2.4

(2) Let \((X,D)\) be a metric space and \((Y,D|Y)\) be a metric subspace of \(X\). Then the following statements are true.

a) A subset \(W\) of \(Y\) is an open set in \(Y\) if and only if \(W = Y \cap U\) for some open set \(U \subset X\).

b) A subset \(C\) of \(Y\) is closed in \(Y\) if and only if \(C = Y \cap F\), where \(F\) is a closed subset of \(X\).

c) If \(Y\) is an open subset of \(X\), then a subset of \(Y\) is open in \(Y\) if and only if it is open in \(X\).

d) If \(Y\) is a closed subset of \(X\), then a subset of \(Y\) is closed in \(Y\) if and only if it is closed in \(X\).

e) A subset of \(Y\) may be open or closed in \(Y\) without being open or closed in \(X\).

(5)

Proposition. Let \(\{F_i\}, i \in I\), be a family of closed subsets of a metric space \((X,D)\) with the property that given any \(x \in X\), there is \(p > 0\) such that \(N(x,p)\) intersects finitely many of the \(F_i\). Then \(\bigcup_i F_i\) is closed.

2.5

(1) The convergence of each of the following sequences will be addressed.

a) \(s_n = 1 + 1/n\) in \(\mathbb{R}\) with the absolute value metric

b) \(s_n = (2, 2)\) in \(\mathbb{R}^2\)

c) \(s_n = (2, n)\) in \(\mathbb{R}^2\) with the maximum metric

d) Let the sequence \(s_n\) be defined as

\[
s_n(x) = \begin{cases} 
0 & \text{if } x \leq 1/n, \\
1 & \text{if } x > 1/n,
\end{cases}
\]

in the space \((X,D)\) from Example 5 of Chapter 2 of the text. This sequence does not converge. The function that most resembles \(s_n(x)\) for very large numbers \(n\) is

\[
f(x) = \begin{cases} 
0 & \text{if } x = 0, \\
1 & \text{if } 0 < x \leq 1.
\end{cases}
\]

However, \(f(x)\) is not the limit of \(s_n(x)\), for let \(p = 1/2\) and consider \(x = 1/n\). Observe that \(f(1/n) = 1\) but \(s_n(1/n) = 0\) for all \(n \in \mathbb{N}\). This implies that \(f(x) \notin N(s_n(x), 1/2)\) because \(D(f(x), s_n(x)) = 1\).

e) The sequence \(s_n(x) = (1/n)x\), also in \((X,D)\) as above, converges to \(g(x) = 0\). Choose \(p > 0\) and let \(M = 1/p\). Then \(n > M\) implies \(1/n < 1/M = p\). Note that \(|s_n(x) - g(x)|\) achieves its maximum value when \(x = 1\). Since \((1/n)(1) < 0 + p\), we can see that \(s_n(1) < g(1) + p\) for all \(n > M\). Therefore, \(s_n(1) \in N(g(x), p)\) and \(s_n(x) \to g(x)\).

(3)

Proposition. Let \((X,D)\) be a metric space and \((Y,D|Y)\) be a subspace of \(X\). If some sequence \(s_n\) in \(Y\) converges to \(y \in Y\), then \(s_n\) considered as a sequence in \(X\) also converges to \(y\).
Proof. We assume \( s_n \) in \( Y \) converges to \( y \). This means, by Proposition 7 of the text, that every open set with \( y \) as an element contains all but finitely many of the \( s_n \). Let \( U \) be an open set in \( X \) such that \( y \in U \). It is impossible for \( U \) to be smaller than every open set in \( Y \) containing \( y \) because \( X \supset Y \). So \( U \) contains all but finitely many of the \( s_n \) considered in \( X \). Therefore, by the backward direction of Proposition 7, \( s_n \) converges to \( y \). □

Contrastingly, consider the following: let \( X = \mathbb{R} \) and \( Y = [0, 1) \), both spaces with the absolute value metric. Let \( s_n = 1 - 1/n \). This sequence, even though all of its terms are in \( Y \), does not have a limit in \( Y \). However, \( s_n \) in \( X \) does have a limit, namely 1.

