Section 1.3 (p. 79-80)

Let $X \subset \mathbb{R}^3$ be the union of a sphere and a diameter. We will construct a simply-connected covering space of $X$.

Using the theorem from class (3/7) and the fact that $\pi_1(X) = \mathbb{Z}$, we know $X$ will need countably many copies of each cell in $X$.

Below is a picture of $	ilde{X}$ where the covering map $p$ naturally sends the surface to the surface of $X$ and the segments to the diameter:

![Diagram of a covering space]

We know $\tilde{X}$ is simply-connected because it is the wedge sum of simply-connected components (line segments and $S^1$). We must check that $p$ is indeed a covering map.

Consider the open cover of $X$ consisting of an open neighborhood about the interior of $ab$ and an open neighborhood about the sphere.

![Diagram of the open cover]

We see that $p^{-1}$ of each open set is a disjoint union of open sets in $\tilde{X}$. Moreover, each preimage is homeomorphic to its image under $p$.

So $\tilde{X}$ is a simply-connected covering space of $X$. 

\[ \text{Homework 5} \]

\[ \frac{20 + 3}{20} \text{ extra points} \]

Due: 2/14
Let $Y \subseteq \mathbb{R}^3$ be the union of a sphere and a circle intersecting it in two points. We will construct a simply-connected covering space of $Y$.

As before, we use the theorem (2/7) to construct $\tilde{Y}$ pictured below where $q$ is the obvious covering map:

$\tilde{Y}$ is simply-connected because it is the wedge sum of line segments and copies of $S^2$. We must check that $q$ is a covering map. This follows from the consideration of the open cover of three small neighborhoods — one about the sphere, one about $a+b$ and one about $b+a$.

Since the preimages of each open set under $q$ is a disjoint union of open sets in $\tilde{Y}$ that each map homeomorphically to the image in $Y$ under $q$, $\tilde{Y}$ is a simply-connected covering space of $Y$. 
Proposition: Let $X$ be path-connected, locally path-connected, with $\pi_1(X)$ finite. Then every map $X \to S^1$ is nullhomotopic.

Proof: Based on the hint, we will use $p: \mathbb{R} \to S^1$ the standard universal cover of the circle.

Let $f: X \to S^1$ be any (continuous) map. This map induces a homomorphism $f_*: \pi_1(X) \to \pi_1(S^1)$. Thus $f_*(\pi_1(X)) \subset \pi_1(S^1) \cong \mathbb{Z}$.

Since $\pi_1(X)$ is finite, its image under $f_*$ must also be finite. So $f_*(\pi_1(X))$ is a finite subgroup of $\mathbb{Z}$.

Claim: the only finite subgroup of $\mathbb{Z}$ is $\{0\}$. Suppose to the contrary that the finite subgroup contains $0 \neq a \in \mathbb{Z}$. Then by closure $\langle a \rangle$ is contained in the subgroup. But $\langle a \rangle$ is infinite — a contradiction.

So $f_*(\pi_1(X)) = \{0\}$. Therefore $f_*(\pi_1(X)) \subset p_*(\pi_1(\mathbb{R}))$ and by Lemma 1 we have a lift $\tilde{f}: X \to \mathbb{R}$.

Since $\mathbb{R}$ is simply-connected, $\tilde{f}$ is nullhomotopic. By composing this nullhomotopy with $p$ we conclude that $f$ is nullhomotopic. \qed
11) Consider the graphs below:

\[ X_1 \quad \text{and} \quad X_2 \]

They are both covered by the same space \( \tilde{X} \). Observe:

\[ \tilde{X}_1 \cong \tilde{X}_2 \]

For the open covers, we would use a small neighborhood around each vertex and then a neighborhood around the interior of each edge. So in the case of \( X_1 \), all preimages of vertex neighborhoods are:

\[ \tilde{x}_1 \text{ or } \tilde{x}_2 \]

which are disjoint and homeomorphic.

Similarly for \( X_2 \),

\[ \tilde{x}_2 \text{ or } \tilde{x}_3 \]

However, it will always be the case for any graphs \( X_1 \) and \( X_2 \) that they both cover a common space, namely, \( X_1 \sqcup X_2 \).

\( X_1 \) covers \( X_1 \sqcup X_2 \) via the inclusion map, and \( X_2 \) covers \( X_1 \sqcup X_2 \) via its inclusion map. (There is no requirement that covering maps be surjective according to Hatcher, so the preimage of the other component of the disjoint union is \( \emptyset \).) Neither does the problem statement require connectivity.
Additional Problem: We know that \( \pi_1(\mathbb{R}P^3 \# \mathbb{R}P^3) \cong \langle a, b \mid a^2 = b^2 = 1 \rangle \).

This group contains a subgroup of index two, namely, \( H = \langle ab \rangle \cong \mathbb{Z} \).

We will show that the double cover corresponding to this subgroup is \( S^2 \times S^1 \).

View \( \mathbb{R}P^3 \) as a solid sphere with antipodal points identified. If we remove a 3-ball from the interior of two 3D projective planes and attach along the boundaries of the balls, then we are essentially placing one \( \mathbb{R}P^3 \) into the "ball-shaped hole" in the interior of the other \( \mathbb{R}P^3 \).

Since \( S^1 \) is homeomorphic to \([0, 1]\) with \( 0 \sim 1 \), we will define \( p : S^2 \times I \to \mathbb{R}P^3 \# \mathbb{R}P^3 \) in the following way: a point \( x \in S^2 \) corresponds with a point on the surface (prior to identification) in \( \mathbb{R}P^3 \), and thus defines a diameter \( \gamma_x \) in \( \mathbb{R}P^3 \). Let \( p(x, t) = \gamma_x(t) \); that is, \( p(x, t) \) is the point at position \( t \) of the path \( \gamma_x \) from antipodal \(-x\) of \( \mathbb{R}P^3 \) to \( x \). O.K.

This is a covering map because a neighborhood in \( \mathbb{R}P^3 \# \mathbb{R}P^3 \) results from a "tight bundle" of diameters and a small interval within \( I \). Thus the preimage under \( p \) is a neighborhood of \( S^2 \) over similar time intervals in \( I \) and so is homeomorphic to the original neighborhood. Moreover, note that \( p(x, 0) = p(x, 1) \) by the identification of \( \mathbb{R}P^3 \) but also \( p(x, t) = p(-x, 1-t) \) by the definition of \( p \). So each neighborhood in \( \mathbb{R}P^3 \# \mathbb{R}P^3 \) has two preimage open sets in \( S^2 \times I \). Therefore, we see that \( S^2 \times I \cong S^2 \times S^1 \) is a double cover of \( \mathbb{R}P^3 \# \mathbb{R}P^3 \).

Why does it correspond to \( H \)? (1)