REASONING-AND-PROVING IN GEOMETRY TEXTBOOKS: WHAT IS BEING PROVED?[1]

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As calls are made for reasoning-and-proving to permeate school mathematics, several textbook analyses have been conducted to identify reasoning-and-proving opportunities outside of high-school geometry. This study looked within geometry, examining six geometry textbooks and characterizing not only the justifications given and the reasoning-and-proving activities expected of students but also the nature of the mathematical statements around which reasoning-and-proving takes place. The majority of reasoning-and-proving exercises focused on particular mathematical statements, whereas the majority of expository mathematical statements were general in nature. Although reasoning-and-proving opportunities were numerous, it remained rare for reasoning-and-proving to be made an explicit object of reflection.

Background

Mathematicians and mathematics educators are calling for reasoning-and-proving[2] to become a central component of the mathematical experiences of students (Hanna, 2000; Stylianou, Blanton, & Knuth, 2009). One argument behind this call is that reasoning-and-proving is integral to the discipline of mathematics and thus an essential piece of an "intellectually honest" (Bruner, 1960) mathematics education. Such a perspective is reflected in the mathematical practices of the *Common Core State Standards for Mathematics* (2010), which include abstract reasoning, the construction of viable arguments, and the critique of others' reasoning. Another argument for the inclusion of reasoning-and-proving throughout the school mathematics curriculum is that, by reasoning through and proving mathematical results, students can develop deeper conceptual understanding of mathematical ideas as well as greater procedural fluency (de Villiers, 1995; Dreyfus, 1999; National Council of Teachers of Mathematics, 2009).

In the United States, however, reasoning-and-proving has not been ubiquitous in school mathematics but has traditionally been confined to a single geometry course in high school (Herbst, 2002). To document this current landscape and to prepare the way for a more comprehensive treatment of reasoning-and-proving, researchers have recently been studying the opportunities that exist for reasoning-and-proving in curriculum materials other than geometry. Kristen Bieda (personal communication, January 18, 2011) is in the process of coding a variety of elementary-level textbooks and Stylianides (2009) has examined an NSF-funded middle school series. At the high-school level, Davis (2010) explored an integrated textbook series and Senk, Thompson, and Johnson (2008) analyzed non-geometry courses, such as algebra, advanced algebra, and precalculus, from six different textbook series. Davis found that 12% of the problems in a particular integrated textbook related to reasoning-and-proving. Senk and her colleagues found this number to be only 6% in the textbooks they analyzed, even though their analysis focused on the chapters where reasoning-and-proving seemed most likely to occur. Senk, Thompson, and Johnson noted that the exposition sections of non-geometry textbooks gave more attention to reasoning-and-proving than the exercises; nevertheless, 40–50% of the stated mathematical properties were left unjustified.

In this study we contribute to the efforts described above by characterizing the nature of reasoning-and-proving opportunities in geometry textbooks themselves. Although we agree with the premise that it is important to understand the current state of reasoning-and-proving outside of geometry as efforts are undertaken to integrate reasoning-and-proving into those domains, we would add that it is equally important to understand reasoning-and-proving opportunities in geometry, where they are most plentiful. In other words, we should strive to understand and reflect upon the way we are handling reasoning-and-proving in geometry so that we may inform the process of expanding it to other courses and grade levels.

Analytic Framework

All of the curriculum analyses cited above have focused on the types of reasoning-and-proving activities that students are expected to perform, presenting data on how often students are asked to notice patterns, make conjectures, test conjectures, or develop arguments. Although many of the documented difficulties that students have with reasoning-and-proving (see Harel & Sowder, 2007, for a review) may be attributed to insufficient opportunity to engage in such practices, it also seems to be the case that students have fundamental misunderstandings of reasoning-and-proving, even after significant exposure. For example, Chazan (1993) found that some geometry students do not understand what has been proven by a deductive argument. Soucy McCrone and Martin (2009) reported on students, also from geometry, who viewed the purpose of proof to be the mere application of recently learned theorems, similar to the way in which recently learned formulas are applied in subsequent student exercises.

