Homework 10

Section 2.1

Proposition A \( H_0(X, A) = 0 \) if and only if \( A \) meets each path-component of \( X \).

Proof: We begin with a fact that will be used later several times:

Let \( B \) be a path-component of \( X \) and \( b_1, b_2 \in B \). Then there exists a path \( \gamma \) from \( b_1 \) to \( b_2 \). So \( \gamma \) is a singular 1-simplex with \( \partial \gamma = b_2 - b_1 \). Hence \([b_1] = [b_2]\) in \( H_0(X)\). More generally, there is one homology class of \( H_0(X) \) for each path-component of \( X \).

(\(\Rightarrow\)) By theorem A (3/4), there exists a long exact sequence:

\[ \cdots \rightarrow H_0(A) \overset{i_*}{\rightarrow} H_0(X) \overset{j_*}{\rightarrow} H_0(X, A) \rightarrow \cdots \rightarrow \]

We assume \( H_0(X, A) = 0 \). Hence \( \text{im } j_* = 0 \) and \( \text{ker } j_* = H_0(X) \) because everything must be sent to 0. By exactness, \( \text{im } i_* = \ker j_* = H_0(X) \) so we see that \( i_* \) is surjective. So the injection \( i \) induces a homomorphism \( i_* \) that reaches all homology classes of \( H_0(X) \), and thus by (\(\ast\)), all path-components of \( X \).

(\(\Leftarrow\)) Let \([\alpha] \in H_0(X)\) and assume \([\alpha] = [\beta] \) for some \( \beta \in A \). We have \( i_\alpha(\alpha) = [\alpha] = [\beta] \). Thus \( i_\alpha \) is surjective. So \( \text{im } i_\alpha = H_0(X) \) and by exactness \( H_0(X) = \ker j_* \). This means \( j_* \) is injective. So \( \text{im } j_* \rightarrow O \); that is, \( \text{im } j_* = 0 \). By exactness, the kernel of the map from \( H_0(X, A) \) to \( O \) is \( \text{im } j_* = 0 \). So \( H_0(X, A) \rightarrow O \) is an injective map, implying \( H_0(X, A) = 0 \).

Proposition B \( H_1(X, A) = 0 \) if and only if \( i: H_1(A) \rightarrow H_1(X) \) is surjective and each path-component of \( X \) contains at most one path-component of \( A \).

Proof: (\(\Rightarrow\)) Assume \( H_1(X, A) = 0 \). So we have the long exact sequence:

\[ \cdots \rightarrow H_1(A) \overset{i_*}{\rightarrow} H_1(X) \overset{j_*}{\rightarrow} H_1(X, A) \rightarrow \cdots \rightarrow \]

We see that \( \ker j_* = H_1(X) \) and so \( \text{im } i_* = H_1(X) \). Hence \( i_* \) is surjective. Since \( \text{im } j_* = 0 \) we get \( \ker j_* = 0 \) and hence \( j_* \) is injective. 

Now suppose to the contrary that \( A_0 \) and \( A_1 \) are two path-components of \( A \) that are contained in a path-component \( X_0 \) of \( X \). We know that \( H_1(A) \) is the direct sum of the homology groups of its path-components. So let

\[
\begin{align*}
I_x & : H_0(A) \oplus H_0(A) \rightarrow H_0(X), \\
\text{where } \triangleleft H_0(X)
\end{align*}
\]
By (**) every point in \( X_0 \) represents the same homology class in \( H_0(X) \) while points in \( A_0 \) and \( A_1 \) represent distinct homology classes in \( H_0(A_0) \oplus H_0(A_1) \). This implies \( I_x \) is not injective when restricted to \( A_0 \) and \( A_1 \), and so \( I_x \) is not injective overall — a contradiction. Therefore, a path-component of \( X \) can contain at most one path-component of \( A \).

