9.1 (6)

Consider $X = \mathbb{R} - \{0\}$ with the subspace topology. This space has the property that given any $x \in X$ and any $N(x,p)$ there exists $N(x,q)$ such that the closure of $N(x,q)$ is a proper subset of $N(x,p)$. To demonstrate this property, let $x \in X$ and let $p > 0$. We have $N(x,p) \cap X$ an open neighborhood of $x$ in $X$. Put $q = \frac{1}{2} \cdot \min\{|x|, p\}$. Observe that $q > 0$ because $x \neq 0$ and $p > 0$. Then $N(x,q)$ is an open neighborhood of $x$ in $X$ whose closure is strictly contained in $N(x,p)$.

But $X$ is a disconnected space because

$$X = (-\infty, 0) \cup (0, \infty).$$

So we see that the property mentioned above does not imply connectedness. What happens if we alter the conditions on $U$ or $V$? The first inclination may be to require that a connected $V$ exists, but we see that the above remains a counterexample (which is why we used $q$ as we did, instead of simply $q = p/2$). However, we do have the following:

Let $X$ be a topological space with the property that given any $x \in X$ and any open neighborhood $U$ of $x$ there exists a neighborhood $V$ of $x$ such that $\overline{V} \subset U$ and $X - V$ is connected. Then $X$ is a connected space.

Suppose to the contrary that $X$ has the aforementioned property but is not connected, so $X = Y \cup Z$ where $Y$ and $Z$ are open, non-empty, disjoint sets. Let $x \in X$. Then $Y$ or $Z$ is a neighborhood of $x$. Assume without loss that $x \in Y$. By assumption, there exists a neighborhood $V$ of $x$ such that the closure of $V$ is a proper subset of $Y$. This implies $V$ is a proper subset of $Y$, so there exists some $y \in Y - V$. Now, we assume $X - V$ is connected and by Proposition 4 it must lie entirely in $Y$ or entirely in $Z$. Since $Z$ is in the complement of $V$ (because $V$ is in $Y$), it must be the case that $X - V$ is in $Z$. Consider the element $y$. We know $y \in Y$ so $y \notin Z$. But $y$ is in the complement of $V$ so $y \in Z$. This is a contradiction. Therefore, $X$ must be connected.
Let
\[ Y = \{(x, y) \mid x > 0, y = \sin(1/x)\} \cup \{(0, 0)\} \subset \mathbb{R}^2. \]

We will show that \( Y \) is not path connected.

Suppose to the contrary that \( Y \) is path connected. Then for any two points \( x, y \in Y \) there is a path in \( Y \) joining \( x \) and \( y \). In particular, there must be a path \( A \subset Y \) connecting \((0,0)\) to \((1,\sin(1))\). By definition of a path, there is a continuous function \( g \) from \([0,1]\) onto \( A \). Without loss of generality, let \( g(0) = (0,0) \) and let \( g(1) = (1,\sin(1)) \).

Note that we are in a Hausdorff space and \([0,1]\) is compact. Exercise 7.4.10 states that in this situation a function is continuous if and only if its graph is compact in the product space of the domain and range. In our situation, 7.4.10 (\( \Rightarrow \)) implies that the graph of \( g \), namely, \( A \), is compact because \( g \) is continuous. But this graph is identical to the graph of the function \( f \) from Exercise 7.4.5, so the graph of \( f \) is compact. Hence, by 7.4.10 (\( \Leftarrow \)) we have that \( f \) is continuous. This is a contradiction because \( f \) was proved discontinuous in 7.4.5. Therefore, no path exists from \((0,0)\) to \((1,\sin(1))\) and \( Y \) is not path connected.

We will show that given a compact \( A \subset \mathbb{R}^n \), \( n \geq 2 \), it is not necessarily the case that \( \mathbb{R}^n - A \) is connected.

Consider \( A \), the unit square in the space of \( \mathbb{R}^2 \) with vertices \((0,0), (0,1), (1,1)\) and \((1,0)\). We know \( A \) is compact for at least two reasons: first, because it is closed and bounded in Euclidean space, and second, because it is the union of four segments all homeomorphic to \([0,1]\) (a finite union of compact sets is compact by Exercise 7.3.8).

