2.7. Let \( f : (X,A) \to (Y,B) \) be a map such that both \( f : X \to Y \) and the restriction \( f : A \to B \) are homotopy equivalences. Then \( f_* : H_n(X,A) \to H_n(Y,B) \) is an isomorphism for all \( n \).

To prove this, we look at the long exact sequences (Theorem 4.1) that exist for \((X,A)\) and \((Y,B)\):

\[
\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots
\]

\[
\cdots \to H_n(B) \to H_n(Y) \to H_n(Y,B) \to H_{n-1}(B) \to H_{n-1}(Y) \to \cdots
\]

We have the isomorphisms shown in the diagram because homology is homotopy invariant. Hence, by the five-lemma, \( H_n(X,A) \cong H_n(Y,B) \).

Now consider the inclusion \( f : (D^n, S^{n-1}) \to (D^n / S^n, \{0\}) \). We will show that \( f \) is not a homotopy equivalence of pairs.

Suppose to the contrary there was a \( g : (D^n, D^n \setminus S^n) \to (S^{n-1}) \) that was the homotopy equivalence inverse to \( f \). Since this is a homotopy equivalence of pairs, \( g' : D^n \setminus S^n \to S^{n-1} \) must also be a homotopy equivalence. Note that \( g' \) maps all of \( D^n \) except the point \( 0 \) to \( S^{n-1} \). But \( g' \) is continuous; so 0 itself must also be mapped to \( S^{n-1} \) by \( g' \). This means \( g' \) factors through \( D^n \) and

\[
g' : H_{n-1}(D^n \setminus S^n) \to H_{n-1}(S^{n-1})
\]

is the zero map. But \( H_{n-1}(S^{n-1}) \cong \mathbb{Z} \neq 0 \) and so \( g' \) is not an isomorphism. Since homotopy equivalences induce isomorphisms, \( g' \) is not a homotopy equivalence.

This is a contradiction.

Therefore, \( f \) is not a homotopy equivalence of pairs.
29) We will show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same homology groups while their universal covers do not.

By Example 2.3, we know

$$H_n(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n \geq 3 \end{cases}$$

By Corollary 2.25, we know that the inclusions of $S^1$, $S^1$, and $S^2$ into $S^1 \vee S^1 \vee S^2$ induce an isomorphism between

$$\tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \cong \tilde{H}_n(S^1 \vee S^1 \vee S^2).$$

From prior work,

$$\tilde{H}_n(S^1) \cong \begin{cases} \mathbb{Z} & n=1 \\ 0 & n \neq 1 \end{cases} \quad \text{and} \quad \tilde{H}_n(S^2) \cong \begin{cases} \mathbb{Z} & n=2 \\ 0 & n \neq 2 \end{cases}.$$

From the direct sums of these we see that

$$\tilde{H}_n(S^1 \vee S^1 \vee S^2) \cong \begin{cases} 0 & n=0 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n \geq 3 \end{cases}.$$ 

This implies that the unreduced homology groups are

$$H_n(S^1 \vee S^1 \vee S^2) \cong \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n \geq 3 \end{cases}.$$ 

So it is true that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same homology groups.

So we move on to consider the universal covers of these spaces.
From our work in chapter 1, we have $\mathbb{R}^2$ as the universal cover of $S^1 \times S^1$.

And the following infinite tree with balloons as the universal cover of $S^1 \vee S^1 \vee S^2$.

Now, $H_0(\mathbb{R}^2) \cong \mathbb{Z}$ and $H_n(\mathbb{R}^2) = 0$ for $n > 1$. In particular, $H_2(\mathbb{R}^2) = 0$. However, the universal cover $Y$ has $H_2(Y) \neq 0$. We can see this informally because $Y$ contains a subspace that deformation retracts to $S^1$ and $H_2(S^1) \cong \mathbb{Z}$. More formally, #11 of Homework 9 implies that the map $H_n(A) \to H_n(Y)$ induced by the inclusion $A \to Y$ is injective. Since $H_0(A) \cong \mathbb{Z}$, we conclude $H_1(Y) \neq 0$. 
Let \( X \) be the space obtained from a torus \( S^1 \times S^1 \) by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to a circle in the torus. Let \( M \) be the Möbius band.

Note that

\[
\tilde{H}_n(S^1 \times S^1) = \begin{cases} 0 & n = 0 \\ \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}
\quad \text{and} \quad
\tilde{H}_n(M) = \begin{cases} 0 & n = 0 \\ \mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}
\]

because \( M \) deformation retracts to \( S^1 \). Let \( A \) be a neighborhood of the torus and \( B \) a neighborhood of the Möbius band so that \( A \) and \( B \) deformation retract to \( S^1 \times S^1 \) and \( M \), respectively, and \( A' \cup B' = X \). The Mayer-Vietoris theorem gives the following long exact sequence on reduced homology:

\[
\ldots \to \tilde{H}_3(A \cap B) \to \tilde{H}_3(A) \oplus \tilde{H}_3(B) \to \tilde{H}_3(A \cup B) \to \tilde{H}_2(A \cap B) \to \tilde{H}_2(A) \oplus \tilde{H}_2(B) \to \tilde{H}_2(A \cup B) \to \ldots
\]

We have \( \tilde{H}_n(A \cap B) = \tilde{H}_n(S^1) \) for all \( n \) because \( A \cap B \) deformation retracts to the circle along which \( S^1 \times S^1 \) and \( M \) are attached. Hence

\[
\ldots \to 0 \to \tilde{H}_3(A) \to 0 \xrightarrow{\partial_3} \mathbb{Z} \oplus 0 \xrightarrow{\Phi_3} \tilde{H}_2(A) \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\Phi_2} \tilde{H}_1(A) \to 0
\]

