1) Proposition. Let $\mathcal{D} \subset TM$ be a smooth distribution, and let $\mathcal{D}(M) \subset \mathcal{Y}(M)$ denote the space of smooth global sections of $\mathcal{D}$. Then $\mathcal{D}$ is involutive if and only if $\mathcal{D}(M)$ is a Lie subalgebra of $\mathcal{Y}(M)$.

Proof. $(\Rightarrow)$ By definition, the linear subspace $\mathcal{D}(M)$ is a Lie subalgebra of $\mathcal{Y}(M)$ if it is closed under brackets. Since $\mathcal{D}$ is involutive, the bracket of any two smooth sections of $\mathcal{D}$ is itself a section of $\mathcal{D}$. So $\mathcal{D}(M)$ is closed under brackets.

$(\Leftarrow)$ By definition, $\mathcal{D}$ is involutive if the bracket of any two sections of $\mathcal{D}$ is also a section of $\mathcal{D}$.

Since $\mathcal{D}(M)$ is a Lie subalgebra it is closed under brackets. This completes the proof. \qed

Let $U \subset \mathbb{R}^3$ be the subset where all coordinates are positive and let $\mathcal{D}$ be the distribution spanned by the vector fields

$$X = y \frac{\partial}{\partial x} - z \frac{\partial}{\partial y} \quad \text{and} \quad Y = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}.$$ 

Note that

$$[X, Y] = X(z) \frac{\partial}{\partial x} + X(-z) \frac{\partial}{\partial z} - Y(z) \frac{\partial}{\partial y} - Y(-z) \frac{\partial}{\partial y} = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$$

$$= y \frac{\partial}{\partial x} - \frac{xy}{z} \frac{\partial}{\partial z} + \frac{xy}{z} \frac{\partial}{\partial y} - x \frac{\partial}{\partial y} = (\frac{x}{z}) Y + (\frac{y}{z}) X$$

so $[X, Y] \in \mathcal{D}$ and $\mathcal{D}$ is involutive. Thus by the Frobenius theorem there is a flat chart for $\mathcal{D}$ on $U$. Let us find one now.
We define vector fields
\[ V = \frac{1}{z} \quad Y = \frac{2}{\partial x} - \frac{x}{z} \frac{2}{\partial z} \]
\[ W = \frac{1}{z} \quad X = \frac{2}{\partial y} - \frac{y}{z} \frac{2}{\partial z} \]
which also span \( D \). To determine the flow of \( V \)
we look at an integral curve \( V : R \rightarrow R^3 \) of \( V \), so
\( \gamma(t) = (x(t), y(t), z(t)) \). Since \( \dot{V}(t) = V(t) \) we know
\[ x'(t) \frac{\partial}{\partial x}(\ln y(t)) + y'(t) \frac{\partial}{\partial y}(\ln y(t)) + z'(t) \frac{\partial}{\partial z}(\ln y(t)) = \frac{2}{\partial x}(\ln y(t)) - \frac{x(t)}{z(t)} \frac{2}{\partial z}(\ln y(t)) \]
This gives us the system of ordinary differential equations
\[ x'(t) = 1 \quad \text{with solutions} \quad x(t) = t + a \]
\[ y'(t) = 0 \quad \quad \quad \Rightarrow \quad y(t) = b \]
\[ z'(t) = \frac{-x(t)}{z(t)} \quad \Rightarrow \quad z(t) = \sqrt{t^2 - 2at + c} \]
for arbitrary constants \( a, b, c \). Thus \( V \mapsto \alpha_t \) where
\( \alpha_t(x, y, z) = (t + x, y, \sqrt{t^2 - 2at + z^2}) \). For \( W \) the work
is similar with \( x \) and \( y \) interchanged, so \( W \mapsto \beta_t \) where
\( \beta_t(x, y, z) = (x, y + t, \sqrt{t^2 - 2yt + z^2}) \). We can now define the
inverse \( \psi \) of our coordinate map by flowing as follows:
Since \( [V, W] = 0 \)
\[ \frac{\partial}{\partial u} \psi = 0 \]
the order of \( \psi \) and \( \beta_t \) doesn't matter.

