Let \( \Pi: \mathbb{R}^2 \rightarrow \mathbb{T}^2 \) be the covering map defined by \( \Pi(q, \theta) = (e^{iq}, e^{i\theta}) \), and let \( X: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be the immersion defined by \( X(q, \theta) = ((2 + \cos q)\cos \theta, (2 + \cos q)\sin \theta, \sin q) \) from p.156. We will show that \( X \) "descends" via \( \Pi \) to an embedding of \( \mathbb{T}^2 \) into \( \mathbb{R}^3 \).

Since \( \Pi \) is a surjective map, for any \( (x, y) \in \mathbb{T}^2 \) there exists \( (q_x, \theta_y) \in \mathbb{R}^2 \) such that \( x = e^{iq_x} \) and \( y = e^{i\theta_y} \). Note that the \( q_x, \theta_y \) are not unique. This will be addressed shortly.

Define \( \hat{X}: \mathbb{T}^2 \rightarrow \mathbb{R}^3, (x, y) \rightarrow ((2 + \cos q_x)\cos \theta_y, (2 + \cos q_x)\sin \theta_y, \sin q_x) \).

We must verify that this is well-defined. Suppose \( (x, y) = (x_a, y_a) \).
Then the \( q_x, \theta_y \) and \( q_{x_a}, \theta_{y_a} \) may not be equal, but certainly \( e^{iq_x} = e^{iq_{x_a}} \) and \( e^{i\theta_y} = e^{i\theta_{y_a}} \). By Euler's formula,

\[ \cos q_x + i\sin q_x = \cos q_{x_a} + i\sin q_{x_a} \quad \& \quad \cos \theta_y + i\sin \theta_y = \cos \theta_{y_a} + i\sin \theta_{y_a} \]

Since complex numbers are equal if and only if both their real and imaginary parts are equal, we have \( \cos q_x = \cos q_{x_a}, \sin q_x = \sin q_{x_a}, \cos \theta_y = \cos \theta_{y_a}, \) and \( \sin \theta_y = \sin \theta_{y_a} \). Thus \( \hat{X}(x, y) = \hat{X}(x_a, y_a) \) and \( \hat{X} \) is well-defined.

Now \( \hat{X} \) "descends" from \( X \) because \( X = \hat{X} \circ \Pi \) and it is an embedding if it is an immersion and a homeomorphism onto its image. We know \( \hat{X} \) is an immersion because it has the same explicit formula as the immersion \( X \) (example 7.111). Clearly \( \hat{X} \) maps onto its image.

For injectivity, suppose \( \hat{X}(x, y) = \hat{X}(x_a, y_a) \) for some \( (x, y), (x_a, y_a) \in \mathbb{T}^2 \), that is,
\[(2 + \cos \varphi_1) \cos \theta_1, (2 + \cos \varphi_1) \sin \theta_1, \sin \varphi_1) = (2 + \cos \varphi_2) \cos \theta_2, (2 + \cos \varphi_2) \sin \theta_2, \sin \varphi_2) \quad (*)\]

For some \((\varphi_1, \theta_1), (\varphi_2, \theta_2) \in \mathbb{R}^2\). We see immediately that

\[
\sin \varphi_1 = \sin \varphi_2. \quad (1)
\]

By squaring and then adding the first and second components of \((*)\), we get

\[
(2 + \cos \varphi_1)^2 \cos^2 \theta_1 + (2 + \cos \varphi_1)^2 \sin^2 \theta_1 = (2 + \cos \varphi_2)^2 \cos^2 \theta_2 + (2 + \cos \varphi_2)^2 \sin^2 \theta_2
\]
\[
\Rightarrow
\]
\[
(2 + \cos \varphi_1)^2 = (2 + \cos \varphi_2)^2
\]
\[
\Rightarrow
\]
\[
2 + \cos \varphi_1 = \pm (2 + \cos \varphi_2).
\]

But \(2 + \cos \varphi > 0\) for all \(\varphi\), so necessarily \(2 + \cos \varphi_1 = 2 + \cos \varphi_2\) and

\[
\cos \varphi_1 = \cos \varphi_2. \quad (2)
\]