We employ the necessity principle as an interpretive frame to make sense of these phenomena and to guide our analysis. The *necessity principle* (Harel & Tall, 1989) is a standard for pedagogy that involves presenting subject matter in a way that encourages learners to see its intellectual necessity, "[f]or if students do not see the rationale for an idea, the idea would seem to them as being evoked arbitrarily; it does not become a concept *of the students*" (p. 41). The fact that many students do not understand the role of reasoning-and-proving in mathematics and view it as being required of them arbitrarily (Tinto, 1988) is evidence that the necessity principle is being violated with respect to reasoning-and-proving. Furthermore, the well-documented overreliance on empirical forms of argumentation (Harel & Sowder, 2007) suggests at least two possibilities: (a) students do not recognize the limitations of empirical reasoning or the intellectual necessity of deductive reasoning, or (b) students recognize the need for deduction but lack the resources or capabilities to successfully develop such arguments and so give an empirical argument rather than leave an item blank. If the latter is the case, then it is important to continue examining the opportunities that exist for students to engage in various reasoning-and-proving activities. But if it is the former, we must push further and consider whether or not those reasoning-and-proving activities are necessitating deductive reasoning.

With these considerations in mind, we developed our analytic framework by building upon past frameworks (particularly Senk et al., 2008) with the addition of a dimension for the mathematical statement around which the reasoning-and-proving activities are taking place. In particular, we distinguish between general and particular mathematical statements (see Figure 1). Our rationale is that general statements intellectually necessitate deductive forms of reasoning because empirical means cannot establish truth for an infinite class of objects. Thus, having students engage in reasoning-and-proving around general mathematical claims has the potential to better satisfy the necessity principle than particular mathematical claims. (This also happens to better align with the disciplinary practices of mathematicians.) We do not mean to imply that all reasoning-and-proving opportunities should be around general statements or that there is no benefit to exercises of a particular nature; we are simply arguing for the value of including this dimension when examining reasoning-and-proving opportunities in textbooks.

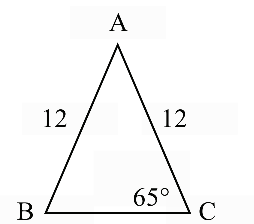
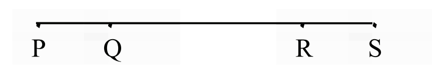
Type of Statement	Definition	Examples
General	A statement made about an infinite class of mathematical objects or an infinite number of mathematical situations.	(1) All isosceles triangles have congruent base angles. (2) Any lines *l* and *m* that are perpendicular to the same line *n* are parallel to one another. (3) Let *a*, *b*, and *c* be natural numbers. Then $a^{b+c} = a^b \cdot a^c$.
Particular	A statement made about a single or finite number of mathematical objects or situations. Note: A particular statement may be easily translated into a general statement, as in (2) to the right which may be reinterpreted as a statement about any such overlapping segments. This awareness of generality is non-trivial, however, and thus statements such as (2) remain categorized as particular.	(1) In the given diagram, angle *ABC* has a measure of 65 degrees.  (2) If *PR* = *QS*, then *PQ* = *RS*.  (3) If 2*x*+*y*=10 and *y*=4, then *x*=3.

Figure 1. General and particular mathematical statements.

Method

This study focused solely on stand-alone high-school geometry textbooks and so did not include analysis of geometry units within integrated textbooks. The six textbooks included in our analysis were *CME Geometry* (CME Project, 2009), Glencoe McGraw-Hill *Geometry* (Carter, Cuevas, Day, Malloy, & Cummins, 2010), Holt McDougal *Geometry* (Burger et al., 2011), Key Curriculum *Discovering Geometry* (Serra, 2008), Prentice Hall *Geometry* (Bass, Charles, Hall, Johnson, & Kennedy, 2009), and UCSMP *Geometry* (Benson et al., 2009). These were chosen to overlap series as much as possible with previous analyses (i.e., Senk, Thompson, & Johnson, 2008) so that comparisons would be possible. The six included series together span nearly 90% of the U.S. high school population (Dossey, Halvorsen, & Soucy McCrone, 2008).

