Topology Homework 4

Section 3.4 - Section 4.1

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3.4

(5)

**Proposition.** Let $\tau$ and $\tau'$ be topologies on $X$. Define $S = \tau \cap \tau'$ and $S' = \tau' \cup \tau'$, and let $\tau_1$ be the topology determined by the subbasis $S$ and $\tau_2$ be the topology determined by the subbasis $S'$. Then $\tau_1$ is the finest topology which is coarser than both $\tau$ and $\tau'$, and $\tau_2$ is the coarsest topology which is finer than $\tau$ and $\tau'$.

**Proof.** We know that $\tau_1 \subset \tau$ and $\tau_1 \subset \tau'$ by Proposition 5 of the text. This means $\tau_1$ is coarser than both $\tau$ and $\tau'$. Suppose $\tau^*$ is another topology coarser than both; that is, $\tau^* \subset \tau$ and $\tau^* \subset \tau'$. Let $U^*$ be an open set in $\tau^*$. Then $U^* \in \tau$ and $U^* \in \tau'$. Thus, $U^* \in \tau \cap \tau' \subset \tau_1$, which implies $\tau^* \subset \tau_1$. So we see that $\tau_1$ is finer than any other topology which is coarser than $\tau$ and $\tau'$.

We know that $\tau \subset \tau_2$ and $\tau' \subset \tau_2$, so $\tau_2$ is finer than both $\tau$ and $\tau'$. Suppose $\tau^*$ be another topology finer than both; that is, $\tau \subset \tau^*$ and $\tau' \subset \tau^*$. Let $U_2$ be an open set in $\tau_2$. Then $U_2 \in \tau$ or $U_2 \in \tau'$. Either case implies $U_2 \in \tau^*$, so $\tau_2 \subset \tau^*$. So $\tau_2$ is coarser than any other topology which is finer than $\tau$ and $\tau'$. □

(8) On a set $X$, let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\tau$ and $\tau'$, respectively. Suppose that each member of $\mathcal{B}'$ contains a member of $\mathcal{B}$. It is not necessarily the case that $\tau$ and $\tau'$ are comparable.

Consider the following counterexample. Let $X = \{a, b, c\}$ and let the bases $\mathcal{B} = \{X, \{a\}\}$, $\mathcal{B}' = \{X, \{a, b\}\}$. Observe that $X$ contains $X$ and $\{a, b\}$ contains $a$, so each $B' \in \mathcal{B}'$ contains a $B \in \mathcal{B}$. The topologies determined by these bases are the bases themselves, $\tau = \mathcal{B}$ and $\tau' = \mathcal{B}'$. However, these topologies are not comparable because $\{a\}$ is open in $\tau$ but not in $\tau'$ and $\{a, b\}$ is open in $\tau'$ but not in $\tau$. Thus, neither is a subset of the other and we cannot discuss their relative coarseness or fineness.

3.5

(1) Let $(X, \tau)$ be a topological space and let $A$ and $B$ be any two subsets of $X$. We will show that it is not generally the case that $A \cap B = A \cap B$.

Consider $X = \mathbb{R}^2$ and $A = \{(x, y) | y > 0\}$, $B = \{(x, y) | y < 0\}$. Then $A \cap B = \emptyset$ and $A \cap B = \{(x, y) | y = 0\} = \emptyset$. But $A \cap B = \{(x, y) | y = 0\} \neq \emptyset$.

(4) Let $(X, \tau)$ and $(X, \tau')$ be topological spaces with $\tau' \subset \tau$. Derived sets will be notated with $'$ if they are with respect to $\tau'$, otherwise they are with respect to $\tau$.

a) $\overline{A} \subset \overline{A}'$ - Let $x \in \overline{A}$. This means $x \in C$ for all closed sets $C$ with respect to $\tau$ containing $A$. But all $C'$ closed with respect to $\tau'$ correspond to some $C$ with respect to $\tau$ because $\tau'$ is coarser than $\tau$. So $x \in C'$ for all closed sets containing $A$. Thus, by definition, $x \in \overline{A}'$. So $\overline{A} \subset \overline{A}'$.

b) $A^o \subset A^o'$ - Let $x \in A^o$. This means $x$ is in some open set with respect to $\tau'$ that is contained within $A$. But all open sets in $\tau'$ are also open sets in $\tau$. So $x$ is necessarily in an open set with respect to $\tau$ that is contained within $A$. Thus, $x \in A^o$ and $A^{o'} \subset A^o$. 

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c) Ext′A ⊂ ExtA - The exterior of a set is the complement of its closure. Since $\overline{A} \subset \overline{A}'$, it follows that $X - \overline{A}' \subset X - \overline{A}$.

d) FrA ⊂ Fr′A - Let $x \in FrA$. By definition, all open sets in $\tau$ containing $x$ meet both $A$ and $X - A$. Note that $A$ and $X - A$ are the same with respect to $\tau$ and $\tau'$. Since $\tau'$ contains no open sets other than those in $\tau$, there can be no new set containing $x$ that is open with respect to $\tau'$. Hence, all open sets with respect to $\tau'$ containing $x$ meet both $A$ and $X - A$. So $x \in Fr'\!A$.

(7) Though it seems counterintuitive, it is possible for two distinct subsets of a topological space to have exactly the same topologically derived sets.