Equations (1) and (2) together imply \(\cos \varphi_1 + i \sin \varphi_1 = \cos \varphi_2 + i \sin \varphi_2\) or equivalently \(e^{i \varphi_1} = e^{i \varphi_2}\). So \(x_1 = x_2\). \(\checkmark\)

Looking at the first component of \((*)\), we know

\[
(2 + \cos \varphi_1) \cos \theta_1 = (2 + \cos \varphi_2) \cos \theta_2.
\]
\[
\Rightarrow
\]
\[
\cos \theta_1 = \cos \theta_2. \quad (3)
\]

Similarly, using the second component of \((*)\),

\[
\sin \theta_1 = \sin \theta_2. \quad (4)
\]

Equations (3) and (4) together imply \(e^{i \theta_1} = e^{i \theta_2}\) and so \(y_1 = y_2\). Hence \((x_1, y_1) = (x_2, y_2)\) and \(X\) is one-one. \(\checkmark\)

**Remark:** \(\mathbb{T}^2\) is compact and \(X\) is one-one immersion $\Rightarrow$ \(X\) embedding. \(\checkmark\) Done.

Continuity of \(X\) follows from the fact that it comprises only continuous functions (e.g., sine and cosine). We know \(X^{-1}\) exists because \(X\) is a bijection. In fact, solving \((x, y, z) = (2 + \cos \varphi) \cos \theta, (2 + \cos \varphi) \sin \theta, \sin \varphi)\) for \(\sin \varphi, \cos \varphi, \sin \theta, \cos \theta\) yields

\[
\sin \varphi = z, \quad \sin \theta = \frac{x}{\sqrt{x^2 + y^2}},
\]
\[
\cos \varphi = \frac{y}{\sqrt{x^2 + y^2}}. \quad \cos \theta = \frac{x}{\sqrt{x^2 + y^2}}.
\]

Note that \(\sqrt{x^2 + y^2} > 0\) because the \(z\)-axis does not intersect \(X(\mathbb{T}^2)\). So \(X^{-1}\) is also continuous and therefore \(X\) is a homeomorphism and an embedding.
2) Define \( F: S^2 \rightarrow \mathbb{R}^4 \) by \( F(x, y, z) = (x^2 - y^2, xy, xz, yz) \).

Let \( p: S^2 \rightarrow \mathbb{R}P^2 \) be the smooth covering map from example 2.5(i) in the text. We will show that \( F \) "descends" via \( p \) to a smooth embedding of \( \mathbb{R}P^2 \) into \( \mathbb{R}^4 \).

Since \( p \) is surjective, for any \((x, y, z) \in \mathbb{R}P^2\) we can choose \((\bar{x}, \bar{y}, \bar{z}) \in S^2\) such that \( p(\bar{x}, \bar{y}, \bar{z}) = (x, y, z) \). Note that this choice is not unique but will be addressed shortly.

Define \( \hat{F}: \mathbb{R}P^2 \rightarrow \mathbb{R}^4 \), \( (x, y, z) \rightarrow (\bar{x}^2 - \bar{y}^2, \bar{x}y, \bar{x}z, \bar{y}z) \).

We must verify that \( \hat{F} \) is well-defined. Suppose \((x, y, z) = (u, v, w)\) in \( \mathbb{R}P^2 \). Then from \( p \) we know \((\bar{x}, \bar{y}, \bar{z}) = (\bar{u}, \bar{v}, \bar{w})\) or \((\bar{x}, \bar{y}, \bar{z}) = (-\bar{u}, -\bar{v}, -\bar{w})\). In the first case, clearly \( \hat{F}(x, y, z) = \hat{F}(u, v, w) \). In the second case,

\[
\begin{align*}
\bar{x}^2 - \bar{y}^2 &= (-\bar{x})^2 - (-\bar{y})^2 = \bar{u}^2 - \bar{v}^2 \\
\bar{x}y &= (-\bar{x})(-\bar{y}) = \bar{u}\bar{v} \\
\bar{x}z &= (-\bar{x})(-\bar{z}) = \bar{u}\bar{w} \quad \checkmark \\
\bar{y}z &= (-\bar{y})(-\bar{z}) = \bar{v}\bar{w}.
\end{align*}
\]

So again \( \hat{F}(x, y, z) = \hat{F}(u, v, w) \) and thus \( \hat{F} \) is well-defined.