(6) Let \( \mathcal{L} \) be the set of lines of \( \mathbb{R}^2 \). A sequence \( \{L_n\}, n \in \mathbb{N} \), in \( \mathcal{L} \) will be said to converge to a line \( L \) if there exist sequences \( s \) and \( t \) such that \( s_n \) and \( t_n \) are distinct points of \( L_n \) for each \( n \) and \( s_n \to x \in L \) and \( t_n \to y \in L \), \( x \neq y \). Suppose \( L_n \) has the equation \( a_nx + b_ny + c_n = 0 \) and \( L \) the equation \( ax + by + c = 0 \).

It is not the case that \( L_n \to L \) if and only if \( a_n \to a \), \( b_n \to b \), and \( c_n \to c \). This is because the backward direction does not guarantee that every equation in the sequence of equations corresponds to an element of \( \mathcal{L} \). Suppose \( a_n = 1/2 - 1/n \), \( b_n = 1/2 - 1/n \), and \( c_n = 1 \). Then \( L \) can be represented by \( x/2 + y/2 + 1 = 0 \) because \( a_n \to 1/2 \), \( b_n \to 1/2 \), and \( c_n \to 1 \), and \( L \) is clearly a line in the real plane. However, it is not the case that \( L_n \) converges to \( L \) because there do not exist sequences \( s \) and \( t \) whose elements are distinct points on \( L_n \) for every \( n \in \mathbb{N} \). This follows from the fact that \( L_2 \) is not a line at all since the equation \( 0x + 0y + 1 = 0 \) is inconsistent and no \( s_2 \) and \( t_2 \) can exist as distinct points on \( L_2 \).

2.6

(2) The continuity of each of the following functions will be addressed.

a) The function \( f(x) = 5x + 7 \) from the standard \( \mathbb{R} \)-space is continuous. This can be shown as follows. Let \( f(a) \in \mathbb{R} \) and \( p > 0 \), and set \( q = p/5 \). Then for any \( x \in N(a, q) \), \( f(x) \in N(f(a), p) \). We know this because \( p/5 > |x - a| \), which implies \( p > 5|x - a| = |5x - 5a| = |5x - 5a + 7 - 7| \). From here, it is clear to see that \( |(5x + 7) - (5a + 7)| < p \). Thus, \( D(f(x), f(a)) < p \) and \( f(x) \in N(f(a), p) \).

b) The function \( f(x, y) = x + y \) from \( (\mathbb{R}^2, \text{"taxi"}) \) to \( (\mathbb{R}, \text{"abs"}) \) is continuous. We use Proposition 9 to demonstrate this fact. Let \( a \in \mathbb{R} \) and \( p > 0 \). Then \( N(a, p) \in \mathbb{R} \) is equal to the open interval \( (a - p, a + p) \). Thus, \( f^{-1}(N(a, p)) = \{(x, y) \in \mathbb{R}^2 | a - p < x + y < a + p \} \), which is the area strictly between two parallel lines. Since this is an open set, \( f \) must be continuous.

c) Let \( f(g) = g(0) \) from the space \( (X, D) \) of Example 5 onto the closed interval \([0, 1]\) considered as a subspace of the standard \( \mathbb{R} \)-space. We know \( f \) is continuous by the definition of continuity. Let \( f(h) \) be any real number in \([0, 1]\), let \( p \) be a positive real number, and set \( q = p \). Choose \( j \in N(h, q) \); that is, \( j(x) \) is never more than \( q \) distant from \( h(x) \). This implies that the distance from \( h(0) \) to \( j(0) \) is less than \( q \) and therefore less than \( p \). Hence, \( f(j) \in N(f(h), p) \).

d) Let \( i \) be the identity function from \( (\mathbb{R}^2, \text{"taxi"}) \) onto \( (\mathbb{R}^2, \text{"max"}) \). This function is continuous by Proposition 8. Let \( U \) be an open set in \( (\mathbb{R}^2, \text{"max"}) \). Then \( i^{-1}(U) = i(U) = U \), now in \( (\mathbb{R}^2, \text{"taxi"}) \). But this is certainly open because any neighborhood with respect to the maximum metric contains the neighborhood with respect to the taxicab metric around the same point and with the same radius.