Consider $A = (0, 1)$ and $B = [0, 1]$ in the topology induced by the absolute value metric on $\mathbb{R}$. We see that $\overline{A} = [0, 1] = \overline{B}$, $A^0 = (0, 1) = B^0$, Ext$A = (-\infty, 0) \cup (1, \infty) = ExtB$, and Fr$A = \{0, 1\} = FrB$. So the derived sets are equal, but $A$ is clearly distinct from $B$.

3.6

(4) Let $\mathbb{R}^2$ be the coordinate plane with the topology induced by the Pythagorean metric. Let $A = \{(x, y) \mid x, y \in \mathbb{Q} \text{ and } x^2 + y^2 < 1\}$. We will determine several derived sets. Note that $x, y \in \mathbb{R}$ unless otherwise indicated.

a) Fr$\overline{A}$ = $\{(x, y) \mid x^2 + y^2 = 1\}$. The closure of $A$ is the closed real unit disc because the rational unit disc is not closed. Clearly, the frontier of the real unit disc is the unit circle.

Fr$\overline{A}$ = $\{(x, y) \mid x^2 + y^2 \leq 1\}$. The frontier of $A$ is the closed real unit disc because all neighborhoods around these points will meet both rationals ($A$) and irrationals ($X - A$). This is a closed set, so its closure is itself.

b) Fr $A'$ = $\{(x, y) \mid x^2 + y^2 = 1\}$. The weak derived set of $A$ is the closed real unit disc because all of these points will have neighborhoods which intersect $A$. This frontier, then, is the same as part (a).

(4 Fr $A'$) = $\{(x, y) \mid x^2 + y^2 \leq 1\}$. As noted, the frontier of $A$ is the closed real unit disc. The weak derived set of this is the unit circle because there are no separated singletons, so it is the same as the closure.

c) ($\overline{A}$)$^0$ = $\{(x, y) \mid x^2 + y^2 < 1\}$. We know $\overline{A}$ is the real unit disc including its boundary. Thus, its interior is the real unit disc without the boundary.

$\overline{A}^0 = \phi$. The interior of $A$ is the union of all open sets contained in $A$. However, all open sets containing points in $A$ will also contain irrational points within the unit disc. Thus, the interior is empty. The smallest closed set containing $\phi$ is $\phi$.

Let $B = \{(x, y) \mid y = 0\}$. Then Fr$\overline{B}$ = $B = Fr\overline{B}$, Fr $B'$ = $B = (Fr B)'$, and ($\overline{B}$)$^0 = \phi = \overline{B}^0$.

4.1

Let $\tau$ be the topology induced on $\mathbb{R}$ by the absolute value metric. We will show that the following subsets of $\mathbb{R}$ have subspace topologies that are not the discrete topology. Define $Z = \{0\}$ and $\tau'$ to be the subspace topology of the respective subsets.

a) $\mathbb{Q} \subset \mathbb{R}$. Clearly, $Z$ is a subset of $\mathbb{Q}$. Suppose that $U \in \tau$ existed so that $Z = \mathbb{Q} \cap U$. Since $U$ is open it must contain a neighborhood in $\mathbb{R}$ around 0, but any such neighborhood would contain other rationals because $\mathbb{Q}$ is dense. Thus, $Z \neq \mathbb{Q} \cap U$ and we conclude that $Z \notin \tau'$. This means $\tau'$ is not the discrete topology because it does not contain every subset of $\mathbb{Q}$.

b) $A = \{x \mid x = 0 \text{ or } x = 1/n, n \in \mathbb{N}\}$. We see that $Z \subset A$. Suppose that $U \in \tau$ existed so that $Z = \mathbb{Q} \cap U$. Again, since $U$ is open it must contain a neighborhood in $\mathbb{R}$ around 0, but any such neighborhood would contain infinitely many elements of $A$ because $A$ considered as a sequence converges to 0. Thus, $U$ does not exist and $Z \notin \tau'$. So $\tau'$ is not the discrete topology.
c) \( B = \{ x \mid x = q\pi, q \in \mathbb{Q} \} \). Note that \( Z \subset B \). The set \( U \in \tau \) cannot exist such that \( Z = B \cap U \) because \( U \) must contain a neighborhood \( N(0,p) \) for \( p > 0 \), but there is a \( q_1 \in \mathbb{Q} \) such that \( q_1\pi \in N(0,p) \) for any \( p > 0 \). Therefore, \( Z \notin \tau' \) and \( \tau' \) is not the discrete topology.

d) In general, any subset of \( \mathbb{R} \) that is somewhere dense will have a subset topology that is not the discrete topology. Let \( C \) be any such subset. Consider a singleton set \( \{ s \} \) from the dense region of \( C \). Then any \( U \in \tau \) containing \( s \) would have to contain \( N(s,p) \) where \( p > 0 \). But \( N(s,p) \) would intersect other elements of \( C \) near \( s \) because \( C \) is dense in this region. Thus, \( \{ s \} \neq C \cap U \) and \( \{ s \} \) is not open in \( \tau' \). Since the discrete topology has every subset open, the subspace topology is not the discrete topology.

(6) The following claim is false: A subset \( A \) of a topological space is nowhere dense if and only if the subspace \( A \) has the discrete topology. The backward direction fails. Consider \( A = X \) where \( X \) is any topological space with the discrete topology \( \tau \). By Example 3 of the text, the subspace \( A \) must also have the discrete topology. Thus, we have fulfilled the hypothesis of the backward direction. Now, \( \overline{A} = A \) because \( A \) is itself a closed set. But \( A = X \), so by definition, \( A \) is dense.