Observe that \( \hat{F} \) indeed "descends" from \( F \) because \( F = \hat{F} \circ p \).

The smoothness of \( \hat{F} \) follows easily from its explicit formula.

We will show that \( \hat{F} \) is a one-one immersion and thus, using theorem 16.1, since \( \mathbb{R}P^2 \) is compact \( \checkmark \) it is necessarily the case that \( \hat{F} \) is a smooth embedding.

For injectivity, we consider several cases. For all of them suppose \((x^2 - y^2, xy, xz, yz) = (u^2 - v^2, uv, uw, vw)\) for some \((x, y, z), (u, v, w) \in \mathbb{R}P^2\).
Case One: \( x = y = 0 \). In this case, we know from \( S^2 \) that 
\[ |z| = 1. \]
Also, we have \((0, 0, 0) \in \mathbb{R}^4\) as the point in question. 
So \( u^2 - v^2 = 0 \) and thus \( u = v = 0 \) which implies \( |w| = 1 \). 
Therefore \((x, y, z) = (u, v, w)\) in \( \mathbb{R}P^2 \).

Case Two: \( x^2 - y^2 = 0 \) but \( x, y \) not both 0. This means \( x^2 = y^2 \) 
and from the first coordinate we also know \( u^2 = v^2 \). Squaring 
the third and fourth coordinates and adding yields 
\[ \frac{x^2 - y^2}{u^2 - v^2} = \frac{x^2 + y^2}{u^2 + v^2}, \]
\[ \frac{x^2 - y^2}{u^2 - v^2} = \frac{2x^2 - 2y^2}{2u^2 - 2v^2}. \]
But observe that squaring the second coordinate gives \( x^2 = u^2 \) 
or equivalently \( 2x^2 = 2u^2 \). And \( x, u \neq 0 \) so the equation 
above, after using the cancellation property, becomes 
\[ z^2 = w^2. \]

Case Three: \( x^2 - y^2 \neq 0 \). Since \( x^2 - y^2 = u^2 - v^2 \neq 0 \) we know 
\[ (i) \ x^2 = u^2 + v^2, \]
\[ (ii) \ x^2 = u^2 - v^2, \]
\[ (iii) \ y^2 = u^2 - v^2, \]
and 
\[ (iv) \ y^2 = u^2 - v^2. \]
from the squaring of the respective coordinates. Plugging (ii) 
into (iii) yields 
\[ \frac{(u^2 - v^2 + v^2) z^2}{u^2 - v^2} = \frac{u^2 v^2}{u^2 - v^2} \]
\[ \frac{u^2 z^2 - v^2 z^2 + v^2 z^2}{u^2 - v^2 + v^2 z^2} = \frac{u^2 v^2}{u^2 - v^2} \]
\[ \frac{u^2 z^2 - v^2 z^2}{u^2 - v^2 + v^2 z^2} = \frac{u^2 v^2}{u^2 - v^2} \]
\[ \frac{z^2}{(u^2 - v^2) z^2} = \frac{(u^2 - v^2) z^2}{(u^2 - v^2) z^2}. \]
Since \( u^2 - v^2 \neq 0 \) we can use the cancellation property to get 
\[ z^2 = w^2. \]

Now, we determined in both case two and case three that \( z^2 = w^2 \).
If \( z \neq 0 \) then \( w \neq 0 \) and by (iv) we have \( y^2 = v^2 \) and 
by (iii) \( x^2 = u^2 \). So \((x, y, z) = (u, v, w)\) in \( \mathbb{R}P^2 \) under these 
conditions as well. We have one final case to check.
Case Four: \( Z = 0 \). Then we are considering the case where 
\[ (x^2 - y^2, xy, 0, 0) = (u^2 - v^2, uv, 0, 0). \]
From \( S^2 \) we also know \( x^2 + y^2 = 1 = u^2 + v^2 \). This implies \( x^2 - u^2 = y^2 - v^2 \).
Also equating the first coordinates above gives \( x^2 - u^2 = y^2 - v^2 \).
So we get
\[ \begin{align*}
\sqrt{y^2 - v^2} &= \sqrt{y^2 - v^2} \\
2y^2 &= 2v^2 \\
&= \sqrt{y^2} \\
&= y^{\frac{3}{2}}.
\end{align*} \]

Using this fact in the previous equation \( x^2 - y^2 = u^2 - v^2 \) gives \( x^2 = u^2 \). So again \( (x, y, z) = (u, v, w) \) in \( \mathbb{R}P^2 \) and we can finally conclude that \( \tilde{F} \) is injective.