\( \Leftarrow \) Assume \( I_x \) is surjective and there is at most one path-component of \( A \) in any path-component of \( X \). Let \( [a, \bar{a}] \in H_0(A) \). If \( I_x([a, \bar{a}]) = I_x([\bar{a}, a]) \) then by (**) \( I(a) = a \) and \( I(\bar{a}) = \bar{a} \) lie in the same path-component of \( X \). Since \( a, \bar{a} \in A \) and only one path-component of \( A \) can be contained in this particular path-component of \( X \), \( a \) and \( \bar{a} \) must also be in the same path-component of \( A \). Thus (**) gives \( [a, \bar{a}] = [\bar{a}, a] \) in \( H_0(A) \). This means \( I_x \) is injective, and it follows that \( \ker I_x = 0 \).

By exactness, \( \ker \partial = 0 \). We assumed \( \text{im } I_x \cong H_1(X) \) and hence \( \ker j_x = H_1(X) \). Since the kernel of \( j_x \) is the entire domain, \( \text{im } j_x = 0 \). By exactness, \( \ker \partial = 0 \). Putting it all together, \( \text{im } \partial = \ker \partial = 0 \) so necessarily \( H_n(X,A) = 0 \). \( \square \)

17. We will compute the homology groups \( H_n(X,A) \) when \( X = S^2 \) and \( A \) is a finite set of points.

Recall that \( H_n(S^2) = \{0\} \) if \( n \neq 0 \). We also know \( H_n(A) = \{0\} \) if \( n > 0 \) where \( k \) is the number of points in \( A \). By theorem A (34) we have

\[
\cdots \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_n(A) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(X,A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots
\]

Substituting what we know gives

\[
\cdots \longrightarrow 0 \longrightarrow H_n(X,A) \longrightarrow 0 \longrightarrow 0 \longrightarrow H_n(X,A) \longrightarrow Z^k \longrightarrow \cdots
\]

Since \( A \) meets each path-component of \( X \), \( H_n(X,A) = 0 \) by \( \# \text{le}(A) \). Within the long exact sequence above we now have the short exact sequence

\[
0 \longrightarrow H_n(X,A) \longrightarrow Z^k \longrightarrow \cdots \longrightarrow 0
\]

From an example in class (34) we know that \( Z \cong \mathbb{Z}/H_n(X,A) \). It follows that \( H_n(X,A) \cong \mathbb{Z}^{k-1} \). From the long exact sequence we conclude that \( \text{im } j_x = 0 \) for \( j_x : 0 \longrightarrow Z^k \) and so \( \ker j_x = 0 \) for \( j_x : Z \longrightarrow H_n(X,A) \). Note that \( \text{im } j_x = \ker \partial = H_0(X,A) \), so by the first isomorphism theorem \( H_0(X,A) \) is isomorphic to \( \mathbb{Z}^{k-1} \). Finally, \( H_n(X,A) = 0 \) for \( n \geq 3 \) because we have \( \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_1(X,A) \longrightarrow 0 \longrightarrow \cdots \) and \( 0 = \text{im } j_x = \ker \partial = 0 \).

Since \( \text{im } \partial = 0 \) also, it must be the case that \( H_n(X,A) = 0 \).
We will compute the homology groups \( H_n(X, A) \) when \( X = S^1 \times S^1 \) and \( A \) is a finite set of points.

Recall that

\[
H_n(S^1 \times S^1) = \begin{cases} 
\mathbb{Z} & \text{if } n = 0, 2 \\
\mathbb{Z}^2 & \text{if } n = 1 \\
0 & \text{if } n \geq 3.
\end{cases}
\]

So we have a long exact sequence

\[
\cdots \to 0 \to H_2(X, A) \to \mathbb{Z} \to H_2(X, A)^2 \to 0 \to \mathbb{Z} \to H_1(X, A) \to \mathbb{Z} \to H_0(X, A) \to 0.
\]

Again by \( \#(A), H_0(X, A) = 0 \) and for the same reason as before, \( H_n(X, A) = 0 \) for \( n \geq 3 \). If we consider the sequence in reduced homology, we see a short exact sequence

\[
0 \to \mathbb{Z}^2 \to H_1(X, A) \to \mathbb{Z} \to 0.
\]