We have left out the \( \tilde{H} \) portion of the sequence, but it is clear that \( \tilde{H}_0(A) = 0 \). We can also see that \( \tilde{H}_n(A) = 0 \) for \( n > 3 \). Let us consider \( \Phi_3 \), which is induced by the inclusions of \( A \cap B \) into \( A \) and \( B \) at the chain complex level. Let \( \alpha \) be the generator of \( \tilde{H}_3(A \cap B) \cong \mathbb{Z} \). With respect to \( A \), \( \alpha \) is sent to an generating loop on the torus. With respect to \( B \), \( \alpha \) gets sent twice the generating loop of the Möbius band because the boundary circle goes around twice. Hence \( \text{im} \Phi_3 = \text{im} \partial_3 = \langle (1,0,2) \rangle \) and clearly \( \ker \Phi_3 = 0 \). By exactness, \( \text{im} \partial_2 = 0 \) and this gives

\[
\mathbb{Z} \xrightarrow{\partial_2} \tilde{H}_2(A) \xrightarrow{\partial_2} 0
\]

which implies \( \tilde{H}_2(A) \cong \mathbb{Z} \). Also

\[
\tilde{H}_1(A) \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle \text{im} \Phi_2 \rangle} \cong \mathbb{Z} \oplus \mathbb{Z} / \langle (1,0,2) \rangle
\]

We can think of \( \mathbb{Z} \oplus \mathbb{Z} \) as \( \langle (a,b,c) \rangle \) and thus \( \tilde{H}_1(A) \) has been quotiented by the relation \( 2a = c \), which has the effect of killing one generator. Therefore \( \tilde{H}_1(X) \cong \mathbb{Z}^2 \).
Let $Y$ be the space obtained by attaching $M$ to $\mathbb{R}P^2$ via a homeomorphism $A$ of its boundary circle to the standard $\mathbb{R}P^1 \subset \mathbb{R}P^2$. Let $A \neq \mathbb{R}P^2$ and $B = M$ such that Mayer-Vietoris applies giving us

$$\to \mathbb{H}_0(A \cap B) \rightarrow \mathbb{H}_0(Y) \rightarrow \mathbb{H}_0(A \cap B) \rightarrow \mathbb{H}_0(A \cap B) \rightarrow \mathbb{H}_0(Y) \rightarrow \mathbb{H}_0(Y) \rightarrow \cdots$$

Note that $\mathbb{H}_n(A) = \mathbb{H}_n(\mathbb{R}P^2) = \mathbb{Z}/2$ for $n = 0, 2, 4$ (1 mod 4) and $A\partial B$ again deformation retracts to $S^1$. So we have

$$\to 0 = 0 \to \mathbb{H}_0(Y) \to 0 \xrightarrow{\varepsilon} 0 \xrightarrow{\varepsilon} \mathbb{H}_0(Y) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{H}_0(Y) \xrightarrow{\varepsilon} 0 \cdots$$

For the same reasons as before, $\mathbb{H}_0(Y) = 0$ and $\mathbb{H}_n(Y) = 0$ for $n \geq 3$.

Again, let $\alpha$ be the generator of $\mathbb{H}_1(A \cap B) \cong \mathbb{Z}$. With respect to $A$, $\alpha$ is mapped to the radial loop $a$ in $\mathbb{R}P^2$ (note $2a = 0$). With respect to $B$, $\alpha$ is mapped to the boundary circle and so twice the generating cycle $b$ of $\mathbb{H}_1(B)$. Hence $\text{im} \varepsilon_1 = \text{im} \{\alpha\} = \{(n, a \beta) \mid n \in \mathbb{Z}\} \cong \langle (1, a) \rangle$. The only cycle in $A \cap B$ sent by $\varepsilon_1$ to $(0, 0)$ is the trivial cycle, so $\text{ker} \varepsilon_1 = 0$.

By exactness, $\text{im} \varepsilon_2 = 0$ and we have $0 \xrightarrow{\varepsilon_2} \mathbb{H}_2(Y) \xrightarrow{\varepsilon_2} 0$ which implies $H_2(Y) = 0$.

We also know

$$H_1(Y) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z} / \langle (1, 2) \rangle.$$

We can represent this as $\langle a, b \mid a + b = 1 + 2, 2a = 0, a + 2b = 0 \rangle$. This group is $\mathbb{Z}_4$ because going once around $\mathbb{R}P^1$ is equivalent to twice around the boundary of $M$. Thus going twice around $\mathbb{R}P^1$, that is, the identity, is equivalent to four times around $M$. Therefore, $H_1(Y) \cong \mathbb{Z}_4$.

This agrees with #21 of Homework #1, in which we found $H_1(Y) \cong \mathbb{Z}_4$.

$$\langle a, b \mid a + b = 1 + 2, 2a = 0, a + 2b = 0 \rangle = \langle b \mid 4b = 0 \rangle.$$
32) Let $X$ be a topological space and $SX$ the suspension of $X$.

Let $A$ be the subspace of $SX$ defined by $X \times [0, \frac{1}{3}]$. Let $B$ be the subspace of $SX$ defined by $X \times [\frac{2}{3}, 1]$. By the definition of $SX$, $A \cap B = X$ and $A, B$ are contractible. So Mayer-Vietoris gives

$$\cdots \to H_n(A) \oplus H_n(B) \to H_n(SX) \to H_{n-1}(A \cap B) \to H_{n-1}(A) \oplus H_{n-1}(B) \to \cdots$$

which we can rewrite as

$$\cdots \to 0 \oplus 0 \to H_n(SX) \to H_{n-1}(X) \to 0 \oplus 0 \to \cdots$$

By the in-class example of $\Sigma I$, this implies $H_n(SX) \cong H_{n-1}(X)$.

Suspension defined on page 8.