(3)

Proposition. Let \( f \) be a function from a metric space \((X, D)\) into the metric space \((Y, D')\) such that \( D(x, x') \geq kD'(f(x), f(x')) \), where \( k \) is a constant positive real number. Then \( f \) is continuous.

Proof. Let \( f(a) \) be an element of \( Y \) and let \( p \) be any positive number. We need to show the existence of \( q > 0 \) so that if \( x \in N(a, q) \), then \( f(x) \in N(f(a), p) \). To do this, set \( q = kp \) and choose \( x \in N(a, q) \). This means that \( kp > D(x, a) \). But \( D(x, a) \geq kD'(f(x), f(a)) \) from our assumption. Hence, \( kp > kD'(f(x), f(a)) \) and \( p > D'(f(x), f(a)) \). Therefore, \( f(x) \in N(f(a), p) \) and, by the definition of continuity, \( f \) is continuous. □

2
Proposition. Let \( f \) be a function from the metric space \((X, D)\) into the metric space \((Y, D')\). Then \( f \) is continuous if and only if given any convergent sequence \( S \) in \( X \), \( f(S) \) is a convergent sequence in \( Y \).

Proof. → We assume \( f \) is continuous, and Proposition 10 tells us that if \( S \) is a sequence in \( X \) that converges to \( y \), say, then \( f(S) \) converges to \( f(y) \) in \( Y \). This certainly implies the weaker formulation found in the proposition here: any convergent sequence in \( X \) has an image that converges in \( Y \).

← We assume that any sequence which converges in \( X \) must have an image that converges in \( Y \). We will prove that \( f \) is continuous by showing that the image of the sequence must converge specifically to the image of the limit from \( X \), thus reducing this direction to the backward direction of Proposition 10.

Let \( S \) be a sequence in \( X \) that converges to \( l \). Suppose \( f(S) \) does not converge to \( f(l) \), but instead to \( y_0 \in Y \). Consider the sequence \( T \) in \( X \) defined by

\[
t_n = \begin{cases} s_n & \text{if } n \text{ is odd,} \\ l & \text{if } n \text{ is even.}
\end{cases}
\]

Then \( T \) converges to \( l \) in \( X \), and by our assumption, \( f(T) \) must converge in \( Y \). But \( f(T) \) will oscillate between \( y_0 \) and \( f(l) \) in its limiting case, and therefore does not converge. This is a contradiction, which tells us that \( f(T) \) must converge to \( f(l) \) and, by Proposition 10, \( f \) is continuous. □

(9) Let \( f \) be a function from \( \mathbb{R}^2 \) into \( \mathbb{R}^2 \).

a) If \( f \) takes any circle onto a circle, is \( f \) necessarily continuous? That is the question. I worked longer on this problem than any other in the assignment, yet it is the furthest from a solution.

At first, I tried to disprove the claim by cleverly concocting a function that would map circles to circles while clearly being discontinuous. My attempt was a “function” where all circles not touching either axes in the real plane are mapped to themselves, while all circles touching an axes are mapped to the circle with the same radius centered at the origin. This “function” does map circles to circles and is not continuous, but it is only a function for circles, not for the actual points in the real plane - points may have more than one image.

Next, I tried to disprove the claim with an even cleverer function. I tried to develop a function that fixed all the points in the plane not on the unit circle, while mapping all the points from the unit circle to the unit circle in some discontinuous fashion. This attempt failed for two reasons: first, a circle intersecting the unit circle would have an image that was broken at a point, and second, I discovered that it was impossible to use a sequence to map the points on the circle because a sequence is at most countable while the points on the unit circle are uncountable.