To check that \( \tilde{F} \) is an immersion we look to \( \tilde{F}_x \) with the matrix representation
\[ F: \mathbb{R}P^2 \rightarrow \mathbb{R}^4. \]

\[
\tilde{F}_x = \begin{bmatrix}
2x & -2y & 0 \\
y & x & 0 \\
z & 0 & 1 \\
0 & x & y
\end{bmatrix}
\]

\[ 2 \times 4 \text{ matrix}. \]

Sadly, the kernel of \( \tilde{F}_x \) includes vectors of the form \((0, 0, 0, c)\) and so \( \tilde{F}_x \) is not injective and \( \tilde{F} \) is not an immersion. There must be a mistake in the work above or with the definition of the map \( \tilde{F} \).
3) Let \( Y: \mathbb{R} \to \mathbb{T}^2 \) be the map defined by \( Y(t) = (e^{2\pi i t}, e^{2\pi i ct}) \) where \( c \) is a fixed irrational number. To show that \( Y(\mathbb{R}) \) is dense in \( \mathbb{T}^2 \) we will show that every point on the torus is a limit point of a sequence in the image of \( Y \).

Let \( p = (e^{2\pi i \alpha}, e^{2\pi i \beta}) \in \mathbb{T}^2 \). We will show that \( \{Y(n\alpha + n\beta) : n \in \mathbb{Z}\} \) has \( p \) as a limit point. Fix \( \epsilon > 0 \). Note that

\[
\begin{align*}
2\pi i (\alpha + n\beta) &= 2\pi i \alpha + 2\pi i n\beta \\
2\pi i (\alpha + n\beta) &= 2\pi i \alpha + 2\pi i n\beta
\end{align*}
\]

for all \( n \). So to show that \( |Y(n\alpha + n\beta) - p| < \epsilon \) for some \( n \), we need only consider the second coordinates, namely,

\[
|e^{2\pi i (\alpha + n\beta)} - e^{2\pi i \beta}| < \epsilon.
\]

**Lemma** The sequence \( \{\alpha n : n \in \mathbb{Z}\} \) is dense in \( \mathbb{R}/\mathbb{Z} \).

**Proof.** See APPENDIX. In this part, we can use a result from the work on equidistribution. Thus here.

By the lemma, there exists \( n_0 \in \mathbb{Z} \) such that

\[
|\alpha n_0 - (\beta - \alpha_0)| < \epsilon,
\]

where \( \epsilon \) is small enough to imply

\[
\left| e^{2\pi i \alpha} - e^{2\pi i (\beta - \alpha_0)} \right| < \frac{\epsilon}{e^{2\pi i \alpha}}
\]

\[
\Rightarrow \quad \left| e^{2\pi i \alpha} < e^{2\pi i (\beta - \alpha_0)} + \frac{\epsilon}{e^{2\pi i \alpha}} \right| \quad (*)
\]

We can now see that

\[
\frac{1}{2}
\]

\[
\begin{align*}
|e^{2\pi i (\alpha + n\beta)} - e^{2\pi i \beta}| &= |e^{2\pi i \alpha} e^{2\pi i n\beta} - e^{2\pi i \beta}| \\
&= |e^{2\pi i \alpha} - e^{2\pi i \beta}| + |e^{2\pi i n\beta}| \\
&= |e^{2\pi i \alpha} - e^{2\pi i \beta}| + 1 \\
&= |e| = \epsilon.
\end{align*}
\]

This means \( |Y(n\alpha + n\beta) - p| < \epsilon \) and so \( p \) is a limit point of \( \{Y(n\alpha + n\beta) : n \in \mathbb{Z}\} \subset Y(\mathbb{R}) \). Since \( p \) was arbitrary, \( Y(\mathbb{R}) \) is dense in \( \mathbb{T}^2 \).
Completed prior to the change in problem statement.

