dt = \int_0^1 c \, dt = c. \tag{1}
\]

Now, let \(\tilde{\gamma}\) be another path from \(p\) to \(q\) (see figure 1).

Figure 1: Two paths from \(p\) to \(q\)
We define $x(t)$ to be the difference between the two paths, that is, $x(t) = \tilde{\gamma}(t) - \gamma(t)$ (see figure 2). Hence $\tilde{\gamma}(t) = \gamma(t) + x(t)$. Note that $x(0) = \tilde{\gamma}(0) - \gamma(0) = 0 = \tilde{\gamma}(1) - \gamma(1) = x(1)$, so $x$ is a closed loop.

Since the norm of a point is necessarily larger than the norm of its first coordinate, we have

$$\left|\dot{\tilde{\gamma}}(t)\right| = \left|\dot{\gamma}(t) + \dot{x}(t)\right| \geq \left|\dot{\gamma}(t)\right|^1 + \left|\dot{x}(t)\right|^1 \geq \left|\dot{\gamma}(t)\right|^1 + (\dot{x}(t))^1.$$ (2)

Now, observe

$$L(\tilde{\gamma}) = \int_0^1 \left|\dot{\tilde{\gamma}}(t)\right| dt \geq \int_0^1 (\dot{\gamma}(t))^1 + (\dot{x}(t))^1 dt = \int_0^1 (\dot{\gamma}(t))^1 dt + \int_0^1 (\dot{x}(t))^1 dt = \int_0^1 c dt + \int_0^1 (\dot{x}(t))^1 dt = FTC \overset{\text{(1)}}{=} c + 0 \overset{\text{(2)}}{=} L(\gamma),$$

where we can apply the fundamental theorem of calculus without hesitation because $(\dot{x}(t))^1$ is merely a function from $\mathbb{R}$ to $\mathbb{R}$. Thus the straight line $\gamma$ is the shortest path. \hfill \Box

### 3

**Proposition.** Let $(M, g)$ be a Riemannian manifold, let $f \in C^\infty(M)$, and let $\nabla f$ be the gradient of $f$ defined by $g(\nabla f, V) = df(V)$ for all vectors $V$. If $q$ is a regular value of $f$ and $M_q$ is the level set of $q$, then

(a) at each $p \in M_q$, $(\nabla f)_p$ is non-zero and perpendicular to $T_p(M_q)$,

(b) $M$ being orientable implies $M_q$ is orientable.

**Proof.** (a) By the definition of a regular value, we know that $f_*$ is surjective at each $p \in M_q$. Suppose to the contrary that $(\nabla f)_p = 0$ for some $p \in M_q$. Let $\{x^i\}$ be coordinates near $p$. This means that locally $\partial f/\partial x^i = 0$ for all $i$. But then $f_* = 0$ and so is not surjective at $p$. This is a contradiction, implying that $(\nabla f)_p \neq 0$ for all $p \in M_q$. 


Since \( q \) is a regular value, the preimage theorem states that \( M_q \) is an embedded submanifold of dimension \( n - 1 \). Choose coordinates \( \{y^i\}^{n-1}_{i=1} \) near a point \( p \in M_q \). Then there exist coordinates \( \{y^j\}^n_{j=1} \) on \( M \) near \( p \) such that \( y^j = y^i \) for \( 1 \leq j \leq n - 1 \). Now, for \( 1 \leq j \leq n - 1 \) we have \( p, y^j \in M_q \) and so

\[
\left. df_p \right|_{\partial/\partial y^j} = \left. \frac{\partial f}{\partial y^j} \right|_p = \lim_{h \to 0} \frac{f(p + hy^j) - f(p)}{h} = \lim_{h \to 0} \frac{q - q}{h} = 0,
\]

because \( p + hy^j \in M_q \) for \( 1 \leq j \leq n - 1 \). For any \( X_p \in T_p(M_q) \) we can write \( X_p \) as a linear combination of \( \partial/\partial y_j \), \( 1 \leq j \leq n - 1 \) (which all get sent to zero by \( df_p \)), and thus

\[
g((\nabla f)_p, X_p) = df_p(X_p) = df_p \left( \sum_{j=1}^{n-1} \alpha^j \frac{\partial}{\partial y^j} \right) = 0.
\]

Therefore, \( (\nabla f)_p \) and \( T_p(M_q) \) are orthogonal.

(b) Assume \( M \) is orientable. Then there exists an orientation form \( \omega \in \Omega^n(M) \). We showed in part (a) that \( \nabla f \) is a non-vanishing orthogonal vector field on \( M_q \). Hence it is pointing either entirely outward or entirely inward. After a possible scaling, we can assume \( \nabla f \) is an outward vector field on \( M_q \).
Define $\tilde{M}_q = \{ p \in M \mid f(p) \leq q \}$. Since $q$ is a regular value, we know that $\tilde{M}_q$ is a manifold with boundary, namely, $\partial \tilde{M}_q = M_q$. Therefore, $\nabla f$ is an outward normal vector field on $\partial \tilde{M}_q$ and by corollary 36.1 from class, $\iota_{\nabla f} \omega|_{M_q}$ is an orientation form on $M_q$. So $M_q$ is orientable.

4

Let $(M, g)$ be an oriented Riemannian manifold. Recall that the Riemannian metric determines a volume form

$$d\text{vol}_g = \sqrt{\det g_{ij}} \, dx^1 \wedge \cdots \wedge dx^n.$$ 

Let $\{x^i\}$ and $\{y_i\}$ be positively oriented charts and let $\phi(y^1, \ldots, y^n) = (x^1, \ldots, x^n)$ be an isomorphism. Then $\det D\phi = 1$ because $\phi$ preserves orientation. Therefore, proposition 12.12 of the text gives

$$\phi^*d\text{vol}_g = \phi^*(\sqrt{\det g_{ij}} \, dx^1 \wedge \cdots \wedge dx^n)$$

$$= \sqrt{\det(g_{ij} \circ \phi)} (\det D\phi) \, dy^1 \wedge \cdots \wedge dy^n$$

$$= \sqrt{\det(g_{ij} \circ \phi)} \, dy^1 \wedge \cdots \wedge dy^n,$$

and we see that $d\text{vol}_g$ has the same expression in $y$-coordinates.