So at this point, I became convinced that the claim was true. There would be no easy way out of this problem. I worked with Ngan and Chris on an idea where we think of all circles in the real plane as the frontier of a neighborhood. Our hope was that neighborhoods necessarily map to neighborhoods, thus taking their frontiers to frontiers. But we soon realized that this was essentially working the wrong direction of the problem. We also tried to construct a sequence of concentric circles converging at their center, but we did not know if the interior of a circle in the domain necessarily mapped to the interior of the circle’s image, though we did feel that the center of a circle must map to the center of the circle’s image.

Finally, I started a proof by contradiction using the negation of Proposition 9; that is, assume \( f \) maps circles to circles and there exists a \( p \)-neighborhood in the range for some \( p > 0 \) around some \((x, y)\) such that \( f^{-1} \) applied to this neighborhood is not open. From here, I felt that I had isolated two points \( f^{-1}(x, y) \) and a point on the boundary of \( f^{-1}(x, y) \) that makes it not open. From here, I hoped to find some third point that would form a circle in the domain but was collinear in the range. Alas, to no avail. I am convinced, though only through intuition, that this is true. When continuity is conceptualized as “preserving nearness” then we can think of three points - \( A, B, \) and \( C \) - on a circle. If we fix \( A \) and \( B \) and consider a point \( C' \) very near \( C \), the result will be a new circle very near the first circle. Consequently, \( f(A), f(B), \) and \( f(C') \) must lie on a circle very near the circle containing \( f(A), f(B), \) and \( f(C) \).
b) If $f$ takes collinear points into collinear points, is $f$ necessarily continuous? I have not solved this problem, but I began an approach that seems promising. Let $S$ be a sequence in the domain such that $s_n \to s$ where $s$ is a point in $\mathbb{R}^2$. Let $T$ be another sequence in the domain such that $t_n, n \in \mathbb{N}$, and let $t_n \to t$. This would mean that $s, t, and a$ are collinear. By assumption, I know that each $f(s_n), f(t_n), and f(a)$ are collinear. Therefore, the sequence of lines formed by $S$ and $T$ along with $a$ in the domain will map to some group of lines emanating from $f(a)$ in the range. Moreover, $f(s), f(t)$, and $f(a)$ must be collinear. But do $f(s_n)$ and $f(t_n)$ necessarily converge to $f(s)$ and $f(t)$, respectively? Do they converge at all? I was hoping to make this determination and, using some form of Proposition 10 or problem (7) above, show that $f$ is continuous.

2.7

(1) We will prove the following properties for closure sets.

a) $A \subset \overline{A}$. We begin with the observation that if $A = \phi$, then $\overline{A} = \phi$, and $\phi \subset \phi$. If $A$ is nonempty, then $D(a, A) = 0$ for all $a \in A$. Thus, $a \in A$ implies $a \in \overline{A}$ and we can conclude that $A \subset \overline{A}$.

b) $\overline{\overline{A}} = \overline{A}$. By Proposition 15, $\overline{\overline{A}}$ is the smallest closed set containing $\overline{A}$. But Proposition 14 tells us that $\overline{A}$ is closed, and it is obviously the smallest set that contains itself, so it must be the case that $\overline{\overline{A}} = \overline{A}$.

c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Let $x \in \overline{A \cup B}$. Then $D(x, A \cup B) = 0$, which implies $D(x, A) = 0$ or $D(x, B) = 0$. Therefore, $x \in \overline{A}$ or $x \in \overline{B}$, which means $x \in \overline{A} \cup \overline{B}$ and $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. On the other hand, let $y \in \overline{A \cup B}$. By definition, $D(y, A) = 0$ or $D(y, B) = 0$. Thus, $D(y, A \cup B) = 0$ and $y \in \overline{A \cup B}$. Since it is now clear that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$, both are subsets of each other and we know $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(3) Proposition. A subset $A$ of a metric space $(X, D)$ is closed if and only if $A = \overline{A}$.

PROOF. $\rightarrow$ Assume $A$ is closed. Then by Proposition 15, which states that the closure of a set is the smallest closed set containing the original set, $A = \overline{A}$.