4. Facts from point-set topology (see Appendix):
   (a) For any map \( f: M \to N \) and any \( A \subseteq N \),
       \[ f^{-1}(A) \subseteq f^{-1}(N) \leq A. \]
   (b) For any map \( f: M \to N \) and any \( B \subseteq M \),
       \[ B \subseteq f^{-1}(f(B)). \]
   (c) Any closed subset of a compact set is compact.

We say that a map \( f: M \to N \) is proper if inverse images of compact sets are compact. We say that an open set \( U \) is a neighborhood of infinity if \( U \) is not compact.

\[ \Rightarrow \text{Let } U = (-\infty, 1), V = (0, \infty). \text{ Then } \overline{U} = (-\infty, 1], \overline{V} = [0, \infty). \]

Proposition: A map \( f: M \to N \) is proper if and only if \( f^{-1}(f(B)) \) is compact in \( M \).

Proof: \((\Rightarrow)\) Suppose to the contrary that \( f \) is proper and there exists a neighborhood of infinity \( B \subseteq M \) such that \( f(B) \subseteq N \) is not a neighborhood of infinity. By definition, \( f(B) \) is compact. Since \( f \) is proper, \( f^{-1}(f(B)) \) is compact in \( M \).

By fact (b) and the fact that \( f(B) \subseteq f(B) \), we have

\[ B \subseteq f^{-1}(f(B)) \subseteq f^{-1}(f(B)). \]

and since \( B \) is the smallest closed set containing \( B \), also

\[ B \subseteq f^{-1}(f(B)). \]

Hence fact (c) implies \( B \) is compact. So by definition, \( B \) is not a neighborhood of infinity — a contradiction.

Therefore, a proper map takes neighborhoods of infinity to neighborhoods of infinity. Then neither \( U \) nor \( V \) is compact in the usual sense. i.e., both \( U \) and \( V \) are neighborhoods of infinity. Then \( U \cap V \) is also a neighborhood of infinity. So, \( (0, 1) = U \cap V \) is a nbhd. of \( \infty \).
(⇐) Contrapositive. Suppose there exists a compact \( A \subseteq N \) such that \( f^{-1}(A) \subseteq M \) is not compact. Since \( A \) is compact and \( N \) is Hausdorff, we know \( A \) is closed and so by continuity \( f^{-1}(A) \) is closed. This means \( f^{-1}(A) = \overline{f^{-1}(A)} \).

Since \( f^{-1}(A) \) is not compact, so also \( \overline{f^{-1}(A)} \) is not compact. By definition, then, \( f^{-1}(A) \) is a neighborhood of infinity in \( M \). Now, fact (a) gives

\[
f(f^{-1}(A)) \subseteq A
\]

\[
\Rightarrow \quad f(\overline{f^{-1}(A)}) \subseteq \overline{A}.
\]

We noted above that \( A \) is closed, so \( \overline{A} = A \) and thus \( \overline{A} \) is compact because \( A \) is. So by fact (c) and (\(*\)), \( f(\overline{f^{-1}(A)}) \) is compact and this means \( f(f^{-1}(A)) \) is not a neighborhood of infinity in \( N \). Therefore \( f \) does not take neighborhoods of infinity to neighborhoods of infinity. \( \blacksquare \)

Ex. A smooth map that is not proper.

\[
f: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \sin x
\]

This map is smooth by the remarks from class (9/5). However, the set \([ -1, 1 ]\) in the image of \( f \) is compact while \( \overline{f([ -1, 1 ])} = \mathbb{R} \) is not compact. So \( f \) is not a proper map.
5) \( F: \mathbb{R}^4 \rightarrow \mathbb{R}^2, (x,y,s,t) \rightarrow (x^2+y, x^2+y^2+s^2+t^2+y) \)

We will show that \((0,1)\) is a regular value of \( F \); that is, for all \( p \in F^{-1}(0,1) \) the map \( F_p: T_p \mathbb{R}^4 \rightarrow T_{(0,1)} \mathbb{R}^2 \) is onto.

