Section 1.1

8) We will show that Borsuk-Ulam does not generalize to the torus.

That is for any continuous \( f : S^1 \times S^1 \to \mathbb{R}^2 \) it is not necessarily the case that there exists \( (x, y) \in S^1 \times S^1 \) such that \( f(x, y) = f(-x, y) \).

Let the torus lie in \( \mathbb{R}^3 \) above the \( xy \)-plane and let \( p \in \mathbb{R}^3 \) be a point higher than the torus and above its hole. Then we can define a map \( F \) that sends a point from the torus to the intersection of the ray through the point emanating from \( p \) and the \( xy \)-plane.

This function is intuitively continuous because it preserves nearness. More formally, it is clear that for any neighborhood \( N \) in the plane we can find a neighborhood \( M \) on the torus such that \( f(M) \subset N \).

Now, suppose \( f(x_1, y_1) = f(x_2, y_2) \) for some \( (x_1, y_1), (x_2, y_2) \in S^1 \times S^1 \). Then \( (x_1, y_1) \) and \( (x_2, y_2) \) both lie on the ray from \( p \) to \( F(x_1, y_1) \). This implies that \( (x_1, y_1) \) and \( (x_2, y_2) \) are on the "same side" of the torus because \( p \) is above the hole. Thus \( y_2 \neq -y_1 \).

So Borsuk-Ulam does not generalize to the torus.
Proposition Let \( A_1, A_2, A_3 \) be compact sets in \( \mathbb{R}^3 \). Then there is a plane \( P \in \mathbb{R}^3 \) that simultaneously divides each \( A_i \) into two pieces of equal measure.

Proof. Let \( a \in A_1 \) and let \( S^2 \) be a sphere centered at \( a \). First, we will identify a plane for each \( s \in S^2 \). Then we will use the Borsuk-Ulam theorem to show that one of these planes divides each \( A_i \) in half.

Step one: For each \( s \in S^2 \) there are a family of planes \( \{P_s^2\}_{s \in S^2} \) containing \( a \) and \( s \). Note that this is the same family of planes as \( \{P_{-s}^2\}_{s \in S^2} \) where \( -s \) is the antipodal point of \( s \).

We continuously identify the sides of a plane \( \alpha \) and \( \beta \) so that \( \alpha s = \beta_{-s} \) and \( \beta s = \alpha_{-s} \) for all \( P \in \{P_s^2\}_{s \in S^2} \) and \( P_{-s}^2 \). In other words, for a plane through \( -s \), \( a \), \( s \), the \( \alpha / \beta \) labels are reversed with respect to \( s \) and \( -s \).

Define \( \alpha_{ps}(A_i) \) to be the measure of \( A_i \) on the \( \alpha \)-side of \( P^s \), and define \( \beta_{ps}(A_i) \) to be the measure of \( A_i \) on the \( \beta \)-side of \( P^s \). Now, define

\[
M: \{P_s^2\}_{s \in S^2} \to \mathbb{R}, \quad P \mapsto \alpha_p(A) - \beta_p(A).
\]

This map is continuous because planes from the same family that are very close together will split the measure of \( A_i \) in very similar ways. Let \( P \in \{P_s^2\}_{s \in S^2} \) and let \( Q \) be \( P \) span \( 180^\circ \) about \( a \). Then \( Q \in \{P_s^2\}_{s \in S^2} \). Note that

\[
M(P) = \alpha_p(A) - \beta_p(A) = -\left(\beta_p(A) - \alpha_p(A)\right) = -\left(\alpha_q(A) - \beta_q(A)\right) \quad \forall \quad P \in \{P_s^2\}_{s \in S^2}.
\]

So by the intermediate value theorem, \( M(R) = 0 \) for some \( R \in \{P_s^2\}_{s \in S^2} \). That is, some plane through \( a \) and \( s \) cuts \( A_i \) in half. By the axiom of choice, choose one such plane for each \( s \in S^2 \) and label it \( P^s \).
Step two: Now that we have a plane $P^x$ corresponding to every $x \in S^2$, we define

$$\mu : S^2 \to \mathbb{R}^2, \quad \begin{pmatrix} \alpha_x(A_z) - \beta_x(A_z) \\ \alpha_x(A_z) - \beta_x(A_z) \end{pmatrix}$$

This is continuous for the same reason that $M$ was continuous.

By the Borsuk-Ulam theorem, $\mu(x) = \mu(-x)$ for some $x \in S^2$.