Within each chapter of the six student edition textbooks, we randomly selected for analysis a minimum of 30% of the canonical sections (i.e., not special exploration or technology investigation sections). Additionally, chapter review exercises were coded for each chapter as a representation of the textbook authors' own identification of key ideas. This process resulted in an actual sampling of 44% of sections across the textbooks, totaling 285 sections and 12,468 exercises. Within the sampled sections, both exposition and student exercises were coded by the authors using the framework in Figure 2. Double-coding was performed on a 20% sample of the included sections yielding 95% agreement on statement-type and 91% agreement on justification-type within exposition sections, and 92% agreement on statement-type and 93% agreement on activity-type within student exercises.

	Exposition	Student Exercises	
	Properties, Theorems, or Claims	Related to Mathematical Claims	Related to Mathematical Arguments
Mathematical Statement or Situation	• General • Particular	• General • Particular • General with particular instantiation provided	• General • Particular • General with particular instantiation provided
Expected Student Activity		• Make a conjecture or refine a statement • Fill in the blanks of a conjecture or statement • Investigate a conjecture	• Develop a mathematical proof • Develop a rationale or non-proof argument • Outline a proof or develop a proof given an outline • Fill in the blanks of an argument or proof • Evaluate or correct an argument or proof • Find a counterexample
Type of Justification (or environment for exploration)	• Deductive • Empirical • Outline • Past or future • Left to student • None	• Deductive (explicit) • Empirical (explicit) • Implicit	• Deductive (explicit) • Empirical (explicit) • Implicit
	• Statements *about* reasoning-and-proving	• Exercises *about* reasoning-and-proving	

Figure 2. An analytic framework for reasoning-and-proving in geometry textbooks.

Results

As shown in Table 1, student exercises involving reasoning-and-proving were much more prevalent in geometry textbooks than in even the most reasoning-and-proving focused units of non-geometry or integrated high-school textbooks. CME contained the most reasoning-and-proving exercises with nearly 38% falling into at least one of the reasoning-and-proving activity categories from Figure 2. The other geometry textbooks ranged from approximately 20% to 27% of exercises related to reasoning-and-proving.

Table 1. Percent of student exercises involving reasoning-and-proving.

Textbook Series	Geometry		Non-Geometry	
	No. of Exercises Analyzed	Reasoning-and-Proving (%)	No. of Exercises Analyzed	Reasoning-and-Proving (%)
CME	1058	37.8	--	--
Glencoe	2730	24.3	2117[a]	3.7[a]
Holt	2531	23.6	2042[a]	3.7[a]
Key Curriculum	1489	26.7	916[a]	8.0[a]
Prentice Hall	2479	19.5	2446[a]	5.6[a]
UCSMP	2181	27.6	1739[a]	6.2[a]
Core Plus	--	--	1114[b]	12.3[b]

[a] Senk, Thompson, & Johnson (2008).
[b] Davis (2010) differed from the other studies in this table by including patterns as reasoning-and-proving.

The types of reasoning-and-proving activities expected of students are presented in Table 2. In four of the books, 13–15% of the reasoning-and-proving exercises (or 3–5% of the total exercises) involved students *developing a mathematical proof*. In two books, CME and Glencoe, such items comprised 25% and 28%, respectively (or approximately 7% of the total). The most common reasoning-and-proving activities were to *investigate a statement* (i.e., determine the truth-value of a mathematical claim) and to *develop a rationale* (i.e., to explain or justify an answer or result in a manner that is not necessarily a proof).