To get our coordinates for the filet chart we
invent the map \( (x, y, z) = \psi(u, v, w) = (u, \sqrt{w^2 - u^2 - v^2}) \);
that is,
\[ (u, v, w) = \psi^- (x, y, z) = (x, y, \sqrt{x^2 + y^2 + z^2}) \]
\( \beta_t \) and \( \alpha_t \) are (local) flows defined in \( U = \{(x, y, z) \in R^3 : x > 0 \} \)
But, \( (0, 0, w) \not\in U \). Try \((x, y, w)\) for some fixed \( x, y, z > 0 \).
And then, to see if your solution is correct, check:
\[ \frac{\partial y}{\partial x} = \psi_x (\frac{\partial y}{\partial x}) = \sqrt{2} \quad \text{and} \quad \frac{\partial y}{\partial x} = \psi_x (\frac{\partial y}{\partial x}) = W. \]
3) Proposition. If $F: M \to N$ is a submersion, then the connected components of the level sets of $F$ foliate $M$.

Proof. Since $F$ is a well-defined function, the level sets comprise all of $M$ and are disjoint. Furthermore, since $F$ is a submersion, $F_p$ is surjective at all points in $M$ so every level set is a regular level set. By the preimage theorem, every level set of $F$ is an immersed submanifold $V$ of $M$.

By the notes above, for each $p \in M$ there exists a unique immersed submanifold $L_p \subset M$, namely, the level set containing $p$. We can define a distribution $\mathcal{D}$ on $M$ by $\mathcal{D}_p = T_p \Gamma_p$ for $p \in M$. Then by construction the integral manifolds of $\mathcal{D}$ are precisely the level sets of $F$. Since each $p \in M$ is contained in a level set, each $p \in M$ is contained in an integral manifold of $\mathcal{D}$. Hence $\mathcal{D}$ is an integrable distribution. Every integrable distribution is involutive (Prop 19.3, pg 497) so $\mathcal{D}$ is involutive.

Therefore by the global Frobenius theorem and the connectedness of the $L_p$'s we have the desired foliation of $M$. $\square$
Let $V$ be a vector space with dual space $V^*$. We wish to show that in finite-dimensional cases, $V$ is isomorphic to $V^{**}$ via an explicit map.

(a) We can define a 'natural' map from $V$ to the second dual space $V^{**}$ without using any basis. Define

$$
\phi : V \rightarrow V^{**} \quad X \mapsto \phi(X)
$$

where $\phi(X) : V^* \rightarrow \mathbb{R}$ is defined by $\phi(X)(\omega) = \omega(X)$ for $\omega \in V^*$.

(b) To show that $\phi$ is injective we will show that it has a trivial kernel. Let $X \in V$ be a non-zero vector. Then there exists a basis of $V$ with $X$ as the first vector, say $(X, E_1, \ldots, E_n, \ldots)$. Let $(e_1^*, \ldots, e_n^*)$ be the dual basis for $V^*$. Note that the basis representation of $X$ under the above basis for $V$ is $(1, 0, \ldots, 0, \ldots)$. So we can compute

$$
\phi(X)(e_1^*) = e_1^*(X) = X^* \delta_1 = 1 \neq 0.
$$

Therefore $X \not\in \ker \phi$ and $\phi$ is injective.

(c) If $V$ is finite-dimensional then $V$ and $V^{**}$ have the same dimension (Prop 6.1 on p. 125) and part (b) implies $\phi$ is bijective. For linearity, observe that for $\omega \in V^*$, $X, Y \in V$ scalars $a, b$ we have

$$
\phi(aX + bY)(\omega) = \omega(aX + bY) = a \omega(X) + b \omega(Y) = a \phi(X)(\omega) + b \phi(Y)(\omega)
$$

because $\omega$ is linear by the definition of $V^*$. Therefore $\phi$ is an isomorphism.
5) Let \( M \) be a smooth manifold and let \( f, g \in C^\infty(M) \).

\( \sqrt{a} \) For \( X_p \in T_pM \) and constants \( a,b \),

\[
d(af + bg)_p(X_p) = X_p(af + bg) = X_p(af) + X_p(bg) = aX_p(f) + bX_p(g) = a df_p(X_p) + b dg_p(X_p),
\]

so \( d(af + bg) = a df + b dg \).

\( \sqrt{b} \) Using the product rule for the derivation \( X_p \),

\[
d((fg)_p(X_p) = X_p((fg)) = f(X_p)g(X_p) + g(X_p)f(X_p) = f g(X_p) + g(X_p) df_p(X_p),
\]

so \( d(fg) = fdg + gd f \).