This means that both the first coordinate and second coordinate of $\mu(x)$ and $\mu(-x)$ are equal, in the first case,

$$\alpha_{px}(A_z) - \beta_{px}(A_z) = \alpha_{px}(A_z) - \beta_{px}(A_z)$$

$$= \beta_{px}(A_z) - \alpha_{px}(A_z).$$

This implies that $\alpha_{px}(A_z) = \beta_{px}(A_z)$ and thus $P^x$ cuts $A^2$ in half. In the second case, similarly,

$$\alpha_{px}(A_z) - \beta_{px}(A_z) = \alpha_{px}(A_z) - \beta_{px}(A_z)$$

$$= \beta_{px}(A_z) - \alpha_{px}(A_z).$$

and so $\alpha_{px}(A_z) = \beta_{px}(A_z)$. Hence, $P^x$ cuts $A_3$ in half as well.

Since $P^x$ was chosen because it cuts $A_i$ in half, we can conclude that $P^x$ simultaneously divides each $A_i$ into two pieces of equal measure. □
10) Let $X$ and $Y$ be topological spaces and $(x_0,y_0) \in X \times Y$ such that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1((x_0, y_0)) \cong \pi_1(Y, y_0)$. Let $f$ be a loop contained in $X \times \{y_0\}$ and let $g$ be a loop contained in $\{x_0\} \times Y$. We will show that $f \cdot g \cong g \cdot f$.

Define
\[ f_\varepsilon(s) = \begin{cases} 
  x_0 & 0 \leq s \leq \varepsilon \\
  f(2s) & \frac{\varepsilon}{2} \leq s \leq 1-rac{\varepsilon}{2} \\
  x_0 & \frac{\varepsilon}{2} \leq s \leq 1.
\end{cases} \]

Observe that $f_\varepsilon(s)$ comprises the entire loop $f$ from $0 \leq s \leq \frac{\varepsilon}{2}$ and then $x_0$ for $\frac{\varepsilon}{2} \leq s \leq 1$. Thus, $f_\varepsilon(s) \cong f(s)$. Also $f_\varepsilon(s)$ remains at $x_0$ for $0 \leq s \leq \frac{\varepsilon}{2}$ and then traverses $f$ from $\frac{\varepsilon}{2} \leq s \leq 1$. Thus, $f_\varepsilon(s) \cong f(s)$.

Define
\[ g_\varepsilon(s) = \begin{cases} 
  x_0 & 0 \leq s \leq \frac{1-\varepsilon}{2} \\
  g(2s) & \frac{1-\varepsilon}{2} \leq s \leq 2-\frac{\varepsilon}{2} \\
  x_0 & 2-\frac{\varepsilon}{2} \leq s \leq 1.
\end{cases} \]

Note that $g_\varepsilon(s)$ is composed of $x_0$ for $0 \leq s \leq \frac{\varepsilon}{2}$ and then the loop $g$ for $\frac{\varepsilon}{2} \leq s \leq 1$. Hence, $g_\varepsilon(s) \cong g(s)$. Furthermore, $g_\varepsilon(s)$ is composed of $g$ for $0 \leq s \leq \frac{\varepsilon}{2}$ and then $x_0$ for $\frac{\varepsilon}{2} \leq s \leq 1$. Hence, $g_\varepsilon(s) \cong g(s)$.

Via the isomorphism from the hypothesis, a loop from $X \times \{y_0\}$ and a loop from $\{x_0\} \times Y$ determine a loop in $X \times Y$. Consider $H_\varepsilon(s) = (f_\varepsilon(s), g_\varepsilon(s))$. We have $H_\varepsilon(s) = (f_\varepsilon(s), g_\varepsilon(s)) \cong f \cdot g$ and $H_\varepsilon(s) = (f_\varepsilon(s), g_\varepsilon(s)) \cong g \cdot f$.

Moreover, $H_\varepsilon(s)$ is continuous because of the isomorphism and since $f_\varepsilon(s)$ and $g_\varepsilon(s)$ are continuous. Therefore,
\[
 f \cdot g \cong g \cdot f.
\]
16. We will show that no retractions \( \gamma : X \to A \) exist in the following situations. Recall the proposition from class that states \( i_* : \pi_1(A, a_0) \to \pi_1(X, x_0), [\gamma] \to [i(\gamma)] \) is injective whenever a retraction exists from \( X \) onto \( A \).

(a) \( X = \mathbb{R}^3 \) and \( A \cong S' \). Note that \( \mathbb{R}^3 \) is simply-connected, so \( \pi_1(X, x_0) = \{e\} \) where \( e \) is the constant loop. Thus, \( i_* \) maps all elements of \( \pi_1(A, a_0) \) to \( [e] \). But \( \pi_1(S', a_0) \cong \mathbb{Z} \), and has more than one element, so \( i_* \) is not injective. This means no retraction exists.