So we know \( \mathbb{Z}^2 \cong H_1(X, A) / \mathbb{Z} \) and hence \( H_1(X, A) \cong \mathbb{Z}^{k+1} \). If we look back at the original long exact sequence, we see \( \ker \beta = H_2(X, A) \) and so also \( \text{im } j_\beta = H_2(X, A) \). By the first isomorphism theorem, \( H_2(X, A) \) is isomorphic to \( \mathbb{Z} / \ker j_\beta = \mathbb{Z} / 0 = \mathbb{Z} \), because \( 0 = \text{im } j_\beta = \ker j_\beta \).

b. We will compute the homology groups \( H_n(X, A) \) and \( H_n(X, B) \) with \( X, A, \) and \( B \) as shown. At various points we will use theorem B (36).

\[
\begin{align*}
H_n(X, A) & = H_n(X, B) \\
\end{align*}
\]

since \( (X, A) \) is a good pair.

By \( \#(A), H_0(X, A) = 0 \). By the theorem from (34), \( H_1(X, A) \) is the abelianization of \( \pi_1(X, A) \). Note that \( X/A \) is the wedge sum of two tori, so \( \pi_1(X/\sim) \cong \mathbb{Z}^4 \). The abelianization of \( \pi_1(X, A) \) is \( H_1(X, A) \cong \mathbb{Z}^4 \).

Consider the chain complex

\[
\cdots \to 0 \to C_2(X, A) \xrightarrow{\partial_2} C_1(X, A) \xrightarrow{\partial_1} C_0(X, A) \xrightarrow{\partial_0} 0.
\]

There are no 3-simplices, so \( C_3(X, A) = 0 = \text{im } \partial_3 \). We have \( C_0(X, A) = \langle L, R \rangle \) and \( \partial_1 L = \partial_1 R = 0 \). Hence \( \ker \partial_2 = \langle L, R \rangle \), and

\[
H_2(X, A) = \ker \partial_2 / \text{im } \partial_3 \cong \langle L, R \rangle / 0 \cong \mathbb{Z}^2.
\]

Since \( C_n(X, A) = 0 \) for \( n \geq 3 \), we get

\[
H_n(X, A) = 0 \quad \text{for } n \geq 3.
\]
Note that $X/B$ is homeomorphic to a torus with two points identified. So the hypotheses of \( \pi_1(x,x_0) \) are fulfilled \((k=2)\) and we conclude that \( H_2(X/B) = \mathbb{Z} \), \( H_1(X/B) = \mathbb{Z}^2 \), \( H_0(X/B) = 0 \) for \( n \geq 3 \).

We will compute the homology groups of $X$, the subspace of $I \times I$ consisting of the four boundary edges plus all points in the interior whose first coordinate is rational.

First, observe that $X$ is path-connected. Let \((a,b), (c,d) \in X\). Then the path composed of the line segments \((a,b) \rightarrow (b,c) \rightarrow (c,d)\) lies entirely within $X$ and goes from \((a,b)\) to \((c,d)\). Thus \( H_0(X) = \mathbb{Z} \).

Consider the chain complex $\cdots \rightarrow C(X) \xrightarrow{d_2} C(X) \xrightarrow{d_2} C(X) \rightarrow 0$. There are no $n$-simplices for $n \geq 2$ so \( C_n(X) = 0 \) and \( H_n(X) = 0 \) for $n \geq 2$. For \( C_1(X) \), there are a countably infinite number of generators — one for each rational in $I$ — so \( C_1(X) \cong \mathbb{Z}^\mathbb{Q} \). Thus \( \text{ker } d_1 = \mathbb{Z}^\mathbb{Q} \). There are also \( \mathbb{Z} \) generating cycles in $X$ so \( \mathbb{Z}^\mathbb{N} \leq \text{ker } d_1 \) on the level of cardinality. So by definition \( H_1(X) = \frac{\text{ker } d_1}{\text{im } d_1} \cong \mathbb{Z}^{\mathbb{Q} / \mathbb{Z}} = \mathbb{Z}^\mathbb{Q} \).

Proposition A. If $X$ is a finite CW complex with dimension $n$, then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is a free abelian group.

Proof. We will induct on $n$. Base Case: if $n = 0$ then $H_i(X) = 0$ for $i > 0$ because $C_i(X) = 0$. Also, $H_0(X) = \mathbb{Z}^k$ where $k$ is the number of 0-cells (or components, in this case). We have \( \mathbb{Z}^k \), a free abelian group.