\( c \) We will use part (b) freely. Observe

\[
d\left( \frac{f}{g} \right) = d\left( \frac{f}{g} \cdot \frac{1}{g} \right) = \frac{1}{g} d\left( \frac{f}{g} \right) + \frac{1}{g} d\left( \frac{1}{g} \right) = \frac{\partial f}{\partial g} g^2 d\frac{1}{g}.
\]

\( \sqrt{c} \) Also,

\[
d\left( \frac{1}{g} \right) = d\left( g \cdot \frac{1}{g^2} \right) = g d\left( \frac{1}{g^2} \right) + \frac{1}{g^2} dg = 2 \frac{d}{g^2} + \frac{d g}{g^2}
\]

\[
\implies d\left( \frac{1}{g} \right) = \frac{-d g}{g^2}.
\]

Now we can see that

\[
d\left( \frac{f}{g} \right) = d\left( \frac{f}{g} \cdot \frac{g}{g} \right) = \frac{1}{g} df + f d\left( \frac{1}{g} \right) = \frac{df}{g} + f \left( \frac{-d g}{g^2} \right)
\]

\[
= g \frac{df}{g^2} - f \frac{dg}{g^2} = g df - f \frac{dg}{g^2}.
\]
(d) Let \(J \subset \mathbb{R}\) contain the image of \(f\) and let \(h : J \to \mathbb{R}\) be smooth. Using displayed equation 6.10 on page 123 we have for coordinates \(\{x^i\}\) on \(M\)
\[
\begin{align*}
    d(h \circ f) &= \sum_i \frac{\partial (h \circ f)}{\partial x^i} \, dx^i \\
    &= \sum_i \left( \frac{h' \circ f}{x^i} \right) \, dx^i \\
    &= (h' \circ f) \sum_i \frac{\partial f}{\partial x^i} \, dx^i \\
    &= (h' \circ f) \, df.
\end{align*}
\]

(e) Let \(f\) be a constant function. Then by lemma 3.4 (pg 105), \(X_f = 0\) for any \(p \in M\) and any \(X \in \mathfrak{X}(M)\).

Thus
\[
(d_f)_p (X_p) = X_p f = 0,
\]

so \(df = 0\).

\[
\frac{1}{2} \, a (x, y)
\]

In each case below we will compute the coordinate representation of \(df\) and determine the set of points \(p \in M\) for which \(df_p = 0\).

(a) \(M = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}\) \(f(x,y) = \frac{x}{x^2 + y^2}\) Standard coords,
\[
\begin{align*}
    df &= \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy \\
    &= \frac{x - 2x^2}{(x^2 + y^2)^2} \, dx + \frac{2xy}{(x^2 + y^2)^2} \, dy \\
    &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \, dx + \frac{2xy}{(x^2 + y^2)^2} \, dy. \quad \checkmark
\end{align*}
\]
For what $p \in M$ is $d\mathbf{f}_p = 0$? If $d\mathbf{f}_p = 0$ then $y^2 - x^2 = 0$ and $2xy = 0$. This latter equation implies $y = 0$ (because $x \neq 0$). But then $y = 0$ and so the former equation implies $x = 0$, which is impossible since $x > 0$. Thus for no $p \in M$ is $d\mathbf{f}_p = 0$.

(b) Same as (a) but in polar coordinates.

Since $f(x, y) = \frac{x}{x^2 + y^2}$ we know $f(r, \theta) = \frac{r \cos \theta}{r^2} = \cos \theta \frac{1}{r}$. [Note that $r > 0$ by def of $M$.] Thus

$$d\mathbf{f} = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta$$

$$= \frac{-\cos \theta}{r^2} dr + \frac{-\sin \theta}{r} d\theta.$$  

This is zero if and only if $\cos \theta = \sin \theta = 0$, which never occurs. So $d\mathbf{f}$ is never zero, which agrees with part (a).

(c) $M = S^2 \subset \mathbb{R}^3$  

$f(p) = z$-coord of $p$  

stereographic coords.

Let $(u_1, u_2)$ be the stereographic coordinates of $S^2$. Recall from HW that $S^2 \subset \mathbb{R}^3$ maps these coordinates into the standard coordinates of $S^2$.