(b) \( X = S' \times D^2 \) and \( A = S' \times S' \). Since \( S' \) and \( D^2 \) are path-connected, \( \pi_1(X) \cong \pi_1(S') \times \pi_1(D^2) \cong \mathbb{Z} \times \mathbb{Z} \) and also \( \pi_1(A) \cong \pi_1(S') \times \pi_1(S') \cong \mathbb{Z} \times \mathbb{Z} \). The homomorphism induced by the inclusion map \( i : A \to X \) is \( i_* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \), which is clearly not injective. More geometrically, consider \([f] \in \pi_1(A)\) — the class of loops that go once around the torus hole — and \([g] \in \pi_1(A)\) — the class of loops that go once around the torus hole and once around its thickness. When \([f]\) and \([g]\) are included into \( X \) the loop of \([g]\) around the thickness can be homotoped through the simply-connected \( D^2 \) to the basepoint. Hence, \([i(f)] = [i(g)]\) and \( i_* \) is not injective. This implies no retraction exists.

(c) \( X = S' \times D^2 \) and \( A \) as shown.
It is given that $A \cong S^1$. So $\pi_1(A) \cong \mathbb{Z}$. Let $f$ be a loop in $A$ based at $a_0 \in A$. Now consider $i(f)$, a loop in $X$ based at $a_0 \in X$. Since $A$ does not encircle the hole of the solid torus, neither does $i(f)$. Thus, $i(f)$ can be homotoped to $c$, the constant loop at $a_0$. Since $f$ was arbitrary and $\pi_1(A)$ is non-trivial, it follows that $i_*$ is not injective. So there is no retraction from $X$ to $A$.

(d) $X = D^2 \vee D^2$ and $A = S^1 \vee S^1$. Recall that the wedge sum is formed by taking the disjoint union and identifying one point from each part to a single point.

Let $a_0$ be the wedge point of both $A$ and $X$. Then $\pi_1(A, a_0)$ is non-trivial because loops that completely circle one side or the other are not homotopic to $c$. However, $\pi_1(X, a_0)$ is trivial because all loops can be homotoped through either disk to $c$. This implies that $i_*$ again is not injective.

(e) $X$ is a disk with two boundary points identified and $A$ is its boundary $S^1 \vee S^1$.

I believe $\pi_1(X) \cong \mathbb{Z}$ and $\pi_1(A) \cong \mathbb{Z} \times \mathbb{Z}$ with $i_* : \pi_1(A) \to \pi_1(X)$ defined by $i_*(m, n) = m+n$. If this is the case, then $i_*$ is clearly not injective because, for instance, $i_*(1, 1) = i_*(0, 2) = 2$. But for good measure, here is a homotopic justification.
Consider the loops $f$ and $g$ in $A$. Obviously, $f \not\cong g$ in $A$.

\[ \infty \quad \text{f} \quad \quad \quad \infty \quad \text{g} \]

However, in $X$, $f$ and $g$ are homotopic via the figure:

\[ \text{t=0} \quad \text{t=1} \]

Hence, $[f(t)] = [g(t)]$ and $i_\ast$ is not injective.

(F) $X$ is the Möbius band and $A$ is its boundary circle.

We know $\pi_1(A) \cong \mathbb{Z}$. Since the Möbius band can be deformation-retracted to its midline circle, $\pi_1(X) \cong \pi_1(S^1) \cong \mathbb{Z}$. So there exist isomorphisms $\phi_A : \pi_1(A) \to \mathbb{Z}$ and $\phi_X : \pi_1(X) \to \mathbb{Z}$.

Let $F$ be a loop that goes entirely around $A$ $n$ times. Then $i(F)$ is homotopic to a loop that goes around $X$ $2n$ times. Hence $\phi_A(F) = n$ and $\phi_X(i(F)) = 2n$. Suppose $r : X \to A$ is a retraction.

Then $r(i(F)) = F$ because $r$ is the identity on $A$. So by isomorphism $r(2n) = n$. Since $r$ is a homomorphism,

\[ n = r(2n) = r(n + n) = r(n) + r(n) = 2r(n). \]

It follows that $r(n) = \frac{n}{2}$. But $\frac{n}{2} \not\in \mathbb{Z}$ for all $n \in \mathbb{Z}$, so $r$ cannot exist. Therefore, the Möbius band does not retract to its boundary circle.