$\leftarrow$ Assume $A = \overline{A}$. Then by Proposition 14, which states that any closure of a set is closed, $\overline{A}$ is closed. But this is equal to $A$, so $A$ is also closed. $\square$

(4) Let $A$ and $B$ be subsets of a metric space $(X, D)$. We will prove or disprove the following statements.

a) If $\overline{A} \cap \overline{B} = \phi$, then $D(A, B) \neq 0$. This statement is false. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ with the absolute value metric. Let $A = \{(x, y) \mid y = 0\}$ and $B = \{(x, y) \mid y = 1/x\}$. These subsets are both closed, so $A = \overline{A}$ and $B = \overline{B}$. Since they have no points in common, $\overline{A} \cap \overline{B} = \phi$. But the distance between these two sets is the greatest lower bound of the distances between their individual elements, which is $0$.

b) $D(A, B) = 0$ if and only if some sequence of points in $A$ converges to a point in $B$. This statement is false, though the backward direction holds. We know $(\leftarrow)$ is true because if a sequence $s_n$ from $A$ converges to a point in $B$, then $D(s_n, B)$ converges to $0$. Since each $s_n$ is an element of $A$, it must be the case that $D(A, B) = 0$. Now, to disprove $(\rightarrow)$ consider the $X$ and $B$ from part (a), but define $A = \{(x, y) \mid y = -1/x\}$. The distance between these two sets is $0$ because they become as close together as we would like. However, the closest a sequence from $A$ could get to converging in $B$ is the point $(x_1, 0)$ for some $x_1$, but this point is not in $B$. Similarly, the closest a sequence in $B$ could get to converging in $A$ is the point $(x_2, 0)$ for some $x_2$, but this point is not in $A$. This completes the counterexample.
c) If $A \cap B = \emptyset$, then there are disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$. This statement is false, though it holds for a few special cases. If $A$ and $B$ are closed, then $U$ and $V$ are known to exist from Proposition 13. If $A$ and $B$ are open, then $A = U$ and $B = V$ satisfy. However, if $A$ and $B$ are neither open nor closed, the statement fails. Consider the intervals $[-1, 0)$ and $[0, 1)$ as subsets of $\mathbb{R}$. It is clear that their intersection is empty, but any open interval that contained $[0, 1)$ would necessarily contain elements of $[-1, 0)$ and could therefore never be disjoint with an open interval containing $[-1, 0)$. The statement can also fail if one set is open and the other is closed.

d) If $D(A, B) \neq 0$, then there are disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$. This statement is true. The open sets $U$ and $V$ can be constructed using the union of $p$-neighborhoods around points in $A$ and points in $B$, respectively, where $p = 1/2D(A, B)$.

(6) Let $(X, D)$ be a metric space and $A \subset X$. We shall find the following frontiers.

a) In the set of real numbers with the absolute value metric, $\text{Fr}(0, 1) = \{0, 1\}$. These are the only two points that have a distance of zero from both $(0, 1)$ and $\mathbb{R} - (0, 1)$.

b) In the real plane with the Pythagorean metric, $\text{Fr}\{(x, y) \mid x^2 + y^2 < 1\} = \{(x, y) \mid x^2 + y^2 = 1\}$.

c) In the set of real numbers with the absolute value metric, $\text{Fr} \mathbb{Q} = \mathbb{R}$. This becomes clear when we realize that any real number is either a member of $\mathbb{Q}$ (and thus having a distance of zero to that set) or is so close to a member of $\mathbb{Q}$ that the greatest lower bound of the distances is zero.

d) In the real plane with the discrete metric, $\text{Fr}\{(x, y) \mid x = 3\} = \emptyset$. The only points that have a distance of zero to $\{(x, y) \mid x = 3\}$ under the discrete metric are the points in the set itself, and these are exactly the points that do not have a distance of zero to the complement of $\{(x, y) \mid x = 3\}$ (they have a distance of one). Thus, there are no points with a distance of zero to both.