We calculate the matrix for \( F \) to be:

\[
\begin{bmatrix}
2x & 1 & 0 & 0 \\
2y & 2y+1 & 2s & 2t
\end{bmatrix}
\]

Let \( p = (x,y,s,t) \in F^{-1}(0,1) \). Now, the image of \( F_p \) will be the span of the images of the basis elements of \( T_p \mathbb{R}^4 \). Using the standard basis,

\[
\text{span}\{F_p(1,0,0,0)\} = \text{span}\{(2x, 2x)\} = \text{span}\{(1,1)\}
\]
\[
\text{span}\{F_p(0,1,0,0)\} = \text{span}\{(1, 2y+1)\}
\]
\[
\text{span}\{F_p(0,0,1,0)\} = \text{span}\{(0, 2s)\}
\]
\[
\text{span}\{F_p(0,0,0,1)\} = \text{span}\{(0, 2t)\}
\]

So, the image \( S \) of \( F_p \) is \( \text{span}\{(1,1), (1,2y), (0,2s), (0,2t)\} \).

If \( s = t = 0 \), then we have \( \text{span}\{(1,1), (1,2y)\} \) and \( y^2+s^2+t^2=1 \) implies \( y = \pm 1 \). Thus \((1,1)\) and \((1,2y)\) are linearly independent, \( \forall y \), on the other hand, \( s \neq 0 \) or \( t \neq 0 \) then \((0,2s) \neq (0,0)\) or \((0,2t) \neq (0,0)\), respectively. This means that \((1,1)\) and at least one of \((0,2s)\) and \((0,2t)\) are linearly independent.

In either case, thus we have two linearly independent vectors in the image of \( F_p \) and so \( F_p(\mathbb{R}^4) \) spans two-dimensions. This means the image of \( F_p \) is necessarily all of \( T_{(0,1)} \mathbb{R}^2 \). Therefore \( F_p \) is onto and \((0,1)\) is a regular value of \( F \).
Define \( \Phi : F^1(0,1) \to S^2 \subset \mathbb{R}^3 \) by

\[
\Phi(x,y,z,t) = \begin{cases} 
(x^2, y, z) & \text{if } x \geq 0 \\
(-x^2, y, z) & \text{if } x < 0.
\end{cases}
\]

We will check that \( \Phi \) is well-defined. Note that since \((x,y,z,t) \in F^1(0,1)\), we have \(x^2 + y = 0\) and \(x^2 + y^2 + z^2 + t^2 = 1\). Using the Pythagorean theorem in the latter we see that \(y^2 + z^2 + t^2 = 1\).

But \(y^2 = x^4\). So

\[
(x^2)^2 + z^2 + t^2 = (x^4)^2 + z^2 + t^2 = 1
\]

and thus \(\Phi(x,y,z,t) \in S^2 \subset \mathbb{R}^3\). So \(\Phi\) is well-defined.

We also know that \(\Phi\) is bijective because an inverse exists, namely

\[
\Phi^{-1}(x,y,z) = \begin{cases} 
(\sqrt{x}, y, z) & \text{if } x \geq 0 \\
(-\sqrt{x}, y, z) & \text{if } x < 0.
\end{cases}
\]

Furthermore, \(\Phi\) and \(\Phi^{-1}\) are continuous. This is clear on each of the maps' pieces, and note also that for \((0,0,z,t) \in F^1(0,1)\) we have \((0,0,z,t) = (0,0,z,t)\) and for \((0,y,z) \in S^2\) we have \((0,0,y,z) = (0,0,y,z)\). So they are continuous everywhere.

Thus \(\Phi\) is a homeomorphism between \(F^1(0,1) \subset \mathbb{R}^4\) and \(S^2 \subset \mathbb{R}^3\). Since we are dealing with two-dimensional smooth manifolds, homeomorphic spaces are necessarily diffeomorphic.

In dim 2 or 3, homeo. manifolds automatically differ. But this isn't true for dimensions \(\geq 4\)!

\[\text{Ask Prof. Parker about this!}\]

or if he is busy, TA can work with you.
(ii) Suppose \( t = \frac{1}{27} \).