Inductive Case: Assume for any finite $(n-1)$-dimensional CW complex $Y$ that $H_i(Y) = 0$ for $i > n-1$ and $H_n(Y)$ is a free abelian group.

With regard to $X$, there are no $i$-cells for $i > n$ and so $C_i(X) = 0$ and consequently $H_i(X) = 0$. (By Hatcher's remark, we can view $X/X^{n-1}$ as a wedge, and so a $\Delta$-complex.)
Note that by definition \((\forall n)\), \(X^0 = X^1\) and \(X^{n-1} \subseteq X^n\). We have a long exact sequence.

\[\cdots \rightarrow H_n(X^{n-1}) \rightarrow H_n(X^n) \rightarrow H_n(X^{n-1}) \rightarrow H_n(X^n) \rightarrow H_n(X^{n-1}) \rightarrow \cdots\]

which we can rewrite using the inductive hypothesis and our new knowledge as

\[\cdots \rightarrow 0 \rightarrow 0 \rightarrow H_{n+1}(X^{n-1}) \rightarrow 0 \rightarrow H_n(X^n) \rightarrow j_*H_n(X^{n-1}) \rightarrow G \rightarrow \cdots\]

where \(G\) is a free abelian group.

Now \(\text{im } \iota_n = 0\) and by exactness \(\ker j* = 0\). Thus, the First Isomorphism Theorem gives

\[\text{im } j* = H_n(X^n) / \ker j* = H_n(X^n) / 0 = H_n(X^n).\]

Since \(\text{im } j* = \ker D\), we get \(H_n(X^n) \cong \ker D\). But \(\ker D\) is a subgroup of \(G\) because it is the kernel of a homomorphism, and any subgroup of a free abelian group is free abelian. Therefore, \(\ker D = H_n(X^n)\) is a free abelian group.

\[\square\]

\[\checkmark\] If there are no cells of dimension \((n-1)\) or \((n+1)\), then \(H_n(X)\) is free with basis in bijective correspondence with the \(n\)-cells.

Let \(k\) be the number of \(n\)-cells. The chain complex

\[\cdots \rightarrow C_{n+1}(X) \rightarrow C_n(X) \rightarrow C_{n-1}(X) \rightarrow \cdots\]

becomes

\[\cdots \rightarrow 0 \rightarrow \mathbb{Z}^k \rightarrow 0 \rightarrow \cdots\]

because there are no generators for \((n-1)\) and \((n+1)\) but \(k\) generators for \(C_n(X)\). We see that \(\ker D = \mathbb{Z}^k\) and \(\text{im } D = 0\).

So \(H_n(X) = \ker D / \text{im } D = \mathbb{Z}^k / 0 = \mathbb{Z}^k\). This shows that \(H_n(X)\) is free abelian with \(k\) basis elements that can be put into bijective correspondence with the \(k\) \(n\)-cells.

\[\checkmark\]

By set theory, there exists a bijection between two sets with the same size, i.e., the set of basis elements of \(H_n(X)\) and the set of \(n\)-cells.

**Proposition C.** If \(X\) has \(k\) \(n\)-cells, then \(H_n(X)\) is generated by at most \(k\) elements.

\[\text{Proof.}\] We will induct on \(n\).
Base Case: if $X$ has $k$ $0$-cells, then $H_0(X) = \mathbb{Z}^k$ with $l \leq k$ because there can be at most $k$ components.

Inductive Case: we have a long exact sequence
\[ \cdots \to H_n(X_{n-1}) \to H_n(X^n) \to H_n(X^*/X^{n-1}) \to H_{n-1}(X^{n-1}) \to \cdots \]
and by the inductive hypothesis and part (a),
\[ \cdots \to 0 \to H_n(X^n) \to \mathbb{Z}^k \to \cdots \]
We have $\text{im} j_* = 0$ so $H_n(X^n) \cong \ker j_*$. But $\ker j_*$ is a subgroup of $\mathbb{Z}^k$ and so is free abelian with fewer than $k$ generators. By assumption, $n$ is the dimension of $X$ so $H_n(X) = H_n(X^k) \cong \mathbb{Z}^l$ with $l \leq k$. \qed