We see that \( \mathbb{R}^2 - A \) is not connected because it is the union of open, non-empty, disjoint sets (the space inside the square and the space outside the square).

**Proposition.** If \( X \) is connected, then the one-point compactification of \( X \) is also connected.

**Proof.** Let \( Y \) be the one-point compactification of \( X \). If \( Y = X \), then clearly \( Y \) is connected and we are done. So let \( Y = X \cup \{p\} \). (Note that this means \( X \) is not compact.)

Suppose to the contrary that \( Y \) is not connected; that is, there exist open, non-empty, disjoint sets \( U \) and \( V \) such that \( Y = U \cup V \). We assumed \( X \) to be connected, so by Proposition 4, \( X \subset U \) or \( X \subset V \). Without loss of generality, suppose \( X \subset U \). Since \( V \) is non-empty, it is necessarily the case that \( p \in V \) and in fact \( \{p\} = V \). But the complement of \( \{p\} \) is \( X \) which is not compact, so \( \{p\} \) is not open in the one-point compactification topology. This means \( V = \{p\} \) is not open—a contradiction. Therefore, \( Y \) is connected. \( \square \)
We shall determine whether each of the spaces below is connected or disconnected.

(a) \( X = \{(x, y) \mid y = 1/x\} \cup \{(x, y) \mid y = 0\} \subset \mathbb{R}^2 \). This space is disconnected by Proposition 10. Let \( S = \{(x, y) \mid y = 1/x\} \) and \( T = \{(x, y) \mid y = 0\} \). These are both closed in the plane so

\[ S \cap \overline{T} = \overline{S} \cap T = S \cap T = \emptyset. \]

The fact that \( X = S \cup T \) means \( X \) is disconnected.

(b) \( X = \{(x, y) \mid x^2 + y^2 < 1\} \cup \{(x, y) \mid x = 1\} \). This space is disconnected by Proposition 10. Let \( S \) be the disk and \( T \) the line. Note that they are closed and do not meet because \((0, 1)\) is excluded from the disk. Thus, \( X = S \cup T \) means \( X \) is disconnected.

(c) \( X \) is the set of all functions from \([0, 1]\) into itself with the least upper bound metric. This space is connected. Suppose to the contrary that \( X = U \cup V \) where \( U \) and \( V \) are open, non-empty, disjoint subsets of \( X \). Recall that open sets in this topology are regions formed by unions of \( p \)-collars whose elements (functions) lie entirely within the region. We claim that there must exist some \( x \in [0, 1] \) in the domain such that the regions of \( U \) and \( V \) do not overlap in \( x \times [0, 1] \). Suppose that the regions of \( U \) and \( V \) overlap above every \( x \) in the domain. Then we can construct a function \( h \) (using the Axiom of Choice) by choosing a function value for each \( x \) from the overlapping region above \( x \). This would place \( h \) entirely in the region of \( U \) and entirely in the region of \( V \), contradicting the fact that \( U \) and \( V \) are disjoint. Thus, there is some \( x \in [0, 1] \) such that the regions of \( U \) and \( V \) do not overlap above \( x \). This means \( U \) and \( V \) must partition \( x \times [0, 1] \). But \( x \times [0, 1] \) is homeomorphic to \([0, 1]\) and is connected, so it cannot be partitioned by open sets such as \( U \) and \( V \) (restricted to \( x \)). Therefore, \( X \) is connected.

(d) \( X = \mathbb{R}^2 \) with the point-line topology. This space is connected. Suppose to the contrary that \( X = U \cup V \) where \( U \) and \( V \) are open, non-empty, disjoint subsets of \( X \). Then clearly \( U \) is in the complement of \( V \) because they are disjoint. But by the definition of open sets, \( U \) contains infinitely many lines while \( V \) has finitely many lines in its complement. This is a contradiction. So \( X \) is connected.