Recall that \( F^\prime(t) = \{(x,y) \in \mathbb{R}^2 | x^2 + xy + y^3 = 0\} \),

which is depicted below. 

Then \( F^\prime(\frac{1}{27}) = \{(0,0)\} \subseteq \mathbb{R}^2 \),

which is depicted below.

We can see clearly from the point \((0,0)\) that this is not an embedded submanifold.

The presence of the singleton \((\frac{1}{3}, -\frac{1}{3})\) means this is not an embedded submanifold of different dimension.
Does the converse of the preimage theorem hold? That is, if $F: M \to N$ is smooth, $c \in N$, and $F^{-1}(c)$ is an embedded submanifold of dimension $m - n$, then is $c$ necessarily a regular value of $F$?

No. Consider the following counterexample.

Let $M$ be a 2-torus and let $N = \mathbb{R}$. Define $F$ to be the particular height function displayed below, which is clearly smooth.

![Diagram](attachment:image.png)

If $h$ is the height of the torus (the maximum output of $F$), then $F^{-1}(h)$ is the circle on the top of the torus. This is obviously an embedded submanifold ($S^1 \subset \mathbb{T}^2$) and it is of the appropriate dimension.

However, the tangent space for points in $F^{-1}(h)$ is "horizontal" in our situation, and so does not map onto $T_h \mathbb{R}$. Said another way, all derivatives at points in $F^{-1}(h)$ are 0, so $F_{\ast}$ cannot be an onto map. Therefore, $h$ is not a regular value of $F$. $\sqrt{\text{Good!}}$

(The same argument applies for $F^{-1}(0)$ assuming 0 is the "bottom" of the torus.)
8) Let $S$ be a closed embedded submanifold of $N$. Then there exists an immersion $\sigma : S \to N$ that is a homeomorphism onto its image, let $n = \dim N$ and $k = \dim S$ ($k \leq n$).

$(\forall) \text{ If } f \in C^\infty(S), \text{ then } F \text{ is the restriction of some } \overline{f} \in C^\infty(N).$

For each $p \in S$, let $U_p$ be an open neighborhood of $p$ in $S$. Then $\{U_p : p \in S\}$ is an open cover of $S$ and $\{\sigma(U_p) : p \in S\}$ is an open cover of $\sigma(S)$ in the subspace topology but not open in the topology of $N$. Then let $B_p(\varepsilon)$ be an $(n-k)$-dimensional ball centered at $p$ so that $\{\sigma(U_p) \times B_p(\varepsilon) : p \in S\}$ forms an open cover of $\sigma(S) \subset N$. Since $S$ is closed we know $N \setminus S$ is open and hence $\bigcup_{p \in S} \sigma(U_p) \times B_p(\varepsilon) \subset N \setminus S$ is an open cover of $N$. (called a slice chart of $S$ in $N$.)

By theorem 13.3 there exists a partition of unity $\{\psi_p : N \to \mathbb{R}\}$ subordinate to the open cover above.

For each $p \in S$ we choose a smooth function $\psi_p : \sigma(U_p) \times B_p(\varepsilon) \to \mathbb{R}$ such that $\psi_p$ agrees with $f \circ \sigma^{-1}$ on $\sigma(U_p)$. Define $\psi_p \psi_p$ to be zero on $N \setminus \text{supp } \psi_p$. Then $\psi_p \psi_p$ is defined on all of $N$. Now define

$$F : N \to \mathbb{R}, x \mapsto \sum_{p \in S} \psi_p(x) \psi_p(x).$$

Since the supports of the $\psi_p$ are locally finite, we have the sum defined for all $p$. And $F$ is smooth because it is the sum/product of smooth functions. Moreover, for $x \in \sigma(S)$ we have $F_p(x) = f(\sigma^{-1}(x))$ for all $p \in S$ by construction. So for $x \in \sigma(S)$

$$F(x) = \sum \psi_p(x) \psi_p(x) = \sum \psi_p(x) f(\sigma^{-1}(x)) = (\sum \psi_p(x)) f(\sigma^{-1}(x)) = f(\sigma^{-1}(x)).$$

Thus $F$ restricted to $\sigma(S) \subset N$ is equivalent to $f$ on the preimage of $\sigma(S)$, that is, $S$. 
(b) If \( X \in \mathcal{I}(S) \), then there is \( Y \in \mathcal{I}(N) \) such that \( X = Y |_S \).