Table 2. Nature of reasoning-and-proving activities expected of students.

Textbook Series	Type of Activity by No. of Reasoning-and-Proving Items (%)					
	Develop a Proof	Develop a Rationale	Find a Counter-example	Investigate a Statement	Make a Conjecture	Other
CME	25	36	7	45	16	3
Glencoe	28	48	4	30	17	5
Holt	13	42	4	39	8	16
Key Curriculum	13	52	26	42	20	6
Prentice Hall	15	54	1	46	19	11
UCSMP	14	44	3	36	14	5

Note: Rows sum to more than 100% because exercises often involved more than one activity. “Other” includes fill-in-the-blanks of a proof, provide or argue from a proof outline, and evaluate a given proof.

Table 2 does not answer the question, what sorts of mathematical claims do students have opportunities to reason about or prove? Figure 3 depicts the percentages of reasoning-and-proving exercises that involved a *general mathematical statement*. This graph excludes general statements for which a particular instantiation was provided to the student, because in such cases the student may reason about the given object as they would any particular object without realizing the general implications (as was found by Chazan, 1993). Figure 3 compares the percentages of general statements in exercises with the percentages in textbook exposition.

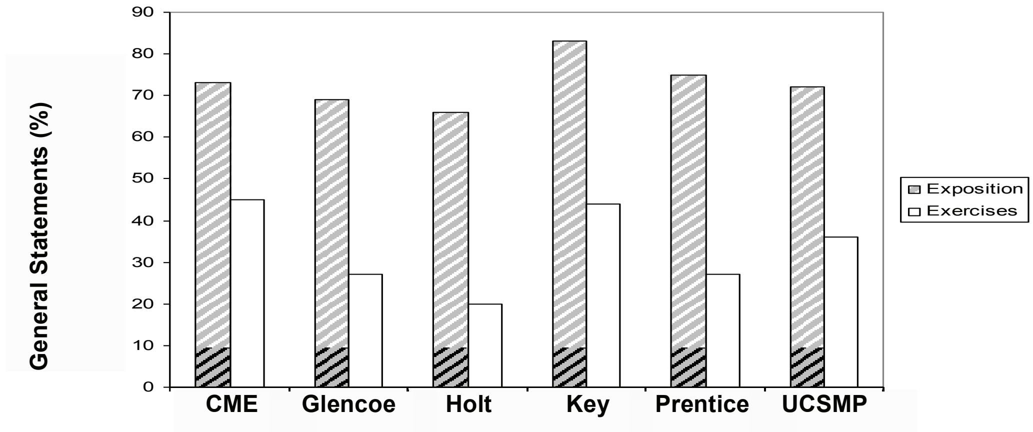


Figure 3. Nature of mathematical statements in exposition versus student exercises.

Most reasoning-and-proving exercises involved particular mathematical statements. In Glencoe, Holt, and Prentice Hall, approximately two-thirds of the reasoning-and-proving exercises were of a particular nature. In UCSMP, the number is 58%. CME and Key Curriculum had lower percentages of particular-type exercises—52% and 48%, respectively—but even in these textbooks, general statements were used in less than half of the reasoning-and-proving exercises. In textbook exposition, on the other hand, at least 66% of the statements containing a mathematical claim or result were of a general nature, with most textbooks falling above 70% (see Figure 3). Expository statements of a particular nature were especially infrequent in Key Curriculum and UCSMP, but were noticeably present in Glencoe (26%), Holt (24%), and Prentice Hall (20%). These particular statements that did appear in the exposition were almost always in the form of "worked examples." From this perspective, particular statements essentially appeared in exposition only when the textbook authors were modeling the behavior of students, for whom reasoning-and-proving exercises around particular statements are common.