Thus, for any $(u_1, u_2) \in S^2$

$$f(u_1, u_2) = \frac{|u|^2 - 1}{|u|^2 + 1}$$

$$= \frac{u_1^2 + u_2^2 - 1}{u_1^2 + u_2^2 + 1}.$$
Now we can compute
\[ df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 \]
\[ = \left( \frac{2u_1(u_1^2+u_2^2+1) - 2u_1(u_1^2+u_2^2-1)}{(lu_1^2 + 1)^2} \right) du_1 + \left( \frac{2u_2(u_1^2+u_2^2-1) - 2u_1(u_1^2+u_2^2-1)}{(lu_1^2 + 1)^2} \right) du_2 \]
\[ = \frac{4u_1}{(lu_1^2+1)^2} du_1 + \frac{4u_2}{(lu_1^2+1)^2} du_2. \]

This is only zero if \( 4u_1 = 4u_2 = 0 \), that is, if \( u_1 = u_2 = 0 \).

The only points of \( S^2 \setminus \{\eta_1\} \) that stereographically project to \( (0,0) \) are the north and south poles, \( N \) and \( S \) resp.

Thus \(\text{df}_p = 0 \).

(d) \( M = \mathbb{R}^n \), \( f(x) = |x|^2 \) standard coords.

For \( (x_1, \ldots, x_n) \in \mathbb{R}^n \), \( f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^2 \). Hence by the power rule
\[ df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \]
\[ = \sum_{i=1}^n 2x_i \, dx_i. \]

This is only zero when \( x_i = 0 \) for all \( i \), so at the origin, this is zero.

For \( c \), \( f: S^2 \to \mathbb{R} \) is given by \( f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 = 1 \).

You chose \( \sigma: S^2 \setminus \{\eta_1\} \to \mathbb{R}^2 \) so that
\[ \sigma(u_1, u_2) = \left( \frac{u_1}{\sqrt{1+u_1^2+u_2^2}} \right), \]
and then showed
\[ df_0 = 0 \Rightarrow (u_1, u_2) = (0, 0). \]

For \( c \), \( f(0, 0) = 1 \), so \( \sigma \) does not map to \( \mathbb{R}^2 \) at \( (0, 0) \).

You need to use different charts of \( S^2 \setminus \{\eta_1\} \to \mathbb{R}^2 \) to conclude: \( f \circ \sigma^{-1} \left( df_0 = 0 \right) = f \circ \sigma^{-1} \left( 0 \right) = 0 \).
Let $\phi: \mathbb{R}^3 \to \mathbb{R}^3$ be the map $\phi(x, y, z) = (u, v, w) = (x e^y, x^2 y z, z \sin x)$. Let $\xi$ and $\eta$ be the 1-forms $\xi = \omega \, dv + u^2 \, dw$ and $\eta = \sqrt{v} \, du + u^3 \, dv + du$, we will compute the pullbacks $\phi^* \xi$ and $\phi^* \eta$.

Using displayed equation 6.15 on page 137,

\begin{align*}
\phi^* \xi &= (u \circ \phi) \, d(v \circ \phi) + (u^2 \circ \phi) \, d(w \circ \phi) \\
&= (z \sin x) \, d(x^2 y z^2) + (x^2 e^{y'}) \, d(z \sin x) \\
&= (z \sin x) \left( (x^2 y dz + x^2 dy) \, e^{y'} + (x^2 y') \, (z \cos x \, dx + \sin x \, dz) \right) \\
&= (z \sin x) \left( (3x^2 y \, dx + x^2 \, dy) \, e^{y'} + 2x^3 y \, dz \right) \\
&= 3x^2 y^2 \sin x \, dx + x^2 \sin x \, dy + 2x^3 y z \, dz \\
&= 3x^2 y^2 \sin x \, dx + x^2 \sin x \, dy + 2x^3 y z \, dz.
\end{align*}

\begin{align*}
\phi^* \eta &= (v \circ \phi) \, d(u \circ \phi) + (u^2 \circ \phi) \, d(w \circ \phi) + d(w \circ \phi) \\
&= (x^3 y^2) \, d(x e^{y'}) + (x e^{y'}) \, d(x^3 y^2) + d(z \sin x) \\
&= (x^3 y^2) \left( e^{y'} \, dx + x \, d(y') \right) + (x e^{y'}) \left( (3x^2 y \, dx + x^2 \, dy) \, e^{y'} + 2x^3 y \, dz \right) + (z \cos x \, dx + \sin x \, dz) \\
&= x^3 y^2 e^{y'} \, dx + x^2 y e^{y'} \, dy + z \cos x \, dz \\
&= x^3 y^2 e^{y'} \, dx + x^2 y e^{y'} \, dy + z \cos x \, dz \\
&= x^3 y^2 \sin x \, dx + x^2 \sin x \, dy + 2x^3 y z \, dz.
\end{align*}