Now we leave out \( \sigma \) and just think of \( S \subset N \). Using the cover
\[ \mathcal{U} = \mathcal{B} \] (c) from part (a), let \( \chi_x \) be a vector field that agrees with \( X \) on \( U_x \). Then
\[ \sqrt{Y} = \sum_{x \in F_x} \chi_x \]

is a global vector field that, when restricted to \( S \), is equal to \( X \). The verification of this is similar to the verification in part (a).

(c) Parts (a) & (b) fail when the closed condition on \( S \) is omitted.

Consider \( (-\pi/2, \pi/2) < \mathbb{R} \) which is certainly not closed but is an embedded submanifold. Then the function
\[ f : (-\pi/2, \pi/2) \to \mathbb{R}, \theta \to \tan \theta \] cannot be extended continuously and therefore cannot be extended smoothly to all of \( \mathbb{R} \).

Also, the vector field on \( (-\pi/2, \pi/2) \) defined by
\[ \theta \mapsto \frac{1}{\theta - \pi/2} \] cannot be extended to include \( \pi/2 \) and thus cannot be extended to a vector field on all of \( \mathbb{R} \).
APPENDIX

4. (a) Let \( y \in f(f^{-1}(A)) \subseteq N \). Then there exists some \( x \in f^{-1}(A) \subseteq M \) such that \( f(x) = y \). By definition of \( f^{-1}(A) \), the image of \( x \) under \( f \) is in \( A \). So \( f(x) = y \in A \).

(b) Let \( x \in B \subseteq M \). Then certainly \( f(x) \in f(B) \). Now simply observe that \( x \in f^{-1}(f(x)) \subseteq f^{-1}(f(B)) \).

(c) This is Lemma A.17(c) of the text.

3. Proof of Lemma. Let \( x \in \mathbb{R}/\mathbb{Z} \). Since \( \{n \in \mathbb{Z} : x - n \in \mathbb{Q} \} \) is an infinite subset of the compact \( \mathbb{R}/\mathbb{Z} \), we know \( \{n \in \mathbb{Z} : x - n \in \mathbb{Q} \} \) has a limit point \( z_0 \in \mathbb{R}/\mathbb{Z} \). Fix \( \varepsilon > 0 \). Then there exist distinct integers \( n_1, n_2 \) such that \( |cn_n - z_0| < \varepsilon/4 \) and \( |cn_2 - z_0| < \varepsilon/4 \). Thus \( |c(n_1 - n_2) - 0| = |cn_2 - cn_1| < \varepsilon/2 \). (This is similar to Example 7.3, p. 157)

Since \( \mathbb{Q}/\mathbb{Z} \) is dense in \( \mathbb{R}/\mathbb{Z} \), we have a \( p/q \) with \( p, q \in \mathbb{Z} \) such that \( |x - p/q| < \varepsilon/2 \). Also, choose \( k \in \mathbb{N} \) such that \( |1/k| < \varepsilon/2 \). Then we have

\[
|c(n_k - n_2) - c| = |c(n_k - n_2) - 0 - c| < \varepsilon.
\]

This implies

\[
|kpc(n_k - n_2)| = |kpc(n_k - n_2) - kpc| < kpe.
\]

We now put this all together to see that for \( kpc(n_k - n_2) \in \mathbb{Z} \),

\[
|x - kpc(n_k - n_2)| = |x - p/q + kpc(n_k - n_2) - kpc| < \varepsilon + kpe.
\]

Since \( \varepsilon \) was arbitrary, we conclude that \( x \in \mathbb{R}/\mathbb{Z} \) is a limit point of \( \{n \in \mathbb{Z} : x - n \in \mathbb{Q} \} \) and thus the sequence is dense. \( \Box \)

But your choice of both \( k \) and \( p \) depended on \( \varepsilon \) so not clear if you can make \( kpe = k(\varepsilon)p(\varepsilon) - \varepsilon \) arbitrarily small.