Finally, we note results with respect to statements and exercises *about* reasoning-and-proving (see Table 3). For example, an exposition section may note that a deductive argument builds upon definitions or previously proved theorems, or an exercise may ask a student to write about the process of proof by contradiction. Within the 285 coded sections (out of 653 total sections), there were only 98 statements that made reasoning-and-proving an explicit object of reflection, and nearly half of these were found in a single book, UCSMP. Of the 12,468 coded exercises, only 67 asked students *about* the reasoning-and-proving process (as opposed to asking them to engage in that process). Therefore, although we saw in Table 1 that reasoning-and-proving is relatively common in geometry textbooks, opportunities are rare even in geometry to step out of the process and reflect on the core mathematical practice of reasoning-and-proving.

Table 3. Reasoning-and-proving not as an activity but as an object of discussion or reflection.

Textbook Series	No. of Statements *about* Reasoning-and-Proving	No. of Exercises *about* Reasoning-and-Proving	Percent of Total Exercises Analyzed
CME	13	4	0.38
Glencoe	8	17	0.62
Holt	15	17	0.67
Key Curriculum	14	13	0.87
Prentice Hall	7	2	0.08
UCSMP	41	14	0.64

Discussion

In this paper, we have presented top-level results of an investigation of the reasoning-and-proving opportunities in six different high-school geometry textbooks. Even in geometry, the traditional home of reasoning-and-proving, students were asked to develop a mathematical proof in less than 7% of the textbook exercises, and statements or questions about reasoning-and-proving as a mathematical practice were rare. The most common reasoning-and-proving activities were to provide a rationale (not necessarily a proof) and to determine the truth-value of a mathematical claim. Interestingly, students were expected to make judgments about truth much more frequently than they were expected to provide deductive arguments—the disciplinary process by which truth is established. With respect to the prominence of rationale exercises, one might contend that an "explain" prompt provides students with an opportunity to develop a proof because a key function of mathematical proofs is explanation (de Villiers, 1995). However, the question remains: Do students realize that a proof would be an effective response to an "explain" exercise? Answering this question would take us beyond the realm of textbook analysis.

The necessity principle (Harel & Tall, 1989) implies that it would be beneficial to help students recognize the intellectual need for deductive forms of reasoning by, for example, providing them with opportunities to reason around general mathematical claims for which empirical arguments falter. Our analysis has revealed that the majority of reasoning-and-proving exercises in geometry textbooks are around particular, not general, mathematical statements. In exposition sections, on the other hand, the majority of mathematical statements are general in nature. This discrepancy may shed light on such phenomena as geometry students believing that proof is merely an application of recently learned theorems (Soucy McCrone & Martin, 2009), because indeed students are applying the theorems presented in exposition sections to prove things about contrived, particular situations, or geometry students believing that mathematical knowledge is created by others and not themselves (Schoenfeld, 1988), because indeed the most significant mathematical results are general in nature and likely found in textbook exposition.

Pursuing these potential connections requires further research, and one might be skeptical of the merits of this course of study. Perhaps it is necessary for key results to be explicated in exposition sections so that they may be officially established in the classroom canon. Moreover, one could argue that it is necessary to provide students with numerous particular statements to prove because practice is essential and there are not enough relevant general statements to allow for this practice. In response to these points, we would again cite the research literature which shows that the status quo of reasoning-and-proving in geometry is not producing the student outcomes the mathematics education community hopes to see. Yes, it is important to establish important results into a collective space, but it is also important to reflect upon how the process of establishing those results may influence students' notions of who is capable of generating mathematical knowledge. Yes, it is important to allow students to practice reasoning-and-proving, but it is also important to consider whether the nature of the practice we afford them aligns with actual mathematical practice. In the end, our goal is for students to have success with reasoning-and-proving but also to see its intellectual necessity and value.

Endnotes

1. This work was supported by a grant from the College of Natural Science at Michigan State University. We thank Kristen Bieda and Sharon Senk for their insightful feedback.
2. We join Stylianides (2009) in hyphenating reasoning-and-proving to emphasize the inseparability of the reasoning process that leads to a proof and the